Bashir Ahmad · Ahmed Alsaedi
Sotiris K. Ntouyas · Jessada Tariboon

Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities

Springer

Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities

Bashir Ahmad • Ahmed Alsaedi • Sotiris K. Ntouyas
Jessada Tariboon

Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities

Springer

Bashir Ahmad
Department of Mathematics
King Abdulaziz University
Jeddah, Saudi Arabia

Ahmed Alsaedi
Department of Mathematics
King Abdulaziz University
Jeddah, Saudi Arabia

Sotiris K. Ntouyas
Department of Mathematics
University of Ioannina
Ioannina, Greece

Jessada Tariboon
Department of Mathematics
King Mongkut's University of Technology
North Bangkok
Bangkok, Thailand

ISBN 978-3-319-52140-4 ISBN 978-3-319-52141-1 (eBook)
DOI 10.1007/978-3-319-52141-1

Library of Congress Control Number: 2016963216

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

The recent studies on fractional differential equations indicate that a variety of interesting and important results concerning existence and uniqueness of solutions, stability properties of solutions, and analytic and numerical methods of solutions for these equations have been obtained, and the surge for investigating more and more results is underway [36, 37]. The tools of fractional calculus have played a significant role in improving the modeling techniques for several real-world problems. Nowadays, fractional-order differential equations appear extensively in a variety of applications such as diffusion processes, chaos, thermo-elasticity, biomathematics, fractional dynamics, etc. [87, 118, 142, 183, 187]. One of the characteristics of operators of fractional order is their nonlocal nature accounting for the hereditary properties of many phenomena and processes involved. For the recent development of the topic, we refer the reader to a series of books and papers [1, 6, 8, 9, 40, 41, 91, 96, 114, 121, 133, 141, 180]. However, it has been noticed that most of the work on the topic is based on Riemann-Liouville, and Caputo-type fractional differential equations. Another kind of fractional derivatives that appears side by side to Riemann-Liouville and Caputo derivatives in the literature is the fractional derivative due to Hadamard, introduced in 1892 [89], which contains logarithmic function of arbitrary exponent in the kernel of the integral appearing in its definition. Hadamard-type integrals arise in the formulation of many problems in mechanics such as in fracture analysis. For details and applications of Hadamard fractional derivative and integral, we refer the reader to the works in [51–53, 94, 96–98].

The main idea for writing this book is to focus on the recent development of fractional differential equations, integrodifferential equations, and inclusions and inequalities involving Hadamard derivative and integral. In precise terms, we address the issues related to initial and boundary value problems involving Hadamard-type differential equations and inclusions as well as their functional counterparts. Much of the material presented in this book is based on the recent research of the authors on the topic.

The book is organized as follows. Chapter 1 contains fundamental concepts of multivalued analysis, differential inclusions, and Hadamard fractional calculus. We also describe a number of fixed-point theorems used to establish the existence results for the proposed problems. Included among the fixed-point theorems recognized by their names are Amini-Harandi, Boyd and Wong, Covitz and Nadler, Dhage, Guo-Krasnosel'skii, Krasnosel'skii, Krasnosel'skii-Zebreiko, Leggett-Williams, Leray-Schauder nonlinear alternative for single and multivalued maps, O'Regan, Petryshyn, and Sadovski.

Chapter 2 is devoted to the study of existence of solutions for initial and boundary value problems of fractional-order Hadamard-type functional and neutral functional differential equations and inclusions with both retarded and advanced arguments.

The objective of Chapter 3 is to investigate fractional integral boundary value problems involving Hadamard fractional derivative and integral for nonlocal fractional differential equations and inclusions. We establish some existence and uniqueness results for the given problems by means of classical fixed-point theorems.

In Chapter 4, we introduce a new class of mixed initial value problems involving Hadamard derivative and Riemann-Liouville fractional integrals. Existence and uniqueness results for the given problems are obtained with the help of standard fixed-point theorems. The purpose of Chapter 5 is to study nonlocal boundary value problems of Riemann-Liouville fractional differential equations and inclusions equipped with Hadamard fractional integral boundary conditions. In Chapter 6, we switch onto the study of coupled systems of Hadamard- and Riemann-Liouville-type fractional differential equations with coupled and uncoupled nonlocal Hadamard fractional boundary conditions.

Chapter 7 studies nonlinear Langevin equations and inclusions involving Hadamard-Caputo-type fractional derivatives with nonlocal fractional integral conditions. Then we extend our study to coupled systems of Langevin equation with fractional integral conditions. In Chapter 8, we investigate a nonlinear boundary value problem of impulsive hybrid multi-orders Caputo-Hadamard fractional differential equations with nonlinear integral boundary conditions. In Chapter 9, we study the existence of solutions for initial and boundary value problems of hybrid fractional differential equations and inclusions of Hadamard type. In Chapter 10, we develop some fractional integral inequalities using the Hadamard fractional integral. Several new integral inequalities are obtained by using Young and weighted AM-GM inequalities. Many special cases are also discussed. Moreover, a Grüss-type Hadamard fractional integral inequality is obtained. Chapter 11 is concerned with the existence criteria of positive solutions for fractional differential equations of Hadamard type with integral boundary condition on infinite intervals.

We are grateful to the senior editor, Marc Strauss, for his continued support and encouragement during the preparation of this text.

Jeddah, Saudi Arabia — Bashir Ahmad
Jeddah, Saudi Arabia — Ahmed Alsaedi
Ioannina, Greece — Sotiris K. Ntouyas
Bangkok, Thailand — Jessada Tariboon
November 2016

Contents

Chapter 1
Preliminaries

1.1 Definitions and Results from Multivalued Analysis

In this section, we introduce notations, definitions and preliminary facts from multivalued analysis, which are used throughout this book.

For a normed space $(X, \|\cdot\|)$, let

$$\begin{aligned}
\mathscr{P}_{cl}(X) &= \{Y \in \mathscr{P}(X) : Y \text{ is closed}\},\\
\mathscr{P}_{b}(X) &= \{Y \in \mathscr{P}(X) : Y \text{ is bounded}\},\\
\mathscr{P}_{cp}(X) &= \{Y \in \mathscr{P}(X) : Y \text{ is compact}\},\\
\mathscr{P}_{cp,c}(X) &= \{Y \in \mathscr{P}(X) : Y \text{ is compact and convex}\}, \text{ and}\\
\mathscr{P}_{cl,b}(X) &= \{Y \in \mathscr{P}(X) : Y \text{ is closed and bounded}\}.
\end{aligned}$$

A multivalued map $G : X \to \mathscr{P}(X)$:

(1) is *convex (closed) valued* if $G(x)$ is convex (closed) for all $x \in X$;
(2) is *bounded* on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in \mathscr{P}_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$);
(3) is called *upper semi-continuous (u.s.c.)* on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathscr{N}_0$ of x_0 such that $G(\mathscr{N}_0) \subseteq N$;
(4) G is *lower semi-continuous (l.s.c.)* if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E;
(5) is said to be *completely continuous* if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathscr{P}_b(X)$;

B. Ahmad et al., *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, DOI 10.1007/978-3-319-52141-1_1

(6) is said to be *measurable* if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable;

(7) *has a fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$.

Definition 1.1 A multivalued map $F : J \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \longmapsto F(t, x)$ is measurable for each $x \in \mathbb{R}$;
(ii) $x \longmapsto F(t, x)$ is upper semicontinuous for almost all $t \in J$;

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \leq \varphi_\alpha(t)$$

for all $\|x\| \leq \alpha$ and for a.e. $t \in J$.

For each $x \in \mathscr{C}$, define the set of selections of F by

$$S_{F,x} := \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, x(t)) \text{ for a.e. } t \in J\}.$$

We define the graph of G to be the set $Gr(G) = \{(x, y) \in X \times Y, y \in G(x)\}$ and recall two useful results regarding closed graphs and upper-semicontinuity.

Lemma 1.1 ([69, Proposition 1.2]) *If $G : X \to \mathscr{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 1.2 ([108]) *Let X be a Banach space. Let $F : [0, T] \times \mathbb{R} \to \mathscr{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$. Then the operator*

$$\Theta \circ S_{F,x} : C(J, X) \to \mathscr{P}_{cp,c}(C(J, X)), \quad x \mapsto (\Theta \circ S_{F,x})(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

Let A be a subset of $J \times \mathbb{R}$. A is $\mathscr{L} \otimes \mathscr{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathscr{J} \times \mathscr{D}$, where $\mathscr{J}$ is Lebesgue measurable in J and $\mathscr{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathscr{A}$ of $L^1(J, \mathbb{R})$ is decomposable if for all $x, y \in \mathscr{A}$ and measurable $\mathscr{J} \subset J = I$, the function $x\chi_{\mathscr{J}} + y\chi_{I-\mathscr{J}} \in \mathscr{A}$, where $\chi_{\mathscr{J}}$ stands for the characteristic function of $\mathscr{J}$.

Definition 1.2 Let Y be a separable metric space and let $N : Y \to \mathscr{P}(L^1(J, \mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semicontinuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F : J \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathscr{F} : PC(J \times \mathbb{R}) \to \mathscr{P}(L^1(J, \mathbb{R}))$ associated with F as

$$\mathscr{F}(x) = \{w \in L^1(J, \mathbb{R}) : w(t) \in F(t, x(t)) \text{ for a.e. } t \in J\},$$

which is known as the Nemytskii operator associated with F.

Definition 1.3 Let $F : J \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathscr{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Lemma 1.3 ([50]) *Let Y be a separable metric space and let $N : Y \to \mathscr{P}(L^1(J, \mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1(J, \mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

For more details on multivalued analysis, we refer to the books of Deimling [69], Gorniewitz [85], Hu and Papageorgiou [93] and Tolstonogov [165].

1.2 Definitions and Results from Fractional Calculus

Definition 1.4 The Riemann-Liouville fractional integral of order $q > 0$ with the lower limit zero for a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$_{RL}I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$, where $\Gamma(\cdot)$ denotes the Euler Gamma function defined by $\Gamma(q) = \int_0^\infty e^{-s} s^{q-1} ds$.

Definition 1.5 The Riemann-Liouville fractional derivative of order $q > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$_{RL}D^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} f(s) ds, \quad n-1 < q < n,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Lemma 1.4 *Let $b, q > 0$ and $x \in C(0, b) \cap L^1(0, b)$. Then the general solution of fractional differential equation*

$$_{RL}D^q x(t) = 0$$

is

$$x(t) = c_1 t^{q-1} + c_2 t^{q-2} + \ldots + c_n t^{q-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and $n - 1 < q < n$.

Lemma 1.5 *Let $q > 0$ and $x \in C(0, b) \cap L^1(0, b)$. Then*

$$_{RL}I^q {}_{RL}D^q x(t) = x(t) + c_1 t^{q-1} + c_2 t^{q-2} + \ldots + c_n t^{q-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and $n - 1 < q < n$.

Definition 1.6 The Hadamard fractional integral of order $q \in \mathbb{R}^+$ of a function $f \in L^p[a, b]$, $0 \leq a \leq t \leq b \leq \infty$, is defined as

$$_H I^q f(t) = \frac{1}{\Gamma(q)} \int_a^t \left(\log \frac{t}{s}\right)^{q-1} f(s) \frac{ds}{s}.$$

Definition 1.7 Let $0 < a < b < \infty$, $\delta = t\frac{d}{dt}$ and $AC_\delta^n[a, b] = \{f : [a, b] \to \mathbb{R} : \delta^{n-1}[f(t)] \in AC[a, b]\}$. The Hadamard derivative of fractional order q for a function $f \in AC_\delta^n[a, b]$ is defined as

$$_H D^q f(t) = \delta^n (I^{n-q})(t) = \frac{1}{\Gamma(n-q)} \left(t\frac{d}{dt}\right)^n \int_a^t \left(\log \frac{t}{s}\right)^{n-q-1} \frac{f(s)}{s} ds,$$

where $n - 1 < q < n, n = [q] + 1$, $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

Recall that the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral in the space $L^p[a, b]$, $1 \leq p \leq \infty$, that is, $_H D^q {}_H I^q g(t) = g(t)$ (Theorem 4.8, [97]).

In [94], Caputo-type modification of the Hadamard fractional derivatives was proposed as follows:

$$^C\mathscr{D}^q g(t) = {}_H D^q \Big[g(s) - \sum_{k=0}^{n-1} \frac{\delta^k g(a)}{k!} \Big(\log \frac{s}{a} \Big)^k \Big](t), \ t \in (a, b). \tag{1.1}$$

Further, it was shown in (Theorem 2.1, [94]) that $^C\mathscr{D}^q g(t) = {}_H I^{n-q} \delta^n g(t)$. For $0 < q < 1$, it follows from (1.1) that

$$ {}^{C}\mathscr{D}^{q}g(t) = {}_{H}D^{q}\Big[g(s) - g(a)\Big](t). $$

Furthermore, it was established in Lemmas 2.4 and 2.5 of [94] respectively that

$$ {}^{C}\mathscr{D}^{q}({}_{H}I^{q}g)(t) = g(t), \ \ {}_{H}I^{q}({}^{C}\mathscr{D}^{q})g(t) = g(t) - \sum_{k=0}^{n-1} \frac{\delta^{k}g(a)}{k!}\Big(\log\frac{t}{a}\Big)^{k}. \tag{1.2} $$

From the second formula in (1.2), one can easily infer that the solution of Hadamard differential equation: ${}^{C}\mathscr{D}^{q}u(t) = \sigma(t)$ can be written as

$$ u(t) = {}_{H}I^{q}\sigma(t) + \sum_{k=0}^{n-1} \frac{\delta^{k}u(a)}{k!}\Big(\log\frac{t}{a}\Big)^{k}, $$

for appropriate function $u(t)$ and $\sigma(t)$ (as required in the above definitions).

Note that the Hadamard integral and derivative defined above are left-sided. One can define the Hadamard right-sided integral and derivative in the same way, for instance, see [94].

Lemma 1.6 ([96, p. 113]) *Let $q > 0$ and $\beta > 0$. Then the following formulas*

$$ {}_{H}I^{q}t^{\beta} = \beta^{-q}t^{\beta} \quad \textit{and} \quad {}_{H}D^{q}t^{\beta} = \beta^{q}t^{\beta} $$

hold.

For Hadamard fractional integrals, the semigroup property holds:

$$ {}_{H}I^{\alpha}{}_{H}I^{\beta}f(t) = {}_{H}I^{\alpha+\beta}f(t), \quad \alpha \geq 0, \ \beta \geq 0, $$

which leads to the commutative property:

$$ {}_{H}I^{\alpha}{}_{H}I^{\beta}f(t) = {}_{H}I^{\beta}{}_{H}I^{\alpha}f(t). $$

Lemma 1.7 ([96, Property 2.24]) *If $a, \alpha, \beta >, 0$ then*

$$ \left({}_{H}D^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{x}{a}\right)^{\beta-\alpha-1}, $$

$$ \left({}_{H}I^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\log\frac{x}{a}\right)^{\beta+\alpha-1}. $$

Lemma 1.8 ([96]) *Let $q > 0$ and $x \in C[1,\infty) \cap L^1[1,\infty)$. Then the solution of Hadamard fractional differential equation ${}_HD^q x(t) = 0$ is given by*

$$x(t) = \sum_{i=1}^{n} c_i (\log t)^{q-i},$$

and the following formula holds:

$${}_HI^q{}_HD^q x(t) = x(t) + \sum_{i=1}^{n} c_i (\log t)^{q-i},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and $n-1 < q < n$.

Remark 1.1 In the subsequent work, we denote by $\mathscr{C}^n[a,b]$ the space of functions $x(t)$, which have continuous δ-derivative ($\delta = t\frac{d}{dt}$) of order $n-1$ on $[a,b]$ and $\delta^n x(t)$ on $[a,b]$ such that $\delta^n x(t) \in C[a,b]$.

The theoretical development of fractional calculus and fractional differential equations has deeply been given in excellent monographs, for instance, by Miller and Ross [121], Podlubny [141], Kilbas et al. [96], Lakshmikantham et al. [107], Diethelm [75] and Samko et al. [145].

1.3 Fixed Point Theorems

Fixed point theorems play a major role in establishing the existence theory for initial and boundary value problems. We collect here some well-known fixed point theorems used in this book.

Theorem 1.1 (Contraction Mapping Principle [70]) *Let E be a Banach space, $D \subset E$ be closed and $F : D \to D$ a strict contraction, i.e. $|Fx - Fy| \le k|x - y|$ for some $k \in (0,1)$ and all $x, y \in D$. Then F has a unique fixed point.*

Theorem 1.2 (Krasnoselskii's Fixed Point Theorem [101]) *Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that (a) $Ax + By \in M$ whenever $x, y \in M$; (b) A is a compact and continuous; (c) B is a contraction mapping. Then there exists $z \in M$ such that $z=Az+Bz$.*

Theorem 1.3 (Leray-Schauder Alternative [86], p. 4) *Let $F : E \to E$ be a completely continuous operator (i.e., a map restricted to any bounded set in E is compact). Let*

$$\mathscr{E}(F) = \{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}.$$

Then either the set $\mathscr{E}(F)$ is unbounded, or F has at least one fixed point.

Next, we state the Leray-Schauder's nonlinear alternative. By $\bar{U}$ and ∂U, we denote the closure and the boundary of U, respectively.

Theorem 1.4 (Nonlinear Alternative for Single-Valued Maps [86]) *Let E be a Banach space, C be a closed, convex subset of E, U be an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) F has a fixed point in $\overline{U}$, or
(ii) there is a $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda F(u)$.

Theorem 1.5 ([153]) *Suppose that $A : \bar{\Omega} \to E$ is a completely continuous operator. If one of the following conditions is satisfied:*

(i) (Altman) $\|Ax - x\|^2 \geq \|Ax\|^2 - \|x\|^2$, for all $x \in \partial\Omega$,
(ii) (Rothe) $\|Ax\| \leq \|x\|$, for all $x \in \partial\Omega$,
(iii) (Petryshyn) $\|Ax\| \leq \|Ax - x\|$, for all $x \in \partial\Omega$,

then $\deg(I - A, \Omega, \theta) = 1$, and hence A has at least one fixed point in Ω.

The next fixed point theorem is due to O'Regan.

Theorem 1.6 ([131]) *Denote by O an open set in a closed, convex set K of a Banach space X. Assume that $0 \in O$. Also assume that $F(\bar{O})$ is bounded and that $F : \bar{O} \to K$ is given by $F = F_1 + F_2$, in which $F_1 : \bar{O} \to K$ is continuous and completely continuous and $F_2 : \bar{O} \to K$ is nonlinear contraction (that is, there exists a nonnegative nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$, such that $\|F_2x - F_2y\| \leq \phi(\|x - y\|)$ for all $x, y \in O$). Then, either*

(C1) F has a fixed point $u \in \bar{O}$; or
(C2) there exist a point $u \in \partial O$ and $\lambda \in (0, 1)$ with $u = \lambda F(u)$, where $\bar{O}$ and ∂O, respectively, represent the closure and boundary of O.

Following is a hybrid fixed point theorem for two operators in a Banach algebra due to Dhage.

Theorem 1.7 ([73]) *Let S be a nonempty, closed convex and bounded subset of the Banach algebra E and let $A : E \to E$ and $B : S \to E$ be two operators satisfying:*

(a) *A is Lipschitzian with Lipschitz constant δ,*
(b) *B is completely continuous,*
(c) *$x = AxBy \Rightarrow x \in S$ for all $y \in S$,*
(d) *$\delta M < 1$, where $M = \|B(S)\| = \sup\{\|B(x)\| : x \in S\}$.*

Then the operator equation $x = AxBx$ has a solution in S.

Theorem 1.8 ([73]) *Let X be a Banach algebra and let $A : X \to X$ be a single valued operator and $B : X \to \mathscr{P}_{cp,c}(X)$ be a multivalued operator satisfying:*

(a) A is single-valued Lipschitz with a Lipschitz constant k,
(b) B is compact and upper semi-continuous,
(c) $2Mk < 1$, where $M = \|B(X)\|$.

Then either

(i) *the operator inclusion* $x \in AxBx$ *has a solution, or*
(ii) *the set* $\mathscr{E} = \{u \in X | \mu u \in AuBu,\ \mu > 1\}$ *is unbounded.*

Theorem 1.9 ([74]) *Let* M *be a non-empty, closed, convex and bounded subset of the Banach space* X *and let* $A : X \to X$ *and* $B : M \to X$ *be two operators such that*

(i) A *is a contraction,*
(ii) B *is completely continuous, and*
(iii) $x = Ax + By$ *for all* $y \in M \ \Rightarrow\ x \in M$.

Then the operator equation $Ax + Bx = x$ *has a solution.*

Theorem 1.10 (Krasnoselskii-Zabreiko's Fixed Point Theorem [102]) *Let* $(X, \|\cdot\|)$ *be a Banach space, and* $\mathscr{K} : X \to X$ *be a completely continuous operator. Assume that* $\mathscr{L} : X \to X$ *is a bounded linear operator such that* 1 *is not an eigenvalue of* $\mathscr{L}$ *and*

$$\lim_{\|x\|\to\infty} \frac{\|\mathscr{K}x - \mathscr{L}x\|}{\|x\|} = 0.$$

Then $\mathscr{K}$ *has a fixed point in* X.

Definition 1.8 Let E be a Banach space and let $\mathscr{A} : E \to E$ be a mapping. $\mathscr{A}$ is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Psi(0) = 0$ and $\Psi(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$ with the property:

$$\|\mathscr{A}x - \mathscr{A}y\| \le \Psi(\|x - y\|), \qquad \forall x, y \in E.$$

Theorem 1.11 (Boyd and Wong [49]) *Let* E *be a Banach space and let* $\mathscr{A} : E \to E$ *be a nonlinear contraction. Then* $\mathscr{A}$ *has a unique fixed point in* E.

Definition 1.9 ([86]) Let $\Phi : D(\Phi) \subseteq X \to X$ be a bounded and continuous operator on a Banach space X. Then Φ is called a condensing map if $\alpha(\Phi(B)) < \alpha(B)$ for all bounded sets $B \subset D(\Phi)$, where α denotes the Kuratowski measure of noncompactness.

Theorem 1.12 ([178]) *The map* $K_1 + K_2$ *is a k-set contraction with* $0 \le k < 1$, *and thus also condensing if the following conditions hold:*

(i) $K_1, K_2 : D \subseteq X \to X$ *are operators on the Banach space* X;
(ii) K_1 *is k-contractive, that is,* $\|K_1x - K_1y\| \le k\|x - y\|$ *for all* $x, y \in D$ *and fixed* $k \in [0, 1)$;
(iii) K_2 *is compact.*

Theorem 1.13 (Sadovski Fixed Point Theorem [144]) *Let* B *be a convex bounded and closed subset of a Banach space* X *and* $\Phi : B \to B$ *be a condensing map. Then* Φ *has a fixed point.*

Theorem 1.14 ([86]) *Suppose that* $A : \bar{\Omega} \to E$ *is a completely continuous operator and that*

$$Ax \neq \lambda x, \qquad \forall x \in \partial\Omega, \lambda \geq 1.$$

Then $\deg(I - A, \Omega, \theta) = 1$ *and* A *has at least one fixed point in* $\bar{\Omega}$.

The next fixed point theorem is concerned with multivalued mappings and is known as nonlinear alternative of Leray-Schauder for multivalued maps.

Theorem 1.15 (Nonlinear Alternative for Kakutani Maps [86]) *Let* E *be a Banach space,* C *a closed convex subset of* E*,* U *an open subset of* C *and* $0 \in U$*. Suppose that* $F : \overline{U} \to \mathscr{P}_{cp,c}(C)$ *is a upper semicontinuous compact map. Then either*

(i) F *has a fixed point in* $\overline{U}$*, or*
(ii) there is a $u \in \partial U$ *and* $\lambda \in (0, 1)$ *with* $u \in \lambda F(u)$.

Now, we state Krasnoselskii's multivalued fixed point theorem.

Theorem 1.16 (Krasnoselskii's Fixed Point Theorem [139]) *Let* X *be a Banach space,* $Y \in \mathscr{P}_{b,cl,c}(X)$ *and* $A, B : Y \to \mathscr{P}_{cp,c}(X)$ *be two multivalued operators. If the following conditions are satisfied:*

(i) $Ay + By \subset Y$ *for all* $y \in Y$*;*
(ii) A *is contraction;*
(iii) B *is u.s.c and compact,*

then, there exists $y \in Y$ *such that* $y \in Ay + By$.

The next fixed point theorem deals with multivalued mappings and is known as nonlinear alternative for contractive maps [140, Corollary 3.8].

Theorem 1.17 ([140]) *Let* X *be a Banach space, and* D *a bounded neighborhood of* $0 \in X$*. Let* $Z_1 : X \to \mathscr{P}_{cp,c}(X)$ *and* $Z_2 : \bar{D} \to \mathscr{P}_{cp,c}(X)$ *be two multivalued operators satisfying*

(a) Z_1 *is contraction, and*
(b) Z_2 *is u.s.c and compact.*

Then, if $G = Z_1 + Z_2$*, either*

(i) G *has a fixed point in* $\bar{D}$ *or*
(ii) there is a point $u \in \partial D$ *and* $\lambda \in (0, 1)$ *with* $u \in \lambda G(u)$.

Before stating the next fixed point theorem, we recall some preliminaries.

Let (X, d) be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider the Pompeiu-Hausdoff metric $H_d : \mathscr{P}(X) \times \mathscr{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\},$$

where $d(A, b) = \inf_{a\in A} d(a; b)$ and $d(a, B) = \inf_{b\in B} d(a; b)$. Then $(\mathscr{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathscr{P}_{cl}(X), H_d)$ is a generalized metric space (see [99]).

Definition 1.10 A multivalued operator $\mathscr{H} : X \to \mathscr{P}_{cl}(X)$ is called:

(a) κ-Lipschitz if and only if there exists $\kappa > 0$ such that

$$H_d(\mathscr{H}(x), \mathscr{H}(y)) \leq \kappa d(x, y) \quad \text{for each} \quad x, y \in X;$$

(b) a contraction if and only if it is κ-Lipschitz with $\kappa < 1$.

Now, we state a fixed point theorem due to Covitz and Nadler for multivalued contractions.

Theorem 1.18 (Covitz and Nadler [64]) *Let (X, d) be a complete metric space. If $N : X \to \mathscr{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Before stating endpoint fixed point theorem due to Amini-Harandi [33], we define some related concepts.

An element $x \in X$ is called *an endpoint of a multifunction $F : X \to \mathscr{P}(X)$* whenever $Fx = \{x\}$ [33]. Also, we say that F *has an approximate endpoint property* whenever $\inf_{x\in X} \sup_{y\in Fx} d(x, y) = 0$ [33].

A function $f : \mathbb{R} \to \mathbb{R}$ is called upper semi-continuous whenever $\limsup_{n\to\infty} f(\lambda_n) \leq f(\lambda)$ for all sequence $\{\lambda_n\}_{n\geq 1}$ with $\lambda_n \to \lambda$.

Theorem 1.19 ([33]) *Let $\psi : [0, \infty) \to [0, \infty)$ be an upper semi-continuous function such that*

$$\psi(t) < t \ \text{ and } \ \liminf_{t\to\infty}(t - \psi(t)) > 0 \ \text{ for all } \ t > 0,$$

(X, d) is a complete metric space and $S : X \to \mathscr{P}_{cl,b}(X)$ is a multi-function such that

$$H_d(Sx, Sy) \leq \psi(d(x, y)) \ \text{ for all } \ x, y \in X.$$

Then S has a unique endpoint if and only if S has approximate endpoint property.

The following Leggett-Williams fixed point theorem is useful in proving the existence of at least three positive solutions.

Definition 1.11 A continuous mapping $\theta : P \to [1, \infty)$ is said to be a nonnegative continuous concave functional on the cone P of a real Banach space E provided that

$$\theta(\lambda u + (1 - \lambda)v) \geq \lambda\theta(u) + (1 - \lambda)\theta(v)$$

for all $u, v \in P$ and $\lambda \in [0, 1]$.

Let $a, b, d > 0$ be constants. We define $P_d = \{u \in P : \|u\| < d\}$, $\overline{P}_d = \{u \in P : \|u\| \leq d\}$ and $P(\theta, a, b) = \{u \in P : \theta(u) \geq a, \|u\| \leq b\}$.

Theorem 1.20 ([109]) *Let P be a cone in the real Banach space E and $c > 0$ be a constant. Assume that there exists a concave nonnegative continuous functional θ on P with $\theta(u) \leq \|u\|$ for all $u \in \overline{P}_c$. Let $T : \overline{P}_c \to \overline{P}_c$ be a completely continuous operator. Suppose that there exist constants $0 < a < b < d \leq c$ such that the following conditions hold:*

(i) $\{u \in P(\theta, b, d) : \theta(u) > b\} \neq \emptyset$ *and* $\theta(Tu) > b$ *for* $u \in P(\theta, b, d)$;
(ii) $\|Tu\| < a$ *for* $u \leq a$;
(iii) $\theta(Tu) > b$ *for* $u \in P(\theta, b, c)$ *with* $\|Tu\| > d$.

Then T has at least three fixed points u_1, u_2 and u_3 in $\overline{P}_c$. Furthermore, $\|u_1\| < a$, $b < \theta(u_2)$, $a < \|u_3\|$ with $\theta(u_3) < b$.

The following Guo-Krasnoselskii fixed point theorem is used to prove the existence of at least one positive solution.

Theorem 1.21 ([88]) *Let E be a Banach space, and let $P \subset E$ be a cone. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that:*

(i) $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$; *or*
(ii) $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Chapter 2
Initial and Boundary Value Problems of Fractional Order Hadamard-Type Functional Differential Equations and Inclusions

2.1 Introduction

Functional and neutral functional differential equations arise in a variety of areas of biological, physical, and engineering applications, see, for example, the books [90, 100] and the references therein. Fractional functional differential equations involving Riemann-Liouville and Caputo type fractional derivatives have been studied by several researchers [1, 3, 4, 45, 46, 68, 78, 106, 175].

In this chapter, we discuss the existence of solutions for initial and boundary value problems of Hadamard-type functional and neutral functional differential equations and inclusions involving retarded as well as advanced arguments.

2.2 Functional and Neutral Fractional Differential Equations

This section deals with the existence of solutions for initial value problems (IVP for short) of fractional order functional and neutral functional differential equations. In the first problem, we consider fractional order functional differential equations:

$$D^{\alpha}y(t) = f(t, y_t), \text{ for each } t \in J = [1, b], \quad 0 < \alpha < 1, \;\; b > 1, \tag{2.1}$$

$$y(t) = \phi(t), \;\; t \in [1 - r, 1], \quad {}_HJ^{1-\alpha}y(t)|_{t=1} = 0, \tag{2.2}$$

where D^{α} is the Hadamard fractional derivative, $f : J \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ is a given continuous function and $\phi \in C([1 - r, 1], \mathbb{R})$ with $\phi(1) = 0$ and ${}_HJ^{(\cdot)}$ is the

B. Ahmad et al., *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, DOI 10.1007/978-3-319-52141-1_2

Hadamard fractional integral. For any function y defined on $[1-r, b]$ and any $t \in J$, we denote by y_t the element of $C([-r, 0], \mathbb{R})$ and define it as

$$y_t(\theta) = y(t+\theta), \quad \theta \in [-r, 0].$$

Notice that $y_t(\cdot)$ represents the history of the state from time $t-r$ up to the present time t.

The second problem is concerned with fractional neutral functional differential equations:

$$D^\alpha[y(t) - g(t, y_t)] = f(t, y_t), \ t \in J, \tag{2.3}$$

$$y(t) = \phi(t), \ t \in [1-r, 1], \quad {}_HJ^{1-\alpha} y(t)|_{t=1} = 0, \tag{2.4}$$

where f and ϕ are the same as defined in problem (2.1)–(2.2), and $g : J \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ is a given function such that $g(1, \phi) = 0$.

Theorem 2.1 ([96, p. 213]) *Let $\alpha > 0$, $n = -[-\alpha]$ and $0 \leq \gamma < 1$. Let G be an open set in $\mathbb{R}$ and let $f : (a, b] \times G \to \mathbb{R}$ be a function such that: $f(x, y) \in C_{\gamma,\log}[a, b]$ for any $y \in G$. Then the following problem*

$$D^\alpha y(t) = f(t, y(t)), \ \ \alpha > 0, \tag{2.5}$$

$${}_HJ^{\alpha-k} y(a+) = b_k, \ \ b_k \in \mathbb{R}, \ \ (k = 1, \dots, n, \ n = -[-\alpha]), \tag{2.6}$$

satisfies the Volterra integral equation:

$$y(t) = \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha - j + 1)} \left(\log \frac{t}{a}\right)^{\alpha - j} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, y(s)) \frac{ds}{s}, \ \ t > a > 0, \tag{2.7}$$

that is, $y(t) \in C_{n-\alpha,\log}[a, b]$ satisfies the relations (2.5)–(2.6) if and only if it satisfies the Volterra integral equation (2.7).

In particular, if $0 < \alpha \leq 1$, the problem (2.5)–(2.6) is equivalent to the following equation:

$$y(t) = \frac{b}{\Gamma(\alpha)} \left(\log \frac{t}{a}\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, y(s)) \frac{ds}{s}, \ s > a > 0. \tag{2.8}$$

Further details can be found in [96].

2.2.1 Functional Differential Equations

By $C(J,\mathbb{R})$, we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\},$$

where $|\cdot|$ is a suitable complete norm on $\mathbb{R}$. The space $C([-r,0],\mathbb{R})$ is endowed with norm $\|\cdot\|_C$ defined by

$$\|\phi\|_C := \sup\{|\phi(\theta)| : -r \le \theta \le 0\}.$$

Definition 2.1 A function $y \in \mathscr{C}^1([1-r,b],\mathbb{R})$ is said to be a solution of (2.1)–(2.2) if it satisfies the equation $D^\alpha y(t) = f(t, y_t)$ on J, the conditions $y(t) = \phi(t)$ on $[1-r,1]$ and ${}_HJ^{1-\alpha}y(t)|_{t=1} = 0$.

Our first existence result for the IVP (2.1)–(2.2) is based on the Banach's contraction mapping principle.

Theorem 2.2 *Let $f : J \times C([-r,0],\mathbb{R}) \to \mathbb{R}$. Assume that:*

(2.2.1) there exists $\ell > 0$ such that

$$|f(t,u)-f(t,v)| \le \ell\|u-v\|_C, \text{ for } t \in J \text{ and for every } u, v \in C([-r,0],\mathbb{R}).$$

If $\dfrac{\ell(\log b)^\alpha}{\Gamma(\alpha+1)} < 1$, then there exists a unique solution for the IVP (2.1)–(2.2) on the interval $[1-r,b]$.

Proof To transform the problem (2.1)–(2.2) into a fixed point problem, we consider an operator $N : C([1-r,b],\mathbb{R}) \to C([1-r,b],\mathbb{R})$ defined by

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [1-r,1], \\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f(s,y_s)}{s}ds, & \text{if } t \in [1,b]. \end{cases} \tag{2.9}$$

Let $y, z \in C([1-r,b],\mathbb{R})$. Then, for $t \in [1-r,b]$, we have

$$\begin{aligned} |N(y)(t) - N(z)(t| &\le \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}|f(s,y_s) - f(s,z_s)|\frac{ds}{s} \\ &\le \frac{\ell}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\|y_s - z_s\|_C\,\frac{ds}{s} \end{aligned}$$

$$\leq \frac{\ell}{\Gamma(\alpha)} \|y - z\|_{[1-r,b]} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s}$$
$$\leq \frac{\ell(\log t)^\alpha}{\Gamma(\alpha+1)} \|y - z\|_{[1-r,b]}.$$

Consequently,

$$\|N(y) - N(z)\|_{[1-r,b]} \leq \frac{\ell(\log b)^\alpha}{\Gamma(\alpha+1)} \|y - z\|_{[1-r,b]},$$

which implies that N is a contraction as $\dfrac{\ell(\log b)^\alpha}{\Gamma(\alpha+1)} < 1$, and hence the operator N has a unique fixed point by Banach's contraction mapping principle. Therefore, the problem (2.1)–(2.2) has a unique solution on $[1-r, b]$. □

We make use of the nonlinear alternative of Leray-Schauder type to obtain our second existence result for the IVP (2.1)–(2.2).

Theorem 2.3 *Assume that the following hypotheses hold:*

(2.3.1) $f : J \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ *is a continuous function;*
(2.3.2) there exist a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ *and a function* $p \in C([1, b], \mathbb{R}^+)$ *such that*

$$|f(t, u)| \leq p(t)\psi(\|u\|_C) \text{ for each } (t, u) \in [1, b] \times C([-r, 0], \mathbb{R});$$

(2.3.3) there exists a constant $M > 0$ *such that*

$$\frac{M}{\psi(M)\|p\|_\infty \dfrac{(\log b)^\alpha}{\Gamma(\alpha+1)}} > 1.$$

Then the IVP (2.1)–(2.2) has at least one solution on $[1-r, b]$.

Proof We consider the operator $N : C([1-r, b], \mathbb{R}) \to C([1-r, b], \mathbb{R})$ defined by (2.9) and show that it is both continuous and completely continuous.

Step 1: *N is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([1-r, b], \mathbb{R})$. Let $\eta > 0$ such that $\|y_n\|_\infty \leq \eta$. Then

$$|N(y_n)(t) - N(y)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s, y_{ns}) - f(s, y_s)| \frac{ds}{s}$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_1^b \left(\log \frac{t}{s}\right)^{\alpha-1} \sup_{s \in [1,b]} |f(s, y_{ns}) - f(s, y_s)| \frac{ds}{s}$$

$$\leq \frac{\|f(\cdot, y_{n_\cdot}) - f(\cdot, y_\cdot)\|_\infty}{\Gamma(\alpha)} \int_1^b \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s}$$
$$\leq \frac{(\log b)^\alpha \|f(\cdot, y_{n_\cdot}) - f(\cdot, y_\cdot)\|_\infty}{\Gamma(\alpha+1)}.$$

Since f is a continuous function, we have

$$\|N(y_n) - N(y)\|_\infty \leq \frac{(\log b)^\alpha \|f(\cdot, y_{n_\cdot}) - f(\cdot, y_\cdot)\|_\infty}{\Gamma(\alpha+1)} \to 0 \ \ as\ n \to \infty.$$

Step 2: *N maps bounded sets into bounded sets in* $C([1-r, b], \mathbb{R})$.

Indeed, it is enough to show that for any $\eta^* > 0$ there exists a positive constant $\tilde{\ell}$ such that for each $y \in B_{\eta^*} = \{y \in C([1-r, b], \mathbb{R}) : \|y\|_\infty \leq \eta^*\}$, we have $\|N(y)\|_\infty \leq \tilde{\ell}$. By (2.3.2), for each $t \in [1, b]$, we obtain

$$|N(y)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s, y_s)| \frac{ds}{s}$$
$$\leq \frac{\psi(\|y\|_{[1-r,b]})\|p\|_\infty}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s}$$
$$\leq \frac{\psi(\|y\|_{[1-r,b]})\|p\|_\infty}{\Gamma(\alpha+1)} (\log b)^\alpha.$$

Thus

$$\|N(y)\|_\infty \leq \frac{\psi(\eta^*)\|p\|_\infty}{\Gamma(\alpha+1)} (\log b)^\alpha := \tilde{\ell}.$$

Step 3: *N maps bounded sets into equicontinuous sets of* $C([1-r, b], \mathbb{R})$.

Let $t_1, t_2 \in (0, b],\ \ t_1 < t_2,\ B_{\eta^*}$ be a bounded set of $C([1-r, b], \mathbb{R})$ as in Step 2, and let $y \in B_{\eta^*}$. Then

$$|N(y)(t_2) - N(y)(t_1)| \leq \frac{1}{\Gamma(\alpha)} \left| \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-1} \right] f(s, y_s) \frac{ds}{s} \right.$$
$$\left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} f(s, y_s) \frac{ds}{s} \right|$$
$$\leq \frac{\psi(\eta^*)\|p\|_\infty}{\Gamma(\alpha+1)} \Big(2|(\log t_2/t_1)^\alpha| + |(\log t_2)^\alpha - (\log t_1)^\alpha| \Big).$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero, independent of $y \in B_{\eta^*}$. The equicontinuity for the cases $t_1 < t_2 \leq 0$ and $t_1 \leq 0 \leq t_2$ is obvious.

In consequence of Steps 1–3, it follows by the Arzelá-Ascoli theorem that $N : C([1-r,b],\mathbb{R}) \longrightarrow C([1-r,b],\mathbb{R})$ is continuous and completely continuous.

Step 4: *We show that there exists an open set $U \subseteq C([1-r,b],\mathbb{R})$ with $y \neq \lambda N(y)$ for $\lambda \in (0,1)$ and $y \in \partial U$.*

Let $y \in C([1-r,b],\mathbb{R})$ and $y = \lambda N(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in [1,b]$,

$$y(t) = \lambda\left(\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s,y_s)\,\frac{ds}{s}\right).$$

By the assumption (2.3.2), for each $t \in J$, we get

$$\begin{aligned}|y(t)| &\le \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} p(s)\psi(\|y_s\|_C)\frac{ds}{s}\\ &\le \frac{\|p\|_\infty \psi(\|y\|_{[1-r,b]})}{\Gamma(\alpha+1)}(\log b)^\alpha,\end{aligned}$$

which can be expressed as

$$\frac{\|y\|_{[1-r,b]}}{\psi(\|y\|_{[1-r,b]})\|p\|_\infty \dfrac{(\log b)^\alpha}{\Gamma(\alpha+1)}} \le 1.$$

In view of (2.3.3), there exists M such that $\|y\|_{[1-r,b]} \neq M$. Let us set

$$U = \{y \in C([1-r,b],\mathbb{R}) : \|y\|_{[1-r,b]} < M\}.$$

Note that the operator $N : \overline{U} \to C([1-r,b],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda N y$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 1.15), we deduce that N has a fixed point $y \in \overline{U}$ which is a solution of the problem (2.1)–(2.2). This completes the proof. □

2.2.2 *Neutral Functional Differential Equations*

In this subsection, we establish the existence results for the IVP (2.3)–(2.4).

Definition 2.2 A function $y \in \mathscr{C}^1([1-r,b],\mathbb{R})$ is said to be a solution of (2.3)–(2.4) if it satisfies the equation $D^\alpha[y(t) - g(t,y_t)] = f(t,y_t)$ on J, the conditions $y(t) = \phi(t)$ on $[1-r,1]$ and ${}_HJ^{1-\alpha}y(t)|_{t=1} = 0$.

Theorem 2.4 (Uniqueness Result) *Assume that (2.2.1) and the following condition hold:*

(2.4.1) there exists a nonnegative constant c_1 *such that*

$$|g(t,u)-g(t,v)| \leq c_1\|u-v\|_C, \quad \textit{for every } u,v\in C([-r,0],\mathbb{R}).$$

If

$$c_1+\frac{\ell(\log b)^\alpha}{\Gamma(\alpha+1)}<1, \tag{2.10}$$

then there exists a unique solution for the IVP (2.3)–(2.4) on the interval $[1-r,b]$.

Proof Associated with the problem (2.3)–(2.4), we introduce an operator $N_1 : C([1-r,b],\mathbb{R}) \to C([1-r,b],\mathbb{R})$ defined by

$$N_1(y)(t)=\begin{cases}\phi(t), & \text{if } t\in[1-r,1],\\[2ex] g(t,y_t)+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f(s,y_s)}{s}ds, & \text{if } t\in[1,b].\end{cases} \tag{2.11}$$

To show that the operator N_1 is a contraction, let $y,z\in C([1-r,b],\mathbb{R})$. Then, we have

$$\begin{aligned}|N_1(y)(t)-N_1(z)(t)| &\leq |g(t,y_t)-g(t,z_t)|\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_1^t |f(s,y_s)-f(s,z_s)|\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}\\ &\leq c_1\|y_t-z_t\|_C+\frac{\ell}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\|y_s-z_s\|_C\,\frac{ds}{s}\\ &\leq c_1\|y-z\|_{[1-r,b]}+\frac{\ell}{\Gamma(\alpha)}\|y-z\|_{[1-r,b]}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{ds}{s}\\ &\leq c_1\|y-z\|_{[1-r,b]}+\frac{\ell(\log t)^\alpha}{\Gamma(\alpha+1)}\|y-z\|_{[1-r,b]}.\end{aligned}$$

Consequently, we obtain

$$\|N_1(y)-N_1(z)\|_{[1-r,b]}\leq\left[c_1+\frac{\ell(\log b)^\alpha}{\Gamma(\alpha+1)}\right]\|y-z\|_{[1-r,b]},$$

which, in view of (2.10), implies that N_1 is a contraction. Hence N_1 has a unique fixed point by Banach's contraction mapping principle. This, in turn, shows that the problem (2.3)–(2.4) has a unique solution on $[1-r,b]$. □

Theorem 2.5 *Assume that (2.3.1) and (2.3.2) hold. Further, we suppose that:*

(2.5.1) the function g is continuous and completely continuous, and for any bounded set B in $C([1-r,b],\mathbb{R})$, the set $\{t \to g(t,y_t) : y \in B\}$ is equicontinuous in $C([1,b],\mathbb{R})$, and there exist constants $0 \le d_1 < 1,\ d_2 \ge 0$ such that

$$|g(t,u)| \le d_1\|u\|_C + d_2,\ \ t \in [1,b],\ u \in C([-r,0],\mathbb{R});$$

(2.5.2) there exists a constant $M > 0$ such that

$$\frac{(1-d_1)M}{d_2 + \dfrac{\|p\|_\infty \psi(M)}{\Gamma(\alpha+1)}(\log b)^\alpha} > 1.$$

Then the IVP (2.3)–(2.4) has at least one solution on $[1-r,b]$.

Proof Let us show that the operator $N_1 : C([1-r,b],\mathbb{R}) \to C([1-r,b],\mathbb{R})$ defined by (2.11) is continuous and completely continuous.

Using (2.5.1), it suffices to show that the operator $N_2 : C([1-r,b],\mathbb{R}) \to C([1-r,b],\mathbb{R})$ defined by

$$N_2(y)(t) = \begin{cases} \phi(t), & t \in [1-r,1], \\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f(s,y_s)}{s}\,ds, & t \in [1,b], \end{cases}$$

is continuous and completely continuous. The proof is similar to that of Theorem 2.3, so we omit the details.

We now show that there exists an open set $U \subseteq C([1-r,b],\mathbb{R})$ with $y \ne \lambda N_1(y)$ for $\lambda \in (0,1)$ and $y \in \partial U$.

Let $y \in C([1-r,b],\mathbb{R})$ and $y = \lambda N_1(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in [1,b]$, we have

$$y(t) = \lambda\left(g(t,y_t) + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f(s,y_s)}{s}\,ds\right).$$

For each $t \in J$, it follows by (2.3.1) and (2.3.2) that

$$\begin{aligned} |y(t)| &\le d_1\|y_t\|_C + d_2 + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} p(s)\psi(\|y_s\|_C)\frac{ds}{s} \\ &\le d_1\|y_t\|_C + d_2 + \frac{\|p\|_\infty \psi(\|y\|_{[1-r,b]})}{\Gamma(\alpha+1)}, \end{aligned}$$

which yields

$$(1-d_1)\|y\|_{[1-r,b]} \leq d_2 + \frac{\|p\|_\infty \psi(\|y\|_{[1-r,b]})}{\Gamma(\alpha+1)}(\log b)^\alpha.$$

In consequence, we get

$$\frac{(1-d_1)\|y\|_{[1-r,b]}}{d_2 + \dfrac{\|p\|_\infty \psi(\|y\|_{[1-r,b]})}{\Gamma(\alpha+1)}(\log b)^\alpha} \leq 1.$$

In view of (2.5.2), there exists M such that $\|y\|_{[1-r,b]} \neq M$. Let us set

$$U = \{y \in C([1-r,b],\mathbb{R}) : \|y\|_{[1-r,b]} < M\}.$$

Note that the operator $N_1 : \overline{U} \to C([1-r,b],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda N_1 y$ for some $\lambda \in (0,1)$. Thus, by the nonlinear alternative of Leray-Schauder type (Theorem 1.15), we deduce that N_1 has a fixed point $y \in \overline{U}$ which is a solution of the problem (2.3)–(2.4). This completes the proof. □

2.2.3 An Example

Consider the initial value problem for fractional functional differential equations:

$$D^{1/2}y(t) = \frac{\|y_t\|}{2(1+\|y_t\|)} + \frac{1}{3}, \quad t \in J := [1,e], \tag{2.12}$$

$$y(t) = \phi(t), \quad t \in [1-r,1], \qquad {}_HJ^{1/2}y(t)|_{t=1} = 0. \tag{2.13}$$

Let

$$f(t,x) = \frac{x}{2(1+x)}, \quad (t,x) \in [1,e] \times [0,\infty).$$

For $x, y \in [0,\infty)$ and $t \in J$, we have

$$|f(t,x) - f(t,y)| = \frac{1}{2}\left|\frac{x}{1+x} - \frac{y}{1+y}\right| = \frac{|x-y|}{2(1+x)(1+y)} \leq \frac{1}{2}|x-y|.$$

Hence the condition (2.2.1) holds with $\ell = 1/2$. Since $\dfrac{\ell(\log b)^\alpha}{\Gamma(\alpha+1)} = \dfrac{1}{\sqrt{\pi}} < 1$, therefore, by Theorem 2.2, the problem (2.12)–(2.13) has a unique solution on $[1-r,b]$.

2.3 Functional and Neutral Fractional Differential Inclusions

In this section, we study the existence of solutions for initial value problems of functional and neutral functional Hadamard type fractional differential inclusions given by

$$D^{\alpha}y(t) \in F(t, y_t), \text{ for each } t \in J := [1, b], \quad 0 < \alpha < 1, \tag{2.14}$$

$$y(t) = \vartheta(t), \ \ t \in [1-r, 1], \quad {}_HJ^{1-\alpha}y(t)|_{t=1} = 0, \tag{2.15}$$

and

$$D^{\alpha}[y(t) - g(t, y_t)] \in F(t, y_t), \ t \in J, \tag{2.16}$$

$$y(t) = \vartheta(t), \ t \in [1-r, 1], \quad {}_HJ^{1-\alpha}y(t)|_{t=1} = 0, \tag{2.17}$$

where D^{α} is the Hadamard fractional derivative, $F : J \times C([-r, 0], \mathbb{R}) \to \mathscr{P}(\mathbb{R})$ ($\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$) is a given function and $\vartheta \in C([1-r, 1], \mathbb{R})$ with $\vartheta(1) = 0$ and $g : J \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ is a given function such that $g(1, \vartheta) = 0$.

2.3.1 Functional Differential Inclusions

In this section, we establish the existence criteria for the problem (2.14)–(2.15).

Definition 2.3 A function $y \in \mathscr{C}^1([1-r, b], \mathbb{R})$ is called a solution of problem (2.14)–(2.15) if there exists a function $v \in L^1(J, \mathbb{R})$ with $v(t) \in F(t, y_t)$, a.e. on J such that $D^{\alpha}y(t) = v(t)$ for a.e. $t \in J$, $y(t) = \vartheta(t)$, $t \in [1-r, 1]$ and ${}_HJ^{1-\alpha}y(t)|_{t=1} = 0$.

Theorem 2.6 *Assume that:*

(2.6.1) *$F : J \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is L^1-Carathéodory and has nonempty compact and convex values;*

(2.6.2) *there exists a continuous nondecreasing function $\beta : [0, \infty) \to (0, \infty)$ and a function $\zeta \in C(J, \mathbb{R}^+)$ such that*

$$\|F(t, y)\|_{\mathscr{P}} := \sup\{|v| : v \in F(t, y)\} \le \zeta(t)\beta(\|y\|_C),$$

for each $(t, y) \in J \times C([-r, 0], \mathbb{R})$;

(2.6.3) there exists a constant $\sigma > 0$ *such that*

$$\frac{\sigma}{\beta(\sigma)\|\zeta\|_\infty \dfrac{(\log b)^\alpha}{\Gamma(\alpha+1)}} > 1.$$

Then the initial value problem (2.14) and (2.15) has at least one solution on $[1-r, b]$.

Proof Define an operator $\Omega_F : C([1-r, b], \mathbb{R}) \to \mathscr{P}(C([1-r, b], \mathbb{R}))$ by

$$\Omega_F(y) = \left\{ \begin{array}{l} h \in C([1-r, b], \mathbb{R}) : \\ h(t) = \begin{cases} \vartheta(t), & \text{if } t \in [1-r, 1], \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds, & \text{if } t \in [1, b], \end{cases} \end{array} \right\} \tag{2.18}$$

for $v \in S_{F,y}$. It will be shown that the operator Ω_F satisfies the assumptions of Theorem 1.15. Firstly, we observe that Ω_F is convex for each $y \in C([1-r, b], \mathbb{R})$ since $S_{F,y}$ is convex (F has convex values). Next, we show that Ω_F *maps bounded sets into bounded sets in* $C([1-r, b], \mathbb{R})$. For a positive number r, let $B_r = \{y \in C([1-r, b], \mathbb{R}) : \|y\|_{[1-r,b]} \le r\}$ be a bounded ball in $C([1-r, b], \mathbb{R})$. Then, for each $h \in \Omega_F(y), y \in B_r$, there exists $v \in S_{F,y}$ such that

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds.$$

Then, for $t \in J$, we have

$$\begin{aligned} |h(t)| &\le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |v(s)| \frac{ds}{s} \\ &\le \frac{\beta(\|y\|_{[1-r,b]})\|\zeta\|_\infty}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\ &\le \frac{\beta(\|y\|_{[1-r,b]})\|\zeta\|_\infty}{\Gamma(\alpha+1)} (\log b)^\alpha. \end{aligned}$$

Thus

$$\|h\| \le \frac{\beta(r)\|\zeta\|_\infty}{\Gamma(\alpha+1)} (\log b)^\alpha := \tilde{\ell}.$$

Now, we show that Ω_F *maps bounded sets into equicontinuous sets of* $C([1-r, b], \mathbb{R})$. Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $y \in B_r$. For each $h \in \Omega_F(y)$, we obtain

$$|h(t_2) - h(t_1)| \leq \frac{1}{\Gamma(\alpha)} \left| \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{\alpha-1} - \left(\log \frac{t_1}{s}\right)^{\alpha-1} \right] f(s, y_s) \frac{ds}{s} \right.$$
$$\left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} f(s, y_s) \frac{ds}{s} \right|$$
$$\leq \frac{\psi(\eta^*)\|p\|_\infty}{\Gamma(\alpha+1)} \Big(2|(\log t_2/t_1)^\alpha| + |(\log t_2)^\alpha - (\log t_1)^\alpha| \Big).$$

Clearly the right hand side of the above inequality tends to zero independent of $y \in B_r$ as $t_2 - t_1 \to 0$. As Ω_F satisfies the above three assumptions, it follows by the Arzelá-Ascoli Theorem that $\Omega_F : C([1-r, b], \mathbb{R}) \to \mathscr{P}(C([1-r, b], \mathbb{R}))$ is completely continuous.

In our next step, we show that Ω_F is upper semicontinuous. It is known [69, Proposition 1.2] that Ω_F will be upper semicontinuous if we establish that it has a closed graph, since Ω_F is already shown to be completely continuous. Thus, we will prove that Ω_F *has a closed graph.* Let $y_n \to y_*, h_n \in \Omega_F(y_n)$ and $h_n \to h_*$. Then, we need to show that $h_* \in \Omega_F(y_*)$. Associated with $h_n \in \Omega_F(y_n)$, there exists $v_n \in S_{F,y_n}$ such that for each $t \in J$,

$$h_n(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_n(s) \frac{ds}{s}.$$

Thus it suffices to show that there exists $v_* \in S_{F,y_*}$ such that for each $t \in J$,

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_*(s) \frac{ds}{s}.$$

Let us consider the linear operator $\Theta : L^1(J, \mathbb{R}) \to C(J, \mathbb{R})$ given by

$$v \mapsto \Theta(v)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v(s) \frac{ds}{s}.$$

Notice that

$$\|h_n(t) - h_*(t)\| = \left\| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} (v_n(s) - v_*(s)) \frac{ds}{s} \right\| \to 0, \quad \text{as} \quad n \to \infty.$$

Thus, it follows by Lemma 1.2 that $\Theta \circ S_{F,y}$ is a closed graph operator. Further, we have that $h_n(t) \in \Theta(S_{F,y_n})$. Since $y_n \to y_*$, we have

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_*(s) \frac{ds}{s},$$

for some $v_* \in S_{F,y_*}$.

Finally, we show that there exists an open set $U \subseteq C(J, \mathbb{R})$ with $y \notin \Omega_F(y)$ for any $\lambda \in (0,1)$ and all $y \in \partial U$. Let $\lambda \in (0,1)$ and $y \in \lambda \Omega_F(y)$. Then there exists $v \in L^1(J, \mathbb{R})$ with $v \in S_{F,y}$ such that, for $t \in J$, we have

$$y(t) = \lambda \left(\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} v(s) \frac{ds}{s} \right).$$

By the assumption (2.6.2), for each $t \in J$, we get

$$\begin{aligned} |y(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \zeta(s)\beta(\|y_s\|) \frac{ds}{s} \\ &\leq \frac{\|\zeta\|_\infty \beta(\|y\|_{[1-r,b]})}{\Gamma(\alpha+1)} (\log b)^\alpha, \end{aligned}$$

which can be expressed as

$$\frac{\|y\|_{[1-r,b]}}{\beta(\|y\|_{[1-r,b]})\|\zeta\|_\infty \dfrac{(\log b)^\alpha}{\Gamma(\alpha+1)}} \leq 1.$$

In view of (2.6.3), there exists σ such that $\|y\|_{[1-r,b]} \neq \sigma$. Let us set

$$U = \{y \in C([1-r,b], \mathbb{R}) : \|y\|_{[1-r,b]} < \sigma\}.$$

Note that the operator $\Omega_F : \overline{U} \to \mathscr{P}(C([1-r,b], \mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda \Omega_F(y)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 1.15), we deduce that Ω_F has a fixed point $y \in \overline{U}$ which is a solution of the problem (2.14)–(2.15). This completes the proof. □

Next, we prove the existence of solutions for the problem (2.14)–(2.15) with a nonconvex valued right hand side (Lipschitz case) by applying a fixed point theorem for multivalued maps due to Covitz and Nadler (Theorem 1.18).

Theorem 2.7 *Assume that:*

(2.7.1) $F : J \times \mathbb{R} \to \mathscr{P}_{cp}(\mathbb{R})$ *is such that* $F(\cdot, y) : J \to \mathscr{P}_{cp}(\mathbb{R})$ *is measurable for each* $y \in \mathbb{R}$;

(2.7.2) $H_d(F(t,y), F(t,\bar{y})) \leq \ell(t)\|y - \bar{y}\|_C$ *for almost all* $t \in J$ *and* $y, \bar{y} \in C([-r,0], \mathbb{R})$ *with* $\ell \in C(J, \mathbb{R}^+)$ *and* $d(0, F(t,0)) \leq \ell(t)$ *for almost all* $t \in J$.

Then, if $\dfrac{(\log b)^\alpha}{\Gamma(\alpha+1)}\|\ell\|_\infty < 1$, *the initial value problem (2.14)–(2.15) has at least one solution on* $[1-r,b]$.

Proof Observe that the set $S_{F,y}$ is nonempty for each $y \in C([1-r,b],\mathbb{R})$ by the assumption (2.7.1), so F has a measurable selection (see Theorem III.6 [57]). Now, we show that the operator Ω_F, defined by (2.18), satisfies the hypothesis of Theorem 1.18. To show that $\Omega_F(y) \in \mathscr{P}_{cl}(C([1-r,b],\mathbb{R}))$ for each $y \in C([1-r,b],\mathbb{R})$, let $\{u_n\}_{n\geq 0} \in \Omega_F(y)$ be such that $u_n \to u$ $(n\to\infty)$ in $C([1-r,b],\mathbb{R})$. Then $u \in C([1-r,b],\mathbb{R})$ and there exists $v_n \in S_{F,y_n}$ such that, for each $t \in J$,

$$u_n(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v_n(s)\,\frac{ds}{s}.$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1(J,\mathbb{R})$. Thus, $v \in S_{F,y}$ and for each $t \in J$, we have

$$u_n(t) \to u(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v(s)\,\frac{ds}{s}.$$

Hence, $u \in \Omega(y)$.

Next, we show that there exists $\delta < 1$ $\left(\delta := \dfrac{(\log b)^\alpha}{\Gamma(\alpha+1)}\|\ell\|_\infty\right)$ such that

$$H_d(\Omega_F(y),\Omega_F(\bar{y})) \leq \delta\|y-\bar{y}\|_C \ \text{ for each } \ y,\bar{y} \in C([1-r,b],\mathbb{R}).$$

Let $y,\bar{y} \in C([1-r,b],\mathbb{R})$ and $h_1 \in \Omega_F(y)$. Then there exists $v_1(t) \in F(t,y_t)$ such that, for each $t \in J$,

$$h_1(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v_1(s)\,\frac{ds}{s}.$$

By (2.7.2), we have

$$H_d(F(t,y),F(t,\bar{y})) \leq \ell(t)\|y-\bar{y}\|_C.$$

So, there exists $w \in F(t,\bar{y}_t)$ such that

$$|v_1(t)-w| \leq \ell(t)\|y-\bar{y}\|_C, \ \ t \in J.$$

Define $U : J \to \mathscr{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t)-w| \leq \ell(t)\|y-\bar{y}\|_C\}.$$

Since the multivalued operator $U(t)\cap F(t,\bar{y}_t)$ is measurable (Proposition III.4 [57]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t,\bar{y}_t)$ and for each $t \in J$, we have $|v_1(t)-v_2(t)| \leq \ell(t)\|y-\bar{y}\|_C$.

For each $t \in J$, let us define

$$h_2(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_2(s) \frac{ds}{s}.$$

Thus,

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |v_1(s) - v_2(s)| \frac{ds}{s} \\ &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \ell(s) \|y - \bar{y}\|_C \frac{ds}{s} \\ &\leq \frac{1}{\Gamma(\alpha)} \int_1^b \left(\log \frac{t}{s}\right)^{\alpha-1} \ell(s) \|y - \bar{y}\|_{[1-r,b]} \frac{ds}{s} \\ &\leq \frac{(\log b)^\alpha}{\Gamma(\alpha+1)} \|\ell\|_\infty \|y - \bar{y}\|_{[1-r,b]}. \end{aligned}$$

Hence,

$$\|h_1 - h_2\| \leq \frac{(\log b)^\alpha}{\Gamma(\alpha+1)} \|\ell\|_\infty \|y - \bar{y}\|_{[1-r,b]}.$$

Analogously, interchanging the roles of y and $\bar{y}$, we obtain

$$H_d(\Omega_F(y), \Omega_F(\bar{y})) \leq \frac{(\log b)^\alpha}{\Gamma(\alpha+1)} \|\ell\|_\infty \|y - \bar{y}\|_{[1-r,b]}.$$

Since Ω_F is a contraction by the given condition, it follows by Theorem 1.18 that Ω_F has a fixed point y which is a solution of (2.14)–(2.15). This completes the proof. □

2.3.2 *Neutral Functional Differential Inclusions*

This subsection is concerned with the existence of solutions for the problem (2.16)–(2.17).

Definition 2.4 A function $y \in \mathscr{C}^1([1-r,b],\mathbb{R})$ is said to be a solution of (2.16)–(2.17) if there exists a function $v \in L^1([1,b],\mathbb{R})$ with $v(t) \in F(t,y_t)$, a.e. on $[1,b]$ such that $D^\alpha[y(t) - g(t,y_t)] = v(t)$ on J, $y(t) = \vartheta(t)$ on $[1-r,1]$ and ${}_HJ^{1-\alpha}y(t)|_{t=1} = 0$.

Theorem 2.8 *Suppose that (2.5.1), (2.6.1) and (2.6.2) hold. Further it is assumed that:*

(2.8.1) there exists a constant $M > 0$ such that

$$\frac{(1-d_1)M}{d_2 + \dfrac{\|\zeta\|_\infty \beta(M)}{\Gamma(\alpha+1)}(\log b)^\alpha} > 1.$$

Then the IVP (2.16)–(2.17) has at least one solution on $[1-r, b]$.

Proof Define an operator $Q : C([1-r, b], \mathbb{R}) \to \mathscr{P}(C([1-r, b], \mathbb{R})$ by

$$Q(y) = \left\{ \begin{array}{l} h \in C([1-r, b], \mathbb{R}) : \\ h(t) = \begin{cases} \vartheta(t), & \text{if } t \in [1-r, 1], \\ g(t, x_t) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds, & \text{if } t \in [1, b], \end{cases} \end{array} \right\}$$

for $v \in S_{F,y}$.

Using (2.8.1), it suffices to show that the operator $Q_1 : C([1-r, b], \mathbb{R}) \to C([1-r, b], \mathbb{R})$ defined by

$$Q_1(x) = \left\{ \begin{array}{l} h \in C([1-r, b], \mathbb{R}) : \\ h(t) = \begin{cases} \vartheta(t), & \text{if } t \in [1-r, 1], \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds, & \text{if } t \in [1, b], \end{cases} \end{array} \right\}$$

for $v \in S_{F,y}$, is continuous and completely continuous. The proof is similar to that of Theorem 2.6, so, we omit the details.

Next, we show that there exists an open set $U \subseteq C([1-r, b], \mathbb{R})$ with $y \neq \lambda Q(y)$ for $\lambda \in (0, 1)$ and $y \in \partial U$.

Let $y \in C([1-r, b], \mathbb{R})$ be such that $y = \lambda Q(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in [1, b]$, we have

$$y(t) = \lambda \left(g(t, y_t) + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \right).$$

For each $t \in J$, it follows by (2.6.2) and (2.5.1) that

$$\begin{aligned} |y(t)| &\leq d_1 \|y_t\|_C + d_2 + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \zeta(s)\beta(\|y_s\|_C) \frac{ds}{s} \\ &\leq d_1 \|y_t\|_C + d_2 + \frac{\|\zeta\|_\infty \beta(\|y\|_{[1-r,b]})}{\Gamma(\alpha+1)} (\log b)^\alpha, \end{aligned}$$

which yields

$$(1-d_1)\|y\|_{[1-r,b]} \leq d_2 + \frac{\|\zeta\|_\infty \beta(\|y\|_{[1-r,b]})}{\Gamma(\alpha+1)}(\log b)^\alpha.$$

In consequence, we get

$$\frac{(1-d_1)\|y\|_{[1-r,b]}}{d_2 + \frac{\|\zeta\|_\infty \beta(\|y\|_{[1-r,b]})}{\Gamma(\alpha+1)}(\log b)^\alpha} \leq 1.$$

In view of (2.8.1), there exists M such that $\|y\|_{[1-r,b]} \neq M$. Let us set

$$U = \{y \in C([1-r,b],\mathbb{R}) : \|y\|_{[1-r,b]} < M\}.$$

Note that the operator $Q : \overline{U} \to C([1-r,b],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda Qy$ for some $\lambda \in (0,1)$. Thus, by the nonlinear alternative of Leray-Schauder type (Theorem 1.15), we deduce that Q has a fixed point $y \in \overline{U}$ which is a solution of the problem (2.16)–(2.17). This completes the proof. □

Theorem 2.9 *Assume that (2.7.1) and (2.7.2) hold. In addition, we suppose that:*

(2.9.1) there exists a constant $L > 0$ *such that*

$$|g(t,x) - g(t,y)| \leq L\|x-y\|_C, \text{ for all } t \in [1,b] \text{ and } x,y \in C([-r,0],\mathbb{R}).$$

Then, if $L + \frac{(\log b)^\alpha}{\Gamma(\alpha+1)}\|\ell\|_\infty < 1$, *the IVP (2.16)–(2.17) has at least one solution on* $[1-r,b]$.

Proof Since the proof is similar to that of Theorem 2.7, it is omitted. □

2.3.3 Examples

Example 1 For any function $\vartheta \in C([1-r,1],\mathbb{R})$ with $\vartheta(1) = 0$, consider the problem

$$D^\alpha y(t) \in F(t,y_t), \text{ for each } t \in J := [1,e], \quad 0 < \alpha < 1, \tag{2.19}$$

$$y(t) = \vartheta(t), \quad t \in [1-r,1], \quad {}_HJ^{1-\alpha}y(t)|_{t=1} = 0, \tag{2.20}$$

where

$$F(t,y_t)=\Big[\frac{1}{4+e-t}\Big(\frac{|y_t|}{2(1+|y_t|)}+\frac{1}{4}\Big),\frac{1}{16}(1+e^{-t})\Big].$$

Clearly

$$\|F(t,y_t)\|_{\mathscr{P}}:=\sup\{|u|:u\in F(t,y_t)\}\le\frac{1}{4}\Big(\frac{3}{4}\Big),\quad y_t\in\mathbb{R}.$$

With $\zeta(t)=1/4,\ \beta(\|y_t\|)=3/4$, by the condition (2.3.3), we find that

$$M>\frac{3}{16\Gamma(\alpha+1)},\ 0<\alpha<1.$$

Hence, by Theorem 2.6, the problem (2.19)–(2.20) has a solution on $[1-r,e]$.

Example 2 Let us consider the problem (2.19)–(2.20) with

$$F(t,y_t)=\Big[\frac{1}{16},\frac{1}{\pi\sqrt{t+3}}\tan^{-1}(y_t)+\frac{1}{12}\Big]. \tag{2.21}$$

Observe that

$$H_d(F(t,y_t),F(t,\bar{y}_t))\le\frac{1}{\pi\sqrt{t+3}}\|y-\bar{y}\|_C.$$

Letting $\ell(t)=\dfrac{1}{\pi\sqrt{t+3}}$, we find that $d(0,F(t,0))\le\ell(t)$ for almost all $t\in J$ and $\dfrac{(\log b)^\alpha}{\Gamma(\alpha+1)}\|\ell\|_\infty=\dfrac{1}{2\pi\Gamma(\alpha+1)}<1$, for $0<\alpha<1$. Thus all the conditions of Theorem 2.7 are satisfied. Hence, by the conclusion of Theorem 2.7, the problem (2.19)–(2.20) with (2.21) has a solution on $[1-r,e]$.

2.4 Boundary Value Problems of Fractional Order Hadamard-Type Functional Differential Equations and Inclusions with Retarded and Advanced Arguments

In this section, we study Hadamard-type fractional functional differential equations and inclusions involving both retarded and advanced arguments with boundary conditions.

2.4.1 Fractional Order Hadamard-Type Functional Differential Equations

Here, we investigate a boundary value problem of Hadamard-type fractional functional differential equations involving both retarded and advanced arguments given by

$$D^{\alpha}x(t) = f(t, x^t),\ 1 \le t \le e,\ 1 < \alpha < 2, \tag{2.22}$$

$$x(t) = \chi(t),\ 1 - r \le t \le 1, \tag{2.23}$$

$$x(t) = \psi(t),\ e \le t \le e + h, \tag{2.24}$$

where D^{α} is the Hadamard fractional derivative, $f : [1, e] \times C([-r, h], \mathbb{R}) \to \mathbb{R}$ is a given continuous function, $\chi \in C([1 - r, 1], \mathbb{R})$ with $\chi(1) = 0$ and $\psi \in C([e, e + h], \mathbb{R})$ with $\psi(e) = 0$. For any function x defined on $[1-r, e+h]$ and any $1 \le t \le e$, we denote by x^t the element of $C([-r, h], \mathbb{R})$ defined by $x^t(\theta) = x(t + \theta)$ for $-r \le \theta \le h$, where $r,\ h \ge 0$ are constants.

By $C := C([-r, h], \mathbb{R})$, we denote the Banach space of all continuous functions from $[-r, h]$ into $\mathbb{R}$ equipped with the norm

$$\|\chi\|_{[-r,h]} = \sup\{|\chi(\theta)| : -r \le \theta \le h\}$$

and $C([1, e], \mathbb{R})$ is the Banach space endowed with norm $\|x\|_0 = \sup\{|x(t)| : 1 \le t \le e\}$. Also, let $E = C([1 - r, e + h], \mathbb{R})$, $E_1 = C([1 - r, 1], \mathbb{R})$, and $E_2 = C([e, e + h], \mathbb{R})$ be respectively endowed with the norms $\|x\|_{[1-r,e+h]} = \sup\{|x(t)| : 1 - r \le t \le e + h\}$, $\|x\|_{[1-r,1]} = \sup\{|x(t)| : 1 - r \le t \le 1\}$, and $\|x\|_{[e,e+h]} = \sup\{|x(t)| : e \le t \le e + h\}$.

Lemma 2.1 *Given $g \in C([1, e], \mathbb{R})$ and $1 < \alpha \le 2$, the problem*

$$D^{\alpha}u(t) = g(t), \quad 0 < t < 1, \tag{2.25}$$

$$u(1) = u(e) = 0, \tag{2.26}$$

is equivalent to the integral equation

$$u(t) = -\int_1^e G(t, s)\frac{g(s)}{s}ds, \tag{2.27}$$

where

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (\log t)^{\alpha-1}(1 - \log s)^{\alpha-1} - (\log t - \log s)^{\alpha-1}, & 1 \le s \le t \le e, \\ (\log t)^{\alpha-1}(1 - \log s)^{\alpha-1}, & 1 \le t \le s \le e. \end{cases} \tag{2.28}$$

Proof As argued in [96], the solution of Hadamard differential equation (2.25) can be written as

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds + c_1 (\log t)^{\alpha-1} + c_2 (\log t)^{\alpha-2}. \tag{2.29}$$

Using the given boundary conditions, we find that $c_2 = 0$, and

$$c_1 = -\frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds.$$

Substituting the values of c_1 and c_2 in (2.29), we obtain

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds \\ &= -\frac{1}{\Gamma(\alpha)} \int_1^t \left[(\log t)^{\alpha-1}(1-\log s)^{\alpha-1} - (\log t - \log s)^{\alpha-1}\right] \frac{g(s)}{s} ds \\ &\quad - \int_t^e \frac{1}{\Gamma(\alpha)} (\log t)^{\alpha-1}(1-\log s)^{\alpha-1} \frac{g(s)}{s} ds \\ &= -\int_1^e G(t,s) \frac{g(s)}{s} ds, \end{aligned}$$

where $G(t, s)$ is given by (2.28). Converse of the theorem follows by direct computation. This completes the proof. □

By a solution of (2.22)–(2.24), we mean a function $x \in \mathscr{C}^2([1-r, e+h], \mathbb{R})$ that satisfies the equation $D^\alpha x(t) = f(t, x^t)$ on $[1, e]$ and the conditions $x(t) = \chi(t),\ \chi(1) = 0$ on $[1-r, 1]$ and $x(t) = \psi(t),\ \psi(e) = 0$ on $[e, e+h]$.

Theorem 2.10 *Let $f : [1, e] \times C([-r, h], \mathbb{R}) \to \mathbb{R}$ be a continuous function. Assume the following conditions hold:*

(2.10.1) there exist $p \in C(J, \mathbb{R})$ and $\Omega : [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that

$$|f(t, u)| \le p(t)\Omega(\|u\|_{[-r,h]})$$

for all $t \in J$ and all $u \in C([-r, h], \mathbb{R})$;

(2.10.2) there exists a number $K_0 > 0$ such that

$$\frac{K_0}{\frac{2\|p\|_0}{\Gamma(\alpha+1)} \Omega(K_0 + \max\{\|x\|_{[1-r,1]}, \|x\|_{[e,e+h]}\})} > 1.$$

Then the boundary value problem (2.22)–(2.24) has at least one solution on the interval $[1-r, e+h]$.

Proof To transform the problem (2.22)–(2.24) into a fixed point problem, we consider an operator $\mathscr{Q} : C([1-r, e+h], \mathbb{R}) \to C([1-r, e+h], \mathbb{R})$ defined by

$$(\mathscr{Q}x)(t) = \begin{cases} \chi(t), & \text{if } t \in [1-r, 1], \\ \int_1^e G(t,s)\frac{f(s, x^s)}{s}\, ds, & \text{if } t \in [1, e], \\ \psi(t), & \text{if } t \in [e, e+h]. \end{cases} \tag{2.30}$$

Let $u : [1-r, e+h] \to \mathbb{R}$ be a function defined by

$$u(t) = \begin{cases} \chi(t), & \text{if } t \in [1-r, 1], \\ 0, & \text{if } t \in [1, e], \\ \psi(t), & \text{if } t \in [e, e+h]. \end{cases}$$

For each $y \in C([1, e], \mathbb{R})$ with $y(1) = 0$, we denote by z the function defined by

$$z(t) = \begin{cases} 0, & \text{if } t \in [1-r, 1], \\ y(t), & \text{if } t \in [1, e], \\ 0, & \text{if } t \in [e, e+h]. \end{cases}$$

Let us set $x(t) = y(t) + u(t)$ such that $x^t = y^t + u^t$ for every $1 \leq t \leq e$, where

$$x(t) = \int_1^e G(t,s)\frac{f(s, x^s)}{s}\, ds,$$

$$y(t) = \int_1^e G(t,s)\frac{f(s, y^s + u^s)}{s}\, ds.$$

Next, we define $B = \{y \in C([1-r, e+h], \mathbb{R}) : y(1) = 0\}$ and let $\mathfrak{F} : B \to B$ be an operator given by

$$(\mathfrak{F}y)(t) = \begin{cases} 0, & 1-r \leq t \leq 1, \\ \int_1^e G(t,s)\frac{f(s, y^s + u^s)}{s}\, ds, & 1 \leq t \leq e, \\ 0, & e \leq t \leq e+h. \end{cases} \tag{2.31}$$

Then it is enough to show that the operator $\mathfrak{F}$ has a fixed point which will guarantee that the operator $\mathscr{F}$ has a fixed point and in consequence, this fixed point will correspond to a solution of the problem (2.22)–(2.24). In the following three steps, it will be shown that the operator $\mathfrak{F}$ is continuous and completely continuous.

Step 1: *$\mathfrak{F}$ is continuous.*

Let (y_n) be a sequence such that $y_n \to y$ in B. Then, we have

$$
\begin{aligned}
|(\mathfrak{F}y_n)(t) - (\mathfrak{F}y)(t)| &\le \int_1^e G(t,s)|f(s, y^{ns} + u^s) - f(s, y^s + u^s)| \frac{ds}{s} \\
&\le \|f(\cdot, y^{n(\cdot)} + u^{(\cdot)}) - f(\cdot, y^{(\cdot)} + u^{(\cdot)})\|_0 \int_1^e G(t,s) \frac{ds}{s}.
\end{aligned}
$$

Since the function f is continuous, we have

$$
\|\mathfrak{F}y_n - \mathfrak{F}y\|_{[1-r,e+h]} \le \|f(\cdot, y^{n(\cdot)} + u^{(\cdot)}) - f(\cdot, y^{(\cdot)} + u^{(\cdot)})\|_0 \int_1^e G(t,s) \frac{ds}{s} \to 0 \text{ as } n \to \infty.
$$

Step 2: *$\mathfrak{F}$ maps bounded sets into bounded sets in B.*

For any $k > 0$, it is enough to show that there exists a positive constant $\hat{L}$ such that, for each $y \in U_k := \{y \in B : \|y\|_{[1-r,e+h]} \le k\}$, we have $\|\mathfrak{F}y\|_{[1-r,e+h]} \le \hat{L}$. For $y \in B$ and $s \in J$, we have

$$
\|y^s\|_{[-r,h]} = \max_{\theta \in [-r,h]} |y(s+\theta)| \le \max_{t \in [1-r,e+h]} |y(t)| = \|y\|_{[1-r,e+h]}
$$

and

$$
\|y^s + u^s\| \le \|y^s\|_{[-r,h]} + \|u^s\|_{[-r,h]} \le \|y\|_{[-r,h]} + \max\{\|x\|_{[1-r,1]}, \|x\|_{[e,e+h]}\}.
$$

Let $y \in U_k$. Since f is continuous, for $t \in [1,e]$, we have

$$
\begin{aligned}
|(\mathfrak{F}y)(t)| \le &\left| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s, y^s + u^s)}{s} ds \right. \\
&\left. -(\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{f(s, y^s + u^s)}{s} ds \right| \\
\le &\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{p(s)\Omega(\|y^s + u^s\|_{[-r,h]})}{s} ds \\
&+(\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{p(s)\Omega(\|y^s + u^s\|_{[-r,h]})}{s} ds \\
\le &\frac{2\|p\|_0 \Omega(k + \max\{\|x\|_{[1-r,1]}, \|x\|_{[e,e+h]}\})}{\Gamma(\alpha)} \int_1^e \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{1}{s} ds \\
\le &\frac{2\|p\|_0 \Omega(k + \max\{\|x\|_{[1-r,1]}, \|x\|_{[e,e+h]}\})}{\Gamma(\alpha+1)},
\end{aligned}
$$

and so

$$\|\mathfrak{F}y\|_{[1-r,e+h]} \leq \frac{2\|p\|_0\Omega(k+\max\{\|x\|_{[1-r,1]},\|x\|_{[e,e+h]}\})}{\Gamma(\alpha+1)} := \hat{L}.$$

Consequently, $\mathfrak{F}$ maps bounded sets into bounded sets in B.

Step 3: *$\mathfrak{F}$ maps bounded sets into equicontinuous sets of B.*

Let $t_1,\ t_2 \in [1,e]$ with $t_1 < t_2$ and U_k be a bounded set of B as in Step 2. Let $y \in U_k$. Then, we have

$$\begin{aligned}
&|(\mathfrak{F}y)(t_2)-(\mathfrak{F}y)(t_1)| \\
&\leq \int_1^e |G(t_2,s)-G(t_1,s)|\frac{|f(s,y^s+u^s)|}{s}\,ds \\
&\leq \|p\|_0\Omega(k+\max\{\|x\|_{[1-r,1]},\|x\|_{[e,e+h]}\})\int_1^e |G(t_2,s)-G(t_1,s)|\,\frac{ds}{s}.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the last inequality tends to zero, independent of $y \in U_k$. The equicontinuity for the cases $t_1 < t_2 \leq 0$ and $t_1 \leq 0 \leq t_2$ is obvious.

In view of steps 1 to 3, it follows by the Arzelá-Ascoli Theorem that the operator $\mathfrak{F}$ is continuous and completely continuous.

Step 4: *A priori bounds.*

We will show that there exists an open set $U \subset B$ with $y \neq \lambda\mathfrak{F}y$ for $0 < \lambda < 1$ and $y \in \partial U$. Let $y \in B$ and $y = \lambda\mathfrak{F}y$ for some $0 < \lambda < 1$. Thus, for each $t \in [1,e]$, we have

$$y(t) = \lambda\int_1^e G(t,s)f(s,y^s+u^s)\,\frac{ds}{s}.$$

By our assumptions, for each $t \in J$, we get

$$\begin{aligned}
|y(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{p(s)\Omega(\|y^s+u^s\|_{[-r,h]})}{s}ds \\
&\quad +(\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{p(s)\Omega(\|y^s+u^s\|_{[-r,h]})}{s}ds \\
&\leq \frac{2\|p\|_0\Omega(\|y\|_{[-r,h]}+\max\{\|x\|_{[1-r,1]},\|x\|_{[e,e+h]}\})}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}ds \\
&\leq \frac{2\|p\|_0}{\Gamma(\alpha+1)}\Omega(\|y\|_{[-r,h]}+\max\{\|x\|_{[1-r,1]},\|x\|_{[e,e+h]}\}),
\end{aligned}$$

which implies that

$$\frac{\|y\|_{[1-r,e+h]}}{\frac{2\|p\|_0}{\Gamma(\alpha+1)}\Omega(\|y\|_{[-r,h]}+\max\{\|x\|_{[1-r,1]},\|x\|_{[e,e+h]}\})}\leq 1.$$

By (2.10.1), there exists K_0 such that $\|y\|_{[1-r,e+h]} \neq K_0$. Set

$$U=\{y\in B:\|y\|_{[1-r,e+h]}<K_0+1\}.$$

By our choice of U, there is no $y \in \partial U$ such that $y=\lambda\mathfrak{F}y$ for some $0<\lambda<1$. As a consequence of the nonlinear alternative of Leray-Schauder type (Theorem 1.4), we deduce that $\mathfrak{F}$ has a fixed point $y \in \bar{U}$ which is a solution to problem (2.22)–(2.24). □

The next result, concerning the existence of a unique solution of problem (2.22)–(2.24), is based on the Banach's fixed point theorem.

Theorem 2.11 *Let $f:[1,e]\times C([-r,h],\mathbb{R})\to\mathbb{R}$. Assume that there exists $L>0$ such that*

$$|f(t,u(t))-f(t,v(t))|\leq L\|u-v\|_{[-r,h]},$$

for $t\in[1,e]$ and for every $u,\ v\in C([-r,h],\mathbb{R})$.

If

$$\frac{2L}{\Gamma(\alpha+1)}<1,$$

then the BVP (2.22)–(2.24) has a unique solution on the interval $[1-r,e+h]$.

Proof As argued in the proof of the preceding theorem, it will be shown that the operator $\mathfrak{F}:B\to B$ defined by (2.31) is a contraction, where $B=\{y\in C([1-r,e+h],\mathbb{R}):y(1)=0\}$. For that, let $y_1,\ y_2\in B$. Then, for $t\in[1,e]$, we obtain

$$\begin{aligned}|(\mathfrak{F}y_1)(t)-(\mathfrak{F}y_2)(t)| &\leq \int_1^e G(t,s)|f(s,y_1^s+u^s)-f(s,y_2^s+u^s)|\,\frac{ds}{s}\\ &\leq L\int_1^e G(t,s)\|y_1^s-y_2^s\|_{[-r,h]}\,\frac{ds}{s}\\ &\leq \frac{2L}{\Gamma(\alpha)}\|y_1-y_2\|_{[-r,h]}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}ds\\ &\leq \frac{2L}{\Gamma(\alpha+1)}\|y_1-y_2\|_{[1-r,e+h]}.\end{aligned}$$

Consequently, we get

$$\|\mathfrak{F}y_1 - \mathfrak{F}y_2\|_{[1-r,e+h]} \leq \frac{2L}{\Gamma(\alpha+1)}\|y_1 - y_2\|_{[1-r,e+h]},$$

which shows that $\mathfrak{F}$ is a contraction by the given assumption, and hence $\mathfrak{F}$ has a unique fixed point by means of the Banach's contraction mapping principle. This, in turn, implies that the problem (2.22)–(2.24) has a unique solution on the interval $[1-r, e+h]$. □

2.4.2 Fractional Order Hadamard-Type Functional Differential Inclusions

In this subsection, we extend our study initiated for functional fractional differential equations in the last subsection to the multivalued case:

$$D^\alpha x(t) \in F(t, x^t), \ 1 \leq t \leq e, \ 1 < \alpha < 2, \tag{2.32}$$

$$x(t) = \chi(t), \ 1 - r \leq t \leq 1, \tag{2.33}$$

$$x(t) = \psi(t), \ e \leq t \leq e + h, \tag{2.34}$$

where $F : [1, e] \times C([-r, h], \mathbb{R}) \to \mathscr{P}(\mathbb{R})$ is a multivalued map ($\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$), while the rest of the quantities are the same as defined in the problem (2.22)–(2.24).

Theorem 2.12 *Assume that (2.10.2) and the following conditions hold:*

(2.12.1) $F : [1, e] \times C([-r, h], \mathbb{R}) \to \mathscr{P}_{c,cp}(\mathbb{R})$ *is an* L^1*-Carathéodory multivalued map;*

(2.12.2) there exist $p \in C([1, e], \mathbb{R})$ *and a continuous and nondecreasing function* $\Omega : [0, \infty) \to (0, \infty)$ *such that*

$$\|F(t, u)\| := \sup\{|v| : v \in F(t, u)\} \leq p(t)\Omega(\|u\|_{[-r,h]}),$$

for almost all $t \in [1, e]$ *and all* $u \in C([-r, h], \mathbb{R})$.

Then the problem (2.32)–(2.34) has at least one solution on the interval $[1-r, e+h]$.

Proof In relation to the problem (2.32)–(2.34), we introduce an operator $\mathscr{N} : C([1-r, e+h], \mathbb{R}) \longrightarrow \mathscr{P}(C([1-r, e+h], \mathbb{R}))$ as

$$\mathcal{N}(x) := \left\{ \begin{array}{l} h \in C([1-r, e+h], \mathbb{R}) : \\ h(t) = \begin{cases} \chi(t), & \text{if } t \in [1-r, 1], \\ \displaystyle\int_1^e G(t,s)v(s)\frac{ds}{s}, & \text{if } t \in [1, e], \\ \psi(t), & \text{if } t \in [e, e+h], \end{cases} \end{array} \right\}$$

where

$$v \in S_{F,y} = \{v \in L^1([1,e], \mathbb{R}) : v(t) \in F(t, y^t) \ \text{ for a.e. } \ t \in J\}.$$

Observe that the existence of a fixed point of the operator $\mathcal{N}$ implies the existence of a solution to the problem (2.32)–(2.34).

As in the proof of Theorem 2.10, let $B = \{y \in C([1-r, e+h], \mathbb{R}) : y(1) = 0\}$ and let $\mathfrak{T} : B \to \mathscr{P}(B)$ be defined by

$$\mathfrak{T}(y) := \left\{ \begin{array}{l} h \in C([1-r, E+h], \mathbb{R}) : \\ h(t) = \begin{cases} 0, & \text{if } t \in [1-r, 1], \\ \displaystyle\int_1^e G(t,s)v(s)\frac{ds}{s}, & \text{if } t \in [1, e], \\ 0, & \text{if } t \in [e, e+h]. \end{cases} \end{array} \right\}$$

Now, we show that the operator $\mathfrak{T}$ has a fixed point which is equivalent to proving that the operator $\mathcal{N}$ has a fixed point. We do it in several steps.

Claim 1: $\mathfrak{T}(y)$ *is convex for each* $y \in C([1-r, e+h], \mathbb{R})$.

This claim is obvious, since F has convex values.

Claim 2: $\mathfrak{T}$ *maps bounded sets into bounded sets in* $C([1-r, e+h], \mathbb{R})$.

Let $y \in U_k = \{y \in B : \|y\|_{[1-r,e+h]} \leq k\}$. Then, for each $h \in \mathfrak{T}(y)$, there exists $v \in S_{F,y}$ such that

$$h(t) = \int_1^e G(t,s)v(s)\frac{ds}{s}, \quad t \in [1, e],$$

and that

$$\begin{aligned} |h(t)| &\leq \left| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{p(s)\Omega(\|y^s + u^s\|_{[-r,h]})}{s} ds \end{aligned}$$

$$+(\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{p(s)\Omega(\|y^s+u^s\|_{[-r,h]})}{s}ds$$
$$\leq\frac{2\|p\|_0\Omega(k+\max\{\|x\|_{[1-r,1]},\|x\|_{[e,e+h]}\})}{\Gamma(\alpha+1)}.$$

Thus

$$\|h\|_{[1-r,e+h]}\leq\frac{2\|p\|_0\Omega(k+\max\{\|x\|_{[1-r,1]},\|x\|_{[e,e+h]}\})}{\Gamma(\alpha+1)}:=\hat{L}.$$

This shows that $\mathfrak{T}$ maps bounded sets into bounded sets in B.

Claim 3: *$\mathfrak{T}$ maps bounded sets in $C([1-r,e+h],\mathbb{R})$ into equicontinuous sets.*

We consider B_k as in Claim 2 and let $h\in\mathfrak{T}(y)$ for $y\in B_k,\ k>0$. Now let $t_1,t_2\in[1,e]$ with $t_2>t_1$. Then, we have

$$|h(t_2)-h(t_1)|\leq\int_1^e|G(t_2,s)-G(t_1,s)||f(s,y^s+u^s)|\,\frac{ds}{s}$$
$$\leq\|p\|_0\Omega(k+\max\{\|x\|_{[1-r,1]},\|x\|_{[e,e+h]}\})\int_1^e|G(t_2,s)-G(t_1,s)\,\frac{ds}{s}.$$

Clearly the right-hand side of the last inequality tends to zero as $t_1\to t_2$, independently of $y\in B_k$. In view of Claims 2, 3 and the Arzelá-Ascoli Theorem, we conclude that $\mathfrak{T}:B\longrightarrow\mathscr{P}(B)$ is completely continuous.

In our next step, we show that $\mathfrak{T}$ is upper semicontinuous. We are done if we show that the operator $\mathfrak{T}$ has a closed graph, since $\mathfrak{T}$ is already shown to be completely continuous.

Claim 4: *$\mathfrak{T}$ has a closed graph.*

Let $x_n\to x_*, h_n\in\mathfrak{T}(x_n)$ and $h_n\to h_*$. Then, we need to show that $h_*\in\mathfrak{T}(x_*)$. Associated with $h_n\in\mathfrak{T}(x_n)$, there exists $v_n\in S_{F,x_n}$ such that for each $t\in[1,e]$,

$$h_n(t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v_n(s)}{s}ds-(\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v_n(s)}{s}ds.$$

Thus it suffices to show that there exists $v_*\in S_{F,x_*}$ such that for each $t\in[1,e]$,

$$h_*(t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v_*(s)}{s}ds-(\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v_*(s)}{s}ds.$$

Let us consider the linear operator $\Theta : L^1([1,e],\mathbb{R}) \to C([1,e],\mathbb{R})$ given by

$$f \mapsto \Theta(v)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds$$
$$-(\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds.$$

Clearly

$$\|h_n(t) - h_*(t)\| = \left\| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{(v_n(s) - v_*(s))}{s} ds \right.$$
$$\left. -(\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1} \frac{(v_n(s) - v_*(s))}{s} ds \right\| \to 0,$$

as $n \to \infty$. Thus, it follows by Lemma 1.2 that $\Theta \circ S_{F,x}$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, we get

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{v_*(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1} \frac{v_*(s)}{s} ds,$$

for some $v_* \in S_{F,x_*}$.

Claim 5: *We will show that there exists an open set $U \subset B$ with $y \neq \lambda \mathfrak{T} y$ for $0 < \lambda < 1$ and $y \in \partial U$.*

Let $y \in B$ be such that $y \in \lambda \mathfrak{T}(y)$ for some $0 < \lambda < 1$. Then there exists $v \in S_{F,y}$ such that

$$y(t) = \lambda \int_1^e G(t,s) v(s) \frac{ds}{s}, \quad t \in [1,e].$$

By the given assumptions, for each $t \in [1,e]$, we have

$$|y(t)| \le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{p(s)\Omega(\|y^s + u^s\|_{[-r,h]})}{s} ds$$
$$+(\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1} \frac{p(s)\Omega(\|y^s + u^s\|_{[-r,h]})}{s} ds$$
$$\le \frac{2\|p\|_0 \Omega(\|y\|_{[-r,h]} + \max\{\|x\|_{[1-r,1]}, \|x\|_{[e,e+h]}\})}{\Gamma(\alpha)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1} \frac{1}{s} ds$$
$$\le \frac{2\|p\|_0}{\Gamma(\alpha+1)} \Omega(\|y\|_{[1-r,e+h]} + \max\{\|x\|_{[1-r,1]}, \|x\|_{[e,e+h]}\}).$$

Then

$$\frac{\|y\|_{[1-r,e+h]}}{\frac{2\|p\|_0}{\Gamma(\alpha+1)}\Omega(\|y\|_{[1-r,e+h]}+\max\{\|x\|_{[1-r,1]},\|x\|_{[e,e+h]}\})}\leq 1.$$

By (2.12.3), there exists K_0 such that $\|y\|_{[1-r,e+h]}\neq K_0$. Set

$$U=\{y\in C([1-r,e+h],\mathbb{R}):\|y\|_{[1-r,e+h]}<K_0+1\}.$$

From the choice of U there is no $y\in\partial U$ such that $y\in\lambda\mathfrak{T}(y)$ for $\lambda\in(0,1)$. As a consequence of the Leray-Schauder Alternative for Kakutani maps (Theorem 1.15), we deduce that $\mathfrak{T}$ has a fixed point. Thus the problem (2.32)–(2.34) has at least one solution. □

Finally, we present an existence result for the problem (2.32)–(2.34) with nonconvex valued right hand side.

Theorem 2.13 *Suppose that:*

(2.13.1) $F:[1,e]\times C([-r,h],\mathbb{R})\longrightarrow\mathscr{P}_{cp}(\mathbb{R})$ *has the property that* $F(\cdot,y):[1,e]\longmapsto\mathscr{P}_{cp}(\mathbb{R})$ *is measurable for each* $y\in C([-r,h],\mathbb{R})$;

(2.13.2) there exists $\ell\in C(J,\mathbb{R})$ *such that*

$$H_d(F(t,u),F(t,\bar{u}))\leq\ell(t)\|u-\bar{u}\|_{[-r,h]}\quad \textit{for every}\quad u,\bar{u}\in C([-r,h],\mathbb{R}),$$

and

$$d(0,F(0,u))\leq\ell(t),\quad \textit{for a.e.}\quad t\in[1,e].$$

If

$$\frac{2}{\Gamma(\alpha+1)}\|\ell\|_0<1\quad(\|\ell\|_0=\sup_{t\in[1,e]}|\ell(t)|),$$

then there exists at least one solution for the problem (2.32)–(2.34).

Proof Transform the problem (2.32)–(2.34) into a fixed point problem by means of the multivalued operator $\mathfrak{T}:B\to\mathscr{P}(B)$ introduced in Theorem 2.12. We shall show that $\mathfrak{T}$ satisfies the assumptions of Theorem 1.18. The proof will be given in two steps.

Step 1: $\mathfrak{T}(y)\in\mathscr{P}_{cl}(B)$ *for each* $y\in B$.

Indeed, let $(y_n)_{n\geq 0}\in\mathfrak{T}(y)$ such that $y_n\longrightarrow\tilde{y}$ in B. Then $\tilde{y}\in B$ and there exists $g_n\in S_{F,y}$ such that for each $t\in[1,e]$,

$$y_n(t)=\int_1^e G(t,s)g_n(s)\frac{ds}{s}.$$

Using (2.13.1) together with the fact that F has compact values, we may pass onto a subsequence to get that g_n converges weakly to g in $L^1([1,e],\mathbb{R})$. Then, $g \in S_{F,x}$ and for each $t \in [1,e]$, we have

$$y_n(t) \longrightarrow \tilde{y}(t) = \int_1^e G(t,s)g(s)\frac{ds}{s}.$$

So $\tilde{y} \in \mathfrak{T}(y)$.

Step 2: *There exists* $\gamma < 1$ *such that*

$$H_d(\mathfrak{T}(y),\mathfrak{T}(\overline{y})) \leq \gamma\|y-\overline{y}\|_{[1-r,e+h]} \text{ for each } y,\overline{y} \in B.$$

Let $y,\overline{y} \in B$ and $h \in \mathfrak{T}(y)$. Then there exists $g(t) \in F(t,y^t+u^t)$ such that

$$h(t) = \int_1^e G(t,s)g(s)\frac{ds}{s},$$

for each $t \in J$. From (2.13.2), it follows that

$$H_d(F(t,y^t+u^t)),F(t,\overline{y}^t+u^t))) \leq \ell(t)\|y-\overline{y}\|_{[-r,h]}, \quad t \in [1,e].$$

Hence there is $w \in F(t,\overline{y}^t+u^t)$ such that

$$|g(t)-w| \leq \ell(t)\|y-\overline{y}\|_{[-r,h]}, \quad t \in [1,e].$$

Consider $U : [1,e] \to \mathscr{P}(E)$, given by

$$U(t) = \{w \in E : |g(t)-w| \leq \ell(t)\|y-\overline{y}\|_{[-r,h]}\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t,\overline{y}^t+u^t)$ is measurable (see Proposition III.4 in [57]), there exists a function $\overline{g}(t)$, which is a measurable selection for V. So, $\overline{g}(t) \in F(t,\overline{y}^t+u^t)$ and

$$|g(t)-\overline{g}(t)| \leq \ell(t)\|y-\overline{y}\|_{[-r,h]}, \text{ for each } t \in [1,e].$$

Let us define for each $t \in [1,e]$,

$$\overline{h}(t) = \int_1^e G(t,s)\overline{g}(s)\frac{ds}{s}.$$

Then we have

$$\begin{aligned}
|h(t) - \overline{h}(t)| &\le \int_1^e |G(t,s)||g(s) - \overline{g}(s)| \frac{ds}{s} \\
&\le \int_1^e |G(t,s)|\ell(s)\|y - \overline{y}\|_{[-r,h]} \frac{ds}{s} \\
&\le \frac{2\|\ell\|_0\|y - \overline{y}\|_{[-r,h]}}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{1}{s} ds \\
&\le \frac{2}{\Gamma(\alpha+1)} \|\ell\|_0 \|y - \overline{y}\|_{[-r,h]}.
\end{aligned}$$

Thus

$$\|h - \overline{h}\|_{[1-r,e+h]} \le \frac{2}{\Gamma(\alpha+1)} \|\ell\|_0 \|y - \overline{y}\|_{[1-r,e+h]}.$$

Analogously, interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(\mathfrak{T}(y), \mathfrak{T}(\overline{y})) \le \frac{2}{\Gamma(\alpha+1)} \|\ell\|_0 \|y - \overline{y}\|_{[1-r,e+h]}.$$

So, $\mathfrak{T}$ is a contraction and hence, by Theorem 1.18, $\mathfrak{T}$ has a fixed point y, which is a solution to the problem (2.32)–(2.34). □

2.5 Notes and Remarks

We have established several existence results for initial and boundary value problems of Hadamard type fractional order functional and neutral functional differential equations involving both retarded and advanced arguments. Also, we have discussed the multivalued analog of Hadamard type fractional functional and neutral functional equations. Our results rely on the standard tools of the fixed point theory for single and multivalued maps. Our results are not only new in the given setting but also correspond to some new interesting situations for an appropriate choice of r and h. For example, the results for ordinary Hadamard-type fractional differential equations/inclusions follow by taking $r = h = 0$. Our results reduce to the retarded and advanced argument cases for $r > 0; h = 0$ and $r = 0; h > 0$ respectively. The mixed (both retarded and advanced) case follows by choosing $r > 0$ and $h > 0$. The results of this chapter are adapted from the papers [17, 19] and [13].

Chapter 3
Nonlocal Hadamard Fractional Boundary Value Problems

3.1 Introduction

Classical initial and boundary conditions cannot describe some peculiarities of biological, chemical, physical or other processes happening inside the domain. In order to cope with this situation, conditions involving the contributions at intermediate positions of the domain together with its boundary contribution were introduced. Such condition are known as nonlocal conditions and are found to be of great value in modeling many real world phenomena. For example, there are certain problems of thermodynamics, elasticity and wave propagation, where the controllers at the end points of the interval under consideration may dissipate or add energy according to censors located at interior points or segments of the domain. The concept of nonlocal conditions led to the birth of nonlocal multi-point boundary value problems, for instance, see [63, 172, 173].

Integral boundary conditions are found to be of great support in the mathematical modeling of many problems in applied and technical sciences such as blood flow problems, chemical engineering, underground water flow, population dynamics, etc., for example, see [56, 58, 122, 159, 174]. As a matter of fact, integral boundary conditions provide a more realistic alternative for the assumption of 'circular cross-section' throughout the vessels in the study of fluid flow problems. Also, integral boundary conditions help to regularize ill-posed parabolic backward problems in time partial differential equations, see for example, mathematical models for bacterial self-regularization [61]. Nonlocal integral boundary conditions indeed play a key role when it is impossible to directly determine the values of the sought quantity on the boundary and it can be interpreted as the total amount or integral average on space domain, e.g., total energy, average temperature, total mass of impurities. Some results on problems with nonlocal integral boundary conditions for various evolution equations can be found in [38, 48, 120, 143].

The objective of this chapter is to develop the existence theory for a variety of nonlocal nonlinear boundary value problems of fractional differential equations,

B. Ahmad et al., *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, DOI 10.1007/978-3-319-52141-1_3

integro-differential equations and inclusions involving Hadamard fractional derivative and integral. We make use of the standard tools of fixed-point theory for single valued and multivalued maps to establish the existence and uniqueness results for the given problems.

3.2 A Three-Point Hadamard-Type Fractional Boundary Value Problem

In this section, we study the following three-point boundary value problem of Hadamard type fractional differential equations:

$$\begin{cases} D^\alpha x(t) = f(t, x(t)), & 1 < t < e, \quad 1 < \alpha \le 2, \\ x(1) = 0, \quad x(e) = \beta x(\eta), & 1 < \eta < e, \end{cases} \tag{3.1}$$

where D^α is the Hadamard fractional derivative of order α, $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and β is a real constant.

We aim to establish an existence result for the problem (3.1) via Krasnoselskii-Zabreiko's fixed point theorem (Theorem 1.10).

Lemma 3.1 (Auxiliary Lemma) *For* $1 < \alpha \le 2$ *and* $\zeta \in C([1, e], \mathbb{R})$, *the boundary value problem*

$$\begin{cases} D^\alpha x(t) = \zeta(t), & 1 < t < e, \\ x(1) = 0, \quad x(e) = \beta x(\eta), \end{cases} \tag{3.2}$$

is equivalent to the integral equation

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{\zeta(s)}{s} ds \\ &\quad + \frac{(\log t)^{\alpha-1}}{1 - \beta(\log \eta)^{\alpha-1}} \left[\frac{\beta}{\Gamma(\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{\zeta(s)}{s} ds \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{\zeta(s)}{s} ds \right], \quad t \in [1, e], \end{aligned} \tag{3.3}$$

where $\beta(\log \eta)^{\alpha-1} \neq 1$.

Proof As argued in [96], the solution of Hadamard differential equation in (3.2) can be written as

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{\zeta(s)}{s} ds + c_1 (\log t)^{\alpha-1} + c_2 (\log t)^{\alpha-2}. \tag{3.4}$$

Using the given boundary conditions, we find that $c_2 = 0$, and

$$c_1 = \frac{1}{1-\beta(\log\eta)^{\alpha-1}}\left\{\frac{\beta}{\Gamma(\alpha)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{\zeta(s)}{s}ds - \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{\zeta(s)}{s}ds\right\}.$$

Substituting the values of c_1 and c_2 in (3.4), we obtain (3.3). Conversely, by direct computation, it can be established that (3.3) satisfies the problem (3.2). This completes the proof. □

In view of Lemma 3.1, the solution of the problem (3.1) can be written as

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f(s,x(s))}{s}ds \\ &+ \frac{(\log t)^{\alpha-1}}{1-\beta(\log\eta)^{\alpha-1}}\left[\frac{\beta}{\Gamma(\alpha)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{f(s,x(s))}{s}ds\right. \\ &\left. - \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{f(s,x(s))}{s}ds\right], \quad t\in[1,e]. \end{aligned} \tag{3.5}$$

Notation. We denote by $\mathscr{E} = C([1,e],\mathbb{R})$ the Banach space of all continuous functions from $[1,e] \to \mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|x\| = \sup\{|x(t)| : t\in[1,e]\}$.

Theorem 3.1 *Let f be a continuous function, satisfying $f(a,0) \neq 0$ for some $a \in [1,e]$, and*

$$\lim_{|x|\to\infty}\frac{f(t,x)}{x} = \lambda(t), \quad \lambda_{\max} := \max_{t\in[1,e]}|\lambda(t)| < \frac{1}{\delta},$$

with

$$\delta = \frac{1}{\Gamma(\alpha+1)}\left\{1 + \frac{1+\beta(\log\eta)^\alpha}{|1-\beta(\log\eta)^{\alpha-1}|}\right\}.$$

Then the boundary value problem (3.1) has at last one nontrivial solution in $[1,e]$.

Proof Define an operator $F : \mathscr{E} \to \mathscr{E}$ by

$$\begin{aligned} Fx(t) = & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s, x(s))}{s} ds \\ & + \frac{(\log t)^{\alpha-1}}{1 - \beta(\log \eta)^{\alpha-1}} \left[\frac{\beta}{\Gamma(\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{f(s, x(s))}{s} ds \right. \\ & \left. - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{f(s, x(s))}{s} ds \right], \quad t \in [1, e]. \end{aligned} \tag{3.6}$$

It is clear that the mapping F is well defined. By means of Krasnoselsk'ii-Zabreiko's fixed point theorem (Theorem 1.10), we look for fixed points for the operator F in the Banach space $\mathscr{E}$. We split the proof into three steps.

Step 1. *F is continuous.*

Let us consider a sequence $\{x_n\}$ converging to x. For each $t \in [1, e]$, we have

$$\begin{aligned} & |Fx_n(t) - Fx(t)| \\ \leq & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{|f(s, x_n(s)) - f(s, x(s))|}{s} ds \\ & + \frac{1}{|1 - \beta(\log \eta)^{\alpha-1}|} \left[\frac{\beta}{\Gamma(\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{|f(s, x_n(s)) - f(s, x(s))|}{s} ds \right. \\ & \left. + \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{|f(s, x_n(s)) - f(s, x(s))|}{s} ds \right] \\ \leq & \|f(s, x_n(s)) - f(s, x(s))\| \frac{1}{\Gamma(\alpha + 1)} \left\{ 1 + \frac{1 + \beta(\log \eta)^\alpha}{|1 - \beta(\log \eta)^{\alpha-1}|} \right\}. \end{aligned}$$

Thus

$$\|Fx_n - Fx\| \leq \delta \|f(s, x_n(s)) - f(s, x(s))\|.$$

Since the convergence of a sequence implies its boundedness, therefore, there exists a number $k > 0$ such that $\|x_n\| \leq k, \quad \|x\| \leq k$, and hence f is uniformly continuous on the compact set $\{(t, x) : \ t \in [1, e], \ \|x\| \leq k\}$.

Thus $\quad \|Fx_n - Fx\| \leq \varepsilon, \quad \forall n \geq n_0$. This shows that F is continuous.

For any $R > 0$, we consider the closed set $C = \{x \in \mathscr{E} : \|x\| \leq R\}$.

Step 2. *We prove that $F(C)$ is relatively compact in $\mathscr{E}$.*

We set $f_{\max} = \max\limits_{t \in [1, e], \|x\| \leq R} |f(t, x)|$. Then, we have

$$|Fx(t)| \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{|f(s, x(s))|}{s} ds$$

$$+\frac{1}{|1-\beta(\log\eta)^{\alpha-1}|}\left[\frac{\beta}{\Gamma(\alpha)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{|f(s,x(s))|}{s}ds\right.$$
$$\left.+\frac{1}{\Gamma(\alpha)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{|f(s,x(s))|}{s}ds\right]$$
$$\le f_{\max}\frac{1}{\Gamma(\alpha+1)}\left\{1+\frac{1+\beta(\log\eta)^{\alpha}}{|1-\beta(\log\eta)^{\alpha-1}|}\right\}.$$

Thus $\|Fx\| \le f_{\max}\delta$ and consequently $F(C)$ is uniformly bounded. For $\tau_1, \tau_2 \in [1, e]$ with $\tau_1 < \tau_2$, we have

$$|Fx(\tau_2) - Fx(\tau_1)|$$
$$\le \frac{f_{\max}}{\Gamma(\alpha)}\left|\int_1^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{1}{s}ds - \int_1^{\tau_1}\left(\log\frac{\tau_1}{s}\right)^{\alpha-1}\frac{1}{s}ds\right|$$
$$+f_{\max}\left|\frac{(\log\tau_2)^{\alpha-1}-(\log\tau_1)^{\alpha-1}}{1-\beta(\log\eta)^{\alpha-1}}\left[\frac{\beta}{\Gamma(\alpha)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{1}{s}ds\right.\right.$$
$$\left.\left.-\frac{1}{\Gamma(\alpha)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}ds\right]\right|$$
$$\le \frac{f_{\max}}{\Gamma(\alpha)}\left|\int_1^{\tau_1}\left[\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}-\left(\log\frac{\tau_1}{s}\right)^{\alpha-1}\right]\frac{1}{s}ds\right|$$
$$+\frac{f_{\max}}{\Gamma(\alpha)}\left|\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{1}{s}ds\right|$$
$$+f_{\max}\left|\frac{(\log\tau_2)^{\alpha-1}-(\log\tau_1)^{\alpha-1}}{1-\beta(\log\eta)^{\alpha-1}}\left[\frac{\beta}{\Gamma(\alpha)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{1}{s}ds\right.\right.$$
$$\left.\left.-\frac{1}{\Gamma(\alpha)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}ds\right]\right|$$
$$\le \frac{f_{max}}{\Gamma(\alpha+1)}\Big[2(\log(\tau_2/\tau_1))^{\alpha}+|(\log\tau_2)^{\alpha}-(\log\tau_1)^{\alpha}|\Big]$$
$$+f_{\max}\left|\frac{(\log\tau_2)^{\alpha-1}-(\log\tau_1)^{\alpha-1}}{1-\beta(\log\eta)^{\alpha-1}}\left[\frac{\beta(\log\eta)^{\alpha}-1}{\Gamma(\alpha+1)}\right]\right|.$$

Clearly $F(C)$ is equicontinuous, as the right-hand side tends to 0 independent of x as $\tau_1 \to \tau_2$. Thus, by the Arzelá-Ascoli Theorem, the mapping F is completely continuous on $\mathscr{E}$. This completes the proof of Step 2.

Next consider the following boundary value problem

$$\begin{cases} D^\alpha x(t) = \lambda(t)x(t), & 1 < t < e, \\ x(1) = 0, & x(e) = \beta x(\eta). \end{cases} \tag{3.7}$$

Let us define an operator $A : \mathscr{E} \to \mathscr{E}$ by

$$\begin{aligned} Ax(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{\lambda(s)x(s)}{s} ds \\ &\quad + \frac{(\log t)^{\alpha-1}}{1-\beta(\log \eta)^{\alpha-1}} \left[\frac{\beta}{\Gamma(\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{\lambda(s)x(s)}{s} ds \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{\lambda(s)x(s)}{s} ds \right], \quad t \in [1, e]. \end{aligned} \tag{3.8}$$

Obviously A is a bounded linear operator. Furthermore, any fixed point of A is a solution of the boundary value problem (3.7) and vice versa.

Step 3. *We now assert that* 1 *is not an eigenvalue of* A.

Suppose that the boundary value problem (3.7) has a nontrivial solution x. Then

$$\begin{aligned} \|x\| &= \|A(x)\| = \sup_{t\in[1,e]} |Ax(t)| \\ &\le \lambda_{\max} \sup_{t\in[1,e]} \left[\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{|x(s)|}{s} ds \right. \\ &\quad + \frac{1}{|1-\beta(\log \eta)^{\alpha-1}|} \left[\frac{\beta}{\Gamma(\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{|x(s)|}{s} ds \right. \\ &\quad \left.\left. + \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{|x(s)|}{s} ds \right]\right] \\ &\le \lambda_{\max} \frac{1}{\Gamma(\alpha+1)} \left\{ 1 + \frac{1+\beta(\log \eta)^{\alpha}}{|1-\beta(\log \eta)^{\alpha-1}|} \right\} \|x\| \\ &= \lambda_{\max}\, \delta\, \|x\| \\ &< \|x\|. \end{aligned}$$

This contradiction shows that the BVP (3.7) has no nontrivial solution. Thus, 1 is not an eigenvalue of A.

Finally, we establish that

$$\lim_{\|x\|\to\infty} \frac{\|Fx - Ax\|}{\|x\|} = 0.$$

According to $\lim_{|x|\to\infty} \dfrac{f(t,x)}{x} = \lambda(t)$, for any $\varepsilon > 0$, there exists some $M > 0$ such that

$$|f(t,x) - \lambda(t)x| < \varepsilon|x| \quad \text{for} \ \ |x| > M.$$

Set $M^* = \max_{t\in[1,e]}\{\max_{|x|\le M} |f(t,x)|\}$ and select $R' > 0$ such that $M^* + \lambda_{\max} M < \varepsilon R'$. We denote

$$I_1 = \{t \in [1,e] : |x(t)| \le M\}, \qquad I_2 = \{t \in [1,e] : |x(t)| > M\}.$$

For any $x \in \mathscr{E}$ with $\|x\| > R', \quad t \in I_1$, we have

$$\begin{aligned} |f(t,x) - \lambda(t)x| &\le |f(t,x)| + \lambda_{\max}|x| \\ &\le M^* + \lambda_{\max} M \\ &\le \varepsilon R' < \varepsilon\|x\|. \end{aligned}$$

For any $x \in \mathscr{E}$ with $\|x\| > R', \quad t \in I_2$, we have

$$|f(t,x) - \lambda(t)x| < \varepsilon\|x\|.$$

Then for any $x \in \mathscr{E}$ with $\|x\| > R'$, we have

$$|f(t,x) - \lambda(t)x| < \varepsilon\|x\|.$$

Then, we obtain

$$\begin{aligned} \|Fx - Ax\| &= \sup_{t\in[1,e]} |(Fx - Ax)(t)| \\ &\le \sup_{t\in[1,e]} \left[\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{|f(s,x(s)) - \lambda(s)x(s)|}{s} ds \right. \\ &\quad + \frac{(\log t)^{\alpha-1}}{|1 - \beta(\log\eta)^{\alpha-1}|} \left[\frac{\beta}{\Gamma(\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{|f(s,x(s)) - \lambda(s)x(s)|}{s} ds \right. \\ &\quad \left.\left. + \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{|f(s,x(s)) - \lambda(s)x(s)|}{s} ds \right]\right] \\ &< \varepsilon\lambda_{\max} \frac{1}{\Gamma(\alpha+1)} \left\{ 1 + \frac{1 + \beta(\log\eta)^\alpha}{|1 - \beta(\log\eta)^{\alpha-1}|} \right\} \|x\| \\ &= \varepsilon\lambda_{\max}\delta\|x\|, \end{aligned}$$

which, on taking the limit, yields

$$\lim_{\|x\|\to\infty} \frac{\|Fx - Ax\|}{\|x\|} = 0.$$

Consequently, Theorem 1.10 guarantees that the boundary value problem (3.1) has at last one nontrivial solution. □

Example 3.1 Consider the boundary value problem

$$\begin{cases} D^{3/2}x(t) = f(t, x(t)), & 1 < t < e, \\ x(1) = 0, \quad x(e) = \frac{3}{2}x(2). \end{cases} \tag{3.9}$$

Here $\alpha = 3/2,\ \beta = 3/2,\ \eta = 2$ and $\delta \approx 6.3938692$. If $f(t, x) = \frac{1}{20}(t^2 + 1)x(t), t \in [1, e]$, then $\lambda_{\max}\delta \approx 0.4194527 < 1$ and hence by Theorem 3.1 the boundary value problem (3.9) has at least one solution.

3.2.1 The Case of Fractional Integral Boundary Conditions

Here we consider a Hadamard type boundary value problem with fractional integral boundary conditions given by

$$\begin{cases} D^{\alpha}x(t) = g(t, x(t)), & 1 < t < e, \quad 1 < \alpha \le 2, \\ x(1) = 0, \quad x(e) = I^{\beta}x(\eta), & 1 < \eta < e, \end{cases} \tag{3.10}$$

where D^{α} is the Hadamard fractional derivative of order α, I^{β} is the Hadamard fractional integral of order β and $g : [1, e] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Lemma 3.2 *For* $1 < \alpha \le 2$ *and* $\zeta \in C([1, e], \mathbb{R})$, *the boundary value problem*

$$\begin{cases} D^{\alpha}x(t) = \zeta(t), & 1 < t < e, \\ x(1) = 0, \quad x(e) = I^{\beta}x(\eta), \end{cases} \tag{3.11}$$

is equivalent to the integral equation

$$x(t) = I^{\alpha}\zeta(t) + \frac{(\log t)^{\alpha-1}}{\Omega}\left[I^{\beta+\alpha}\zeta(\eta) - I^{\alpha}\zeta(e)\right], \tag{3.12}$$

where

$$\Omega = \frac{1}{1 - \frac{1}{\Gamma(\beta)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\beta-1} \frac{(\log s)^{\alpha-1}}{s} ds} = \frac{1}{1 - \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} (\log \eta)^{\alpha+\beta-1}}, \tag{3.13}$$

with $\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)} (\log \eta)^{\alpha+\beta-1} \neq 1$.

Proof As argued in [96], the solution of Hadamard differential equation in (3.11) can be written as

$$x(t) = I^\alpha \zeta(t) + c_1(\log t)^{\alpha-1} + c_2(\log t)^{\alpha-2}. \tag{3.14}$$

Using the given boundary conditions, we find that $c_2 = 0$, and

$$\begin{aligned} I^\alpha \zeta(e) + c_1 &= I^\beta \left(I^\alpha \zeta(s) + c_1(\log s)^{\alpha-1}\right)(\eta) \\ &= I^{\beta+\alpha}\zeta(\eta) + \frac{c_1}{\Gamma(\beta)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\beta-1} \frac{(\log s)^{\alpha-1}}{s} ds, \end{aligned}$$

which gives

$$c_1 = \frac{1}{1 - \frac{1}{\Gamma(\beta)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\beta-1} \frac{(\log s)^{\alpha-1}}{s} ds} \left[I^{\beta+\alpha}\zeta(\eta) - I^\alpha \zeta(e)\right]. \tag{3.15}$$

Substituting the values of c_1 and c_2 in (3.14), we obtain (3.12). The converse follows dy direct computation. This completes the proof. □

Theorem 3.2 *Let g be a continuous function, satisfying $g(a, 0) \neq 0$ for some $a \in [1, e]$, and*

$$\lim_{|x| \to \infty} \frac{g(t, x)}{x} = \lambda(t), \quad \lambda_{\max} := \max_{t \in [1,e]} |\lambda(t)| < \frac{1}{\delta_1},$$

with

$$\delta_1 = \frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\Omega|}\left(\frac{(\log \eta)^{\beta+\alpha}}{\Gamma(\beta+\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\right).$$

Then the boundary value problem (3.10) has at last one nontrivial solution in $[1, e]$.

Proof In view of Lemma 3.2, lets us define an operator $\mathscr{G} : \mathscr{E} \to \mathscr{E}$ by

$$\begin{aligned}\mathscr{G}x(t) = & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s,x(s))}{s} ds \\ & + \frac{(\log t)^{\alpha-1}}{\Omega} \Bigg[\int_1^\eta \left(\log \frac{\eta}{s}\right)^{\beta+\alpha-1} \frac{g(s,x(s))}{s} ds \\ & - \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{g(s,x(s))}{s} ds \Bigg], \quad t \in [1,e].\end{aligned} \tag{3.16}$$

We omit the further details as the rest of the proof runs parallel to that of Theorem 3.1 with δ_1 in place of δ. □

Example 3.2 Consider the problem

$$\begin{cases} D^{3/2}x(t) = f(t,x(t)), \quad 1 < t < e, \\ \quad x(1) = 0, \quad x(e) = I^{3/2}x(2). \end{cases} \tag{3.17}$$

Here $\alpha = 3/2$, $\beta = 3/2$, $\eta = 2$. With the given values, we have $\Omega \approx 1.27$ and $\delta_1 \approx 1.39$. By taking $f(t,x) = \frac{1}{20}(t^2 + 1)x(t), t \in [1,e]$, it is found that $\lambda_{\max}\delta_1 \approx 0.576 < 1$ and hence by Theorem 3.2 there exists at least one solution for problem (3.17).

3.3 Nonlocal Hadamard BVP of Fractional Integro-Differential Equations

In this section, we study the following boundary value problem

$$\begin{cases} D^\alpha x(t) = f(t,x(t)), \quad 1 < t < e, \quad 1 < \alpha \le 2, \\ x(1) = 0, \quad AI^\gamma x(\eta) + Bx(e) = c, \quad 1 < \eta < e, \end{cases} \tag{3.18}$$

where D^α is the Hadamard fractional derivative of order α, I^γ is the Hadamard fractional integral of order γ, $f : [1,e] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function and A, B, c are real constants.

It is well known that the conserved quantities play a key role in understanding important mathematical and physical concepts such as differential equations, laws of conservation of energy, quantum mechanics. The nonlocal boundary condition given by (3.18) with $B = 0, c = 0$ can be conceived as a conserved boundary condition as the sum (in terms of Hadamard integral) of the values of the continuous unknown function (quantity) over the given interval of arbitrary length is zero. In other words, the accumulative effect of the continuous unknown function over the given interval vanishes.

Lemma 3.3 *Given $y \in \mathscr{E}$, the boundary value problem*

$$\begin{cases} D^{\alpha}x(t) = y(t), \;\; 1 < t < e, \;\; 1 < \alpha \le 2 \\ x(1) = 0, \qquad AI^{\gamma}x(\eta) + Bx(e) = c, \;\; 1 < \eta < e, \end{cases} \tag{3.19}$$

is equivalent to the integral equation

$$x(t) = I^{\alpha}y(t) + (\log t)^{\alpha-1}\frac{c - AI^{\gamma+\alpha}y(\eta) - BI^{\alpha}y(e)}{B + \dfrac{A\Gamma(\alpha)}{\Gamma(\gamma+\alpha)}(\log\eta)^{\gamma+\alpha-1}}, \tag{3.20}$$

with $B + \dfrac{A\Gamma(\alpha)}{\Gamma(\gamma+\alpha)}(\log\eta)^{\gamma+\alpha-1} \neq 0.$

Proof As before the solution of Hadamard differential equation in (3.19) can be written as

$$x(t) = I^{\alpha}y(t) + c_1(\log t)^{\alpha-1} + c_2(\log t)^{\alpha-2}. \tag{3.21}$$

The first boundary condition gives $c_2 = 0$. Note that

$$\begin{aligned} I^{\gamma}x(\eta) &= I^{\gamma+\alpha}y(\eta) + \frac{c_1}{\Gamma(\gamma)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\gamma-1}\frac{(\log s)^{\alpha-1}}{s}ds \\ &= I^{\gamma+\alpha}y(\eta) + c_1\frac{\Gamma(\alpha)}{\Gamma(\gamma+\alpha)}(\log\eta)^{\gamma+\alpha-1}. \end{aligned}$$

Using the second boundary condition, we get

$$AI^{\gamma+\alpha}y(\eta) + Ac_1\frac{\Gamma(\alpha)}{\Gamma(\gamma+\alpha)}(\log\eta)^{\gamma+\alpha-1} + BI^{\alpha}y(e) + Bc_1 = c,$$

which gives

$$c_1 = \frac{c - AI^{\gamma+\alpha}y(\eta) - BI^{\alpha}y(e)}{B + \dfrac{A\Gamma(\alpha)}{\Gamma(\gamma+\alpha)}(\log\eta)^{\gamma+\alpha-1}}.$$

Substituting the values of c_1 and c_2 in (3.21), we obtain (3.20). We can prove the converse of the result by direct computation. This completes the proof. □

In view of Lemma 3.3, the integral solution of the problem (3.18) can be written as

$$
\begin{aligned}
x(t) = & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s))}{s} ds \\
& + \frac{(\log t)^{\alpha-1}}{\Omega} \left\{ c - \frac{A}{\Gamma(\gamma+\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{f(s,x(s))}{s} ds \right. \\
& \left. - \frac{B}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{f(s,x(s))}{s} ds \right\}, \quad t \in [1,e],
\end{aligned} \tag{3.22}
$$

where

$$
\Omega = B + \frac{A\Gamma(\alpha)}{\Gamma(\gamma+\alpha)} (\log \eta)^{\gamma+\alpha-1}. \tag{3.23}
$$

We define an operator $Q : \mathscr{E} \to \mathscr{E}$ by

$$
\begin{aligned}
Qx(t) = & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s))}{s} ds \\
& + \frac{(\log t)^{\alpha-1}}{\Omega} \left\{ c - \frac{A}{\Gamma(\gamma+\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{f(s,x(s))}{s} ds \right. \\
& \left. - \frac{B}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{f(s,x(s))}{s} ds \right\}, \quad t \in [1,e].
\end{aligned} \tag{3.24}
$$

Notice that the problem (3.18) is equivalent to the fixed point operator equation $Qx = x$ and the existence of a fixed point of the operator Q implies the existence of a solution of the problem (3.18).

In the next, we obtain some existence and uniqueness results by using a variety of fixed point theorems.

3.3.1 Existence and Uniqueness Result via Banach's Fixed Point Theorem

For computation convenience, we set:

$$
\omega = \frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\Omega|} \left\{ \frac{|A|(\log \eta)^{\gamma+\alpha}}{\Gamma(\gamma+\alpha+1)} + \frac{|B|}{\Gamma(\alpha+1)} \right\}. \tag{3.25}
$$

Theorem 3.3 *Let $f : [1,e] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the following condition:*

(3.3.1) there exists a constant $L_1 > 0$ such that $|f(t,x) - f(t,y)| \leq L_1|x-y|$, for each $t \in [1,e]$ and $x, y \in \mathbb{R}$.

If

$$L_1\omega < 1, \tag{3.26}$$

then the Hadamard fractional boundary value problem (3.18) has a unique solution on $[1, e]$.

Proof Fixing $\max_{t\in[1,e]} |f(t,0)| = M < \infty$, we define $B_r = \{x \in \mathscr{E} : \|x\| \le r\}$, where $r \ge \dfrac{M\omega + |c|/\Omega}{1 - L_1\omega}$. We show that the set B_r is invariant with respect to the operator Q, that is, $QB_r \subset B_r$. For $x \in B_r$, we have

$$\begin{aligned}
\|Qx\| \le &\max_{t\in[1,e]}\Bigg\{\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|f(s,x(s))|}{s}ds \\
&+\frac{(\log t)^{\alpha-1}}{|\Omega|}\Bigg[|c| + \frac{|A|}{\Gamma(\gamma+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{|f(s,x(s))|}{s}ds \\
&+\frac{|B|}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{|f(s,x(s))|}{s}ds\Bigg]\Bigg\} \\
\le &\max_{t\in[1,e]}\Bigg\{\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{\left(\left|f(s,x(s))-f(s,0)\right|+\left|f(s,0)\right|\right)}{s}ds \\
&+\frac{(\log t)^{\alpha-1}}{|\Omega|}\Bigg[|c| + \frac{|A|}{\Gamma(\gamma+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{\left(\left|f(s,x(s))-f(s,0)\right|+\left|f(s,0)\right|\right)}{s}ds \\
&+\frac{|B|}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{\left(\left|f(s,x(s))-f(s,0)\right|+\left|f(s,0)\right|\right)}{s}ds\Bigg]\Bigg\} \\
\le &(L_1r+M)\max_{t\in[1,e]}\Bigg\{\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{s}ds \\
&+\frac{(\log t)^{\alpha-1}}{|\Omega|}\Bigg[\frac{|A|}{\Gamma(\gamma+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{1}{s}ds + \frac{|B|}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}ds\Bigg]\Bigg\} \\
&+\frac{|c|(\log t)^{\alpha-1}}{|\Omega|} \\
\le &(L_1r+M)\left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\Omega|}\left\{\frac{|A|(\log\eta)^{\gamma+\alpha}}{\Gamma(\gamma+\alpha+1)} + \frac{|B|}{\Gamma(\alpha+1)}\right\}\right] + \frac{|c|}{|\Omega|} \\
= &(L_1r+M)\omega + |c|/|\Omega| \le r,
\end{aligned}$$

which shows that $QB_r \subset B_r$.

Now let $x, y \in \mathscr{E}$. Then, for $t \in [1, e]$, we have

$$|(Qx)(t) - (Qy)(t)|$$

$$
\begin{aligned}
&\leq \max_{t\in[1,e]}\Bigg\{\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|f(s,x(s))-f(s,y(s))|}{s}ds \\
&\quad+\frac{(\log t)^{\alpha-1}}{|\Omega|}\Bigg[\frac{|A|}{\Gamma(\gamma+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{|f(s,x(s))-f(s,y(s))|}{s}ds \\
&\quad+\frac{|B|}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{|f(s,x(s))-f(s,y(s))|}{s}ds\Bigg]\Bigg\} \\
&\leq L_1\|x-y\|\max_{t\in[1,e]}\Bigg\{\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{s}ds \\
&\quad+\frac{(\log t)^{\alpha-1}}{|\Omega|}\Bigg[\frac{|A|}{\Gamma(\gamma+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{1}{s}ds+\frac{|B|}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}ds\Bigg]\Bigg\} \\
&= L_1\omega\|x-y\|.
\end{aligned}
$$

Therefore,

$$\|Qu-Qv\|\leq L_1\omega\|u-v\|.$$

It follows from the assumption (3.26) that Q is a contraction. In consequence, by Banach's fixed point theorem, the operator Q has a fixed point which corresponds to the unique solution of the problem (3.18). This completes the proof. □

Example 3.3 Consider the problem

$$
\begin{cases}
D^{3/2}x(t)=\dfrac{L}{2}\left(\sin x+\dfrac{|x|^3}{1+|x|^3}\right)+\dfrac{\sqrt{t}+1}{e}, \quad 1<t<e, \\
x(1)=0, \quad I^{1/2}x(2)+x(e)=4.
\end{cases}
\tag{3.27}
$$

Here $\alpha=3/2, \gamma=1/2, \eta=2, A=1, B=1, c=4$ and $f(t,x)=\dfrac{L}{2}\left(\sin x+\dfrac{|x|^3}{1+|x|^3}\right)+\dfrac{\sqrt{t}+1}{e}$. With the given values, we find that $\Omega\approx 2.228571$, $\omega\approx 1.197596$, and

$$|f(t,x)-f(t,y)|\leq\frac{L}{2}\left|\sin x+\frac{|x|^3}{1+|x|^3}-\sin y-\frac{|y|^3}{1+|y|^3}\right|\leq L|x-y|.$$

With $L<\dfrac{1}{\omega}\approx 0.835006$, all the assumptions of Theorem 3.3 are satisfied. Hence, the problem (3.27) has a unique solution on $[1,e]$.

3.3.2 Existence Result via Krasnoselskii's Fixed Point Theorem

Theorem 3.4 *Let $f : [1,e] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (3.3.1). In addition, we assume that:*

(3.4.1) $|f(t,x)| \le \mu(t), \quad \forall (t,u) \in [1,e] \times \mathbb{R}$, *and* $\mu \in C([1,e], \mathbb{R}^+)$.

Then the problem (3.18) has at least one solution on $[1,e]$ *if*

$$\frac{1}{|\Omega|}\left\{\frac{|A|(\log\eta)^{\gamma+\alpha}}{\Gamma(\gamma+\alpha+1)} + \frac{|B|}{\Gamma(\alpha+1)}\right\} < 1. \tag{3.28}$$

Proof We define $\sup_{t\in[1,e]}|\mu(t)| = \|\mu\|$ and choose a suitable constant $\bar{r} \ge \|\mu\||\omega| + |c|/|\Omega|$, where ω is defined by (3.25). We define the operators $\mathscr{P}$ and $\mathscr{Q}$ on $B_{\bar{r}} = \{x \in \mathscr{E} : \|x\| \le \bar{r}\}$ as

$$(\mathscr{P}x)(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f(s,x(s))}{s}ds,$$

$$\begin{aligned}(\mathscr{Q}x)(t) = \frac{(\log t)^{\alpha-1}}{\Omega}\Bigg[c - \frac{A}{\Gamma(\gamma+\alpha)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{f(s,x(s))}{s}ds \\ -\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{f(s,x(s))}{s}ds\Bigg].\end{aligned}$$

For $x, y \in B_{\bar{r}}$, we find that

$$\begin{aligned}&\|\mathscr{P}x + \mathscr{Q}y\| \\ &\le \|\mu\|\Bigg\{\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{s}ds \\ &\quad + \frac{(\log t)^{\alpha-1}}{|\Omega|}\Bigg[|c| + \frac{|A|}{\Gamma(\gamma+\alpha)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{1}{s}ds + \frac{|B|}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}ds\Bigg]\Bigg\} \\ &\le \|\mu\|\omega + |c|/|\Omega| \\ &\le \bar{r}.\end{aligned}$$

Thus, $\mathscr{P}x + \mathscr{Q}y \in B_{\bar{r}}$. It follows from the assumption (3.3.1) together with (3.28) that $\mathscr{Q}$ is a contraction. Continuity of f implies that the operator $\mathscr{P}$ is continuous. Also, $\mathscr{P}$ is uniformly bounded on $B_{\bar{r}}$ as $\|\mathscr{P}x\| \le \|\mu\|/\Gamma(\alpha+1)$. Now, we prove the compactness of the operator $\mathscr{P}$.

We define $\sup_{(t,x)\in[1,e]\times B_{\bar{r}}}|f(t,x)| = \bar{f} < \infty$, and let $\tau_1, \tau_2 \in [1,e]$ with $\tau_1 < \tau_2$. Then

$$\begin{aligned}
|(\mathscr{P}x)(\tau_2) - (\mathscr{P}x)(\tau_1)| &\le \frac{\bar{f}}{\Gamma(\alpha)}\left|\int_1^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{1}{s}ds - \int_1^{\tau_1}\left(\log\frac{\tau_1}{s}\right)^{\alpha-1}\frac{1}{s}ds\right| \\
&\le \frac{\bar{f}}{\Gamma(\alpha)}\left|\int_1^{\tau_1}\left[\left(\log\frac{\tau_2}{s}\right)^{\alpha-1} - \left(\log\frac{\tau_1}{s}\right)^{\alpha-1}\right]\frac{1}{s}ds\right| \\
&\quad + \frac{\bar{f}}{\Gamma(\alpha)}\left|\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{1}{s}ds\right| \\
&= \frac{\bar{f}}{\Gamma(\alpha+1)}\Big[|(\log\tau_2)^\alpha - (\log\tau_1)^\alpha| + 2(\log(\tau_2/\tau_1))^\alpha\Big],
\end{aligned}$$

which is independent of x and tends to zero as $\tau_2 - \tau_1 \to 0$. Thus, $\mathscr{P}$ is equicontinuous. So $\mathscr{P}$ is relatively compact on $B_{\bar{r}}$. Hence, by the Arzelá-Ascoli Theorem, $\mathscr{P}$ is compact on $B_{\bar{r}}$. Thus all the assumptions of Theorem 1.2 are satisfied. So the conclusion of Theorem 1.2 implies that the problem (3.18) has at least one solution on $[1, e]$. The proof is completed. □

3.3.3 Existence Result via Leray-Schauder's Nonlinear Alternative

Theorem 3.5 *Let $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that:*

(3.5.1) there exist a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and a function $p \in C([1, e], \mathbb{R}^+)$ such that

$$|f(t, u)| \le p(t)\psi(|u|) \ \textit{for each} \ (t, u) \in [1, e] \times \mathbb{R};$$

(3.5.2) there exists a constant $M > 0$ such that

$$\frac{M}{\psi(M)\|p\|\omega + |c|/|\Omega|} > 1,$$

where Ω and ω are defined by (3.23) and (3.25) respectively.

Then the fractional boundary value problem (3.18) has at least one solution on $[1, e]$.

Proof We complete the proof in several steps. We first show that *Q maps bounded sets (balls) into bounded sets in $\mathscr{E}$*. For a positive number r, let $B_r = \{x \in \mathscr{E} : \|x\| \le r\}$ be a bounded ball in $\mathscr{E}$. Then, for $t \in [1, e]$, we have

$$
\begin{aligned}
|Qx(t)| \le \max_{t\in[1,e]} \Bigg\{ & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{|f(s,x(s))|}{s} ds \\
& + \frac{(\log t)^{\alpha-1}}{|\Omega|} \left[|c| + \frac{|A|}{\Gamma(\gamma+\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{|f(s,x(s))|}{s} ds \right. \\
& \left. + \frac{|B|}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{|f(s,x(s))|}{s} ds \right] \Bigg\} \\
\le \max_{t\in[1,e]} \Bigg\{ & \frac{\psi(\|x\|)\|p\|}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{1}{s} ds \\
& + \frac{(\log t)^{\alpha-1}}{|\Omega|} \left[|c| + \frac{|A|}{\Gamma(\gamma+\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{1}{s} ds \right. \\
& \left. + \frac{|B|}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{1}{s} ds \right] \Bigg\} \\
\le \psi(\|x\|)&\|p\|\omega + |c|/|\Omega|.
\end{aligned}
$$

Consequently

$$\|Qx\| \le \psi(r)\|p\|\omega + |c|/|\Omega|.$$

Next, we show that *Q maps bounded sets into equicontinuous sets of* $\mathscr{E}$. Let $\tau_1, \tau_2 \in [1, e]$ with $\tau_1 < \tau_2$ and $x \in B_r$. Then, we have

$$
\begin{aligned}
& |(Qx)(\tau_2) - (Qx)(\tau_1)| \\
\le & \frac{\psi(r)\|p\|}{\Gamma(\alpha+1)} [|(\log \tau_2)^\alpha - (\log \tau_1)^\alpha| + 2(\log(\tau_2/\tau_1))^\alpha] \\
& + \frac{\psi(r)\|p\||(\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1}|}{|\Omega|} \left[|c| + \frac{|A|}{\Gamma(\gamma+\alpha+1)} (\log \eta)^{\gamma+\alpha} \right. \\
& \left. + \frac{|B|}{\Gamma(\alpha+1)} \right].
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_r$ as $\tau_2 - \tau_1 \to 0$. As Q satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli Theorem that $Q : \mathscr{E} \to \mathscr{E}$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative once, we have shown the boundedness of the set of all solutions to equations $x = \lambda Qx$ for $\lambda \in [0, 1]$.

Let x be a solution. Then, for $t \in [1, e]$, as in the first step, we have

$$\|x\| \le \psi(\|x\|)\|p\|\omega + |c|/|\Omega|,$$

which implies that

$$\frac{\|x\|}{\psi(\|x\|)\|p\|\omega + |c|/|\Omega|} \le 1.$$

In view of (3.5.2), there exists M such that $\|x\| \ne M$. Let us set

$$U = \{x \in \mathscr{E} : \|x\| < M\}.$$

Note that the operator $Q : \overline{U} \to \mathscr{E}$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda Qx$ for some $\lambda \in (0, 1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 1.15), we deduce that Q has a fixed point $x \in \overline{U}$ which is a solution of the problem (3.18). This completes the proof. □

Example 3.4 Consider the problem (3.27) with

$$f(t, x) = \frac{e^{-t}}{3}\Big(\frac{(1 + x)^2}{1 + (1 + x)^2} + x\Big). \tag{3.29}$$

Clearly $|f(t, x)| \le 1/(3e)(1 + \|x\|)$. By the assumption (3.5.2), we find that $M > 2.275971$. Thus, by Theorem 3.5, there exists at least one solution for the problem (3.27) with $f(t, x)$ given by (3.29).

3.3.4 Existence Result via Leray-Schauder's Degree

Theorem 3.6 *Let $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that:*

(3.6.1) there exist constants $0 \le \kappa < \omega^{-1}$ and $M_1 > 0$ such that

$$|f(t, x)| \le \kappa|x| + M_1 \ \textit{for all} \ (t, x) \in [1, e] \times \mathbb{R}.$$

Then the fractional boundary value problem (3.18) has at least one solution on $[1, e]$.

Proof We define an operator $Q : \mathscr{E} \to \mathscr{E}$ as in (3.24) and consider the fixed point problem

$$x = Qx. \tag{3.30}$$

We will show that there exists a fixed point $u \in \mathscr{E}$ satisfying (3.30). It is sufficient to show that $Q : \overline{B}_R \to \mathscr{E}$ satisfies

$$x \neq \lambda Qx, \qquad \forall x \in \partial B_R, \qquad \forall \lambda \in [0,1], \tag{3.31}$$

where $B_R = \{x \in \mathscr{E} : \max_{t\in[1,e]} |x(t)| < R,\ R > 0\}$. We define

$$H(\lambda, x) = \lambda Qx, \qquad u \in \mathscr{E}, \qquad \lambda \in [0,1].$$

As shown in Theorem 3.5, we have that the operator Q is continuous, uniformly bounded and equicontinuous. Then, by the Arzelá-Ascoli Theorem, a continuous map h_λ defined by $h_\lambda(x) = u - H(\lambda, x) = x - \lambda Qx$ is completely continuous. If (3.31) is true, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned}\deg(h_\lambda, B_R, 0) &= \deg(I - \lambda Q, B_R, 0) = \deg(h_1, B_R, 0)\\ &= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \qquad 0 \in B_R,\end{aligned} \tag{3.32}$$

where I denotes the identity operator. By the nonzero property of Leray-Schauder degree, $h_1(x) = x - Qx = 0$ for at least one $x \in B_R$. In order to prove (3.31), we assume that $x = \lambda Qx$ for some $\lambda \in [0,1]$ and for all $t \in [1,e]$. Then, with $\|x\| = \sup_{t\in[1,e]} |x(t)|$, we have

$$\begin{aligned}|Qx(t)| \leq \max_{t\in[1,e]} &\left\{ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{|f(s,x(s))|}{s} ds \right.\\ &+ \frac{(\log t)^{\alpha-1}}{|\Omega|} \left[|c| + \frac{|A|}{\Gamma(\gamma+\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{|f(s,x(s))|}{s} ds \right.\\ &\left.\left. + \frac{|B|}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{|f(s,x(s))|}{s} ds \right]\right\}\\ \leq (\kappa\|x\| + M) &\left\{ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{1}{s} ds \right.\\ &+ \frac{(\log t)^{\alpha-1}}{|\Omega|} \left[\frac{|A|}{\Gamma(\gamma+\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{1}{s} ds \right.\\ &\left.\left. + \frac{|B|}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{1}{s} ds \right]\right\} + \frac{|c|}{|\Omega|}\\ \leq (\kappa\|x\| + M_1)\,&\omega + |c|/|\Omega|,\end{aligned}$$

which, on solving for $\|x\|$, yields

$$\|x\| \leq \frac{M_1\omega + (|c|/|\Omega|)}{1 - \kappa\omega}.$$

If $R = \dfrac{M_1\omega + (|c|/|\Omega|)}{1 - \kappa\omega} + 1$, inequality (3.31) holds. This completes the proof. □

Example 3.5 Consider the problem (3.27) with

$$f(t,x) = \sin(ax) + \sqrt{\log(t) + 3},\ a > 0. \tag{3.33}$$

It is obvious that $|f(t,x)| = |\sin(ax) + \sqrt{\log(t) + 3}| \leq a|x| + 2$. With $a < 1/\omega \approx 0.835006$, the assumptions of Theorem 3.6 are satisfied and in consequence, the problem (3.27) with $f(t,x)$ given by (3.33) has a solution on $[1,e]$.

3.3.5 A Companion Problem

In this section, we consider a companion boundary value problem of (3.18) by replacing the nonlocal integral boundary condition in it by $AI^\gamma x(e) + Bx(\eta) = c$. Precisely, we consider the following problem:

$$\begin{cases} D^\alpha x(t) = f(t,x(t)), & 1 < t < e, \quad 1 < \alpha \leq 2, \\ x(1) = 0, \quad AI^\gamma x(e) + Bx(\eta) = c, & 1 < \eta < e. \end{cases} \tag{3.34}$$

In this case, we obtain an operator $\mathscr{T} : \mathscr{E} \to \mathscr{E}$ defined by

$$\begin{aligned} \mathscr{T}x(t) = {} & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s))}{s} ds \\ & + \frac{(\log t)^{\alpha-1}}{\Theta} \left\{ c - \frac{A}{\Gamma(\gamma+\alpha)} \int_1^e \left(\log\frac{e}{s}\right)^{\gamma+\alpha-1} \frac{f(s,x(s))}{s} ds \right. \\ & \left. - \frac{B}{\Gamma(\alpha)} \int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1} \frac{f(s,x(s))}{s} ds \right\}, \quad t \in [1,e], \end{aligned} \tag{3.35}$$

where

$$\Theta = B(\log\eta)^{\alpha-1} + \frac{A}{\Gamma(\gamma+1)}.$$

The existence results analogue to Theorems 3.3, 3.4, 3.5, 3.6 for problem (3.34) can be obtained in a similar manner with the aid of the operator (3.35) and the constant ω_1 given by

$$\omega_1 = \frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\Theta|}\left\{\frac{|A|}{\Gamma(\gamma+\alpha+1)} + \frac{|B|(\log\eta)^{\alpha}}{\Gamma(\alpha+1)}\right\}. \tag{3.36}$$

3.4 Nonlocal Hadamard BVP of Fractional Integro-Differential Inclusions

In this section, we study the following boundary value problem of fractional differential inclusions with an integral nonlocal boundary condition

$$\begin{cases} D^{\alpha}x(t) \in F(t, x(t)), \quad 1 < t < e, \quad 1 < \alpha \leq 2, \\ x(1) = 0, \quad AI^{\gamma}x(\eta) + Bx(e) = c, \quad 1 < \eta < e, \end{cases} \tag{3.37}$$

where D^{α} is the Hadamard fractional derivative of order α, I^{γ} is the Hadamard fractional integral of order γ, $F : [1, e] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map, $\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and A, B, c are real constants. Further, it is assumed that $B + [A\Gamma(\alpha)(\log\eta)^{\gamma+\alpha-1}/\Gamma(\gamma+\alpha)] \neq 0$.

Definition 3.1 A function $x \in \mathscr{C}^2([1, e], \mathbb{R})$ is called a solution of problem (3.37) if there exists a function $v \in L^1([1, e], \mathbb{R})$ with $v(t) \in F(t, x(t))$, a.e. on $[1, e]$ such that $D^{\alpha}x(t) = v(t)$, $1 < \alpha \leq 2$, a.e. on $[1, e]$ and $x(1) = 0$, $\quad AI^{\gamma}x(\eta) + Bx(e) = c$, $\quad 1 < \eta < e$.

3.4.1 The Carathéodory Case

Theorem 3.7 *Assume that:*

(3.7.1) $F : [1, e] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ *is Carathéodory and has nonempty compact and convex values;*

(3.7.2) there exists a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ *and a function* $p \in C([1, e], \mathbb{R}^+)$ *such that*

$$\|F(t, x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t, x)\} \leq p(t)\psi(\|x\|) \;\; \text{for each} \;\; (t, x) \in [1, e] \times \mathbb{R};$$

(3.7.3) there exists a constant $M > 0$ *such that*

$$\frac{M}{\psi(M)\|p\|\omega + |c|/|\Omega|} > 1,$$

where Ω *and* ω *are defined by (3.23) and (3.25) respectively.*

Then the boundary value problem (3.37) has at least one solution on $[1, e]$.

Proof Define an operator $\Omega_F : \mathscr{E} \to \mathscr{P}(\mathscr{E})$ by

$$\Omega_F(x) = \left\{ \begin{array}{l} h \in \mathscr{E} : \\ h(t) = \left\{ \begin{array}{l} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \\ + \dfrac{(\log t)^{\alpha-1}}{\Omega} \left\{ c - \dfrac{A}{\Gamma(\gamma+\alpha)} \displaystyle\int_1^\eta \left(\log \frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{v(s)}{s} ds \right. \\ \left. - \dfrac{B}{\Gamma(\alpha)} \displaystyle\int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \right\}, \end{array} \right. \end{array} \right\}$$

for $v \in S_{F,x}$. We will show that Ω_F satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps. As a first step, we show that Ω_F *is convex for each* $x \in \mathscr{E}$. This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore, we omit the proof.

In the second step, we show that Ω_F *maps bounded sets (balls) into bounded sets in* $\mathscr{E}$. For a positive number r, let $B_r = \{x \in \mathscr{E} : \|x\| \leq r\}$ be a bounded ball in $\mathscr{E}$. Then, for each $h \in \Omega_F(x), x \in B_r$, there exists $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) = & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \\ & + \frac{(\log t)^{\alpha-1}}{\Omega} \left\{ c - \frac{A}{\Gamma(\gamma+\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{v(s)}{s} ds \right. \\ & \left. - \frac{B}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \right\}. \end{aligned}$$

Then, for $t \in [1, e]$, we have

$$\begin{aligned} |h(t)| \leq & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{|v(s)|}{s} ds \\ & + \frac{(\log t)^{\alpha-1}}{|\Omega|} \left[|c| + \frac{|A|}{\Gamma(\gamma+\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{|v(s)|}{s} ds \right. \\ & \left. + \frac{|B|}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{|v(s)|}{s} ds \right] \\ \leq & \frac{\psi(\|x\|)\|p\|}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{1}{s} ds \\ & + \frac{(\log t)^{\alpha-1}}{|\Omega|} \left[|c| + \frac{|A|\psi(\|x\|)\|p\|}{\Gamma(\gamma+\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{1}{s} ds \right. \end{aligned}$$

$$
\begin{aligned}
&+\frac{|B|\psi(\|x\|)\|p\|}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}ds\Bigg]\\
&\le \psi(\|x\|)\|p\|\left[\frac{1}{\Gamma(\alpha+1)}+\frac{1}{|\Omega|}\left\{\frac{A(\log\eta)^{\gamma+\alpha}}{\Gamma(\gamma+\alpha+1)}+\frac{B}{\Gamma(\alpha+1)}\right\}\right]+\frac{|c|}{|\Omega|}\\
&= \psi(\|x\|)\|p\|\omega+|c|/|\Omega|.
\end{aligned}
$$

Consequently

$$
\|h\|\le\psi(r)\|p\|\omega+|c|/|\Omega|.
$$

Now, we show that Ω_F *maps bounded sets into equicontinuous sets of* $\mathscr{E}$. Let $\tau_1,\tau_2\in[1,e]$ with $\tau_1<\tau_2$ and $x\in B_r$. For each $h\in\Omega_F(x)$, we obtain

$$
\begin{aligned}
&|h(\tau_2)-h(\tau_1)|\\
&\le\frac{\psi(r)\|p\|}{\Gamma(\alpha)}\left|\int_1^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{1}{s}ds-\int_1^{\tau_1}\left(\log\frac{\tau_1}{s}\right)^{\alpha-1}\frac{1}{s}ds\right|\\
&\quad+\frac{\psi(r)\|p\||(\log\tau_2)^{\alpha-1}-(\log\tau_1)^{\alpha-1}|}{|\Omega|}\Bigg[|c|+\frac{|A|}{\Gamma(\gamma+\alpha)}\int_1^{\eta}\left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{1}{s}ds\\
&\quad+\frac{|B|}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}ds\Bigg]\\
&\le\frac{\psi(r)\|p\|}{\Gamma(\alpha+1)}[|(\log\tau_2)^{\alpha}-(\log\tau_1)^{\alpha}|+2(\log(\tau_2/\tau_1))^{\alpha}]\\
&\quad+\frac{\psi(r)\|p\||(\log\tau_2)^{\alpha-1}-(\log\tau_1)^{\alpha-1}|}{|\Omega|}\Bigg[|c|+\frac{|A|}{\Gamma(\gamma+\alpha+1)}(\log\eta)^{\gamma+\alpha}\\
&\quad+\frac{|B|}{\Gamma(\alpha+1)}\Bigg].
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero, independently of $x\in B_r$ as $\tau_2-\tau_1\to 0$. As Ω_F satisfies the above three assumptions, therefore it follows by the Arzelá-Ascoli Theorem that $\Omega_F:\mathscr{E}\to\mathscr{P}(\mathscr{E})$ is completely continuous.

In our next step, we show that Ω_F is upper semicontinuous. By Lemma 1.1, Ω_F will be upper semicontinuous if we establish that it has a closed graph, since Ω_F is already shown to be completely continuous. Thus, we will prove that Ω_F *has a closed graph.* Let $x_n\to x_*$, $h_n\in\Omega_F(x_n)$ and $h_n\to h_*$. Then, we need to show that $h_*\in\Omega_F(x_*)$. Associated with $h_n\in\Omega_F(x_n)$, there exists $v_n\in S_{F,x_n}$ such that for each $t\in[1,e]$,

$$\begin{aligned} h_n(t) = & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_n(s)}{s} ds \\ & + \frac{(\log t)^{\alpha-1}}{\Omega} \left\{ c - \frac{A}{\Gamma(\gamma+\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{v_n(s)}{s} ds \right. \\ & \left. - \frac{B}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v_n(s)}{s} ds \right\}. \end{aligned}$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [1, e]$,

$$\begin{aligned} h_*(t) = & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_*(s)}{s} ds \\ & + \frac{(\log t)^{\alpha-1}}{\Omega} \left\{ c - \frac{A}{\Gamma(\gamma+\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{v_*(s)}{s} ds \right. \\ & \left. - \frac{B}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v_*(s)}{s} ds \right\}. \end{aligned}$$

Let us consider the linear operator $\Theta : L^1([1, e], \mathbb{R}) \to \mathscr{E}$ given by

$$\begin{aligned} f \mapsto \Theta(v)(t) = & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \\ & + \frac{(\log t)^{\alpha-1}}{\Omega} \left\{ c - \frac{A}{\Gamma(\gamma+\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{v(s)}{s} ds \right. \\ & \left. - \frac{B}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \right\}. \end{aligned}$$

Observe that

$$\begin{aligned} \|h_n(t) - h_*(t)\| = & \left\| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{(v_n(s) - v_*(s))}{s} ds \right. \\ & + \frac{(\log t)^{\alpha-1}}{\Omega} \left\{ c - \frac{A}{\Gamma(\gamma+\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{(v_n(s) - v_*(s))}{s} ds \right. \\ & \left. \left. - \frac{B}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{(v_n(s) - v_*(s))}{s} ds \right\} \right\| \to 0, \end{aligned}$$

as $n \to \infty$.

Thus, it follows by Lemma 1.2, that $\Theta \circ S_{F,x}$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v_*(s)}{s}ds + \frac{(\log t)^{\alpha-1}}{\Omega}\left\{c - \frac{A}{\Gamma(\gamma+\alpha)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{v_*(s)}{s}ds - \frac{B}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v_*(s)}{s}ds\right\},$$

for some $v_* \in S_{F,x_*}$.

Finally, we show there exists an open set $U \subseteq \mathscr{E}$ with $x \notin \Omega_F(x)$ for any $\lambda \in (0,1)$ and all $x \in \partial U$. Let $\lambda \in (0,1)$ and $x \in \lambda\Omega_F(x)$. Then there exists $v \in L^1([1,e],\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in [1,e]$, we have

$$x(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds + \frac{(\log t)^{\alpha-1}}{\Omega}\left\{c - \frac{A}{\Gamma(\gamma+\alpha)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{v(s)}{s}ds - \frac{B}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds\right\}.$$

As in the second step, one can have

$$\|x\| \le \psi(\|x\|)\|p\|\omega + |c|/|\Omega|,$$

which implies that

$$\frac{\|x\|}{\psi(\|x\|)\|p\|\omega + |c|/|\Omega|} \le 1.$$

In view of (3.7.3), there exists M such that $\|x\| \ne M$. Let us set

$$U = \{x \in \mathscr{E} : \|x\| < M\}.$$

Note that the operator $\Omega_F : \overline{U} \to \mathscr{P}(\mathscr{E})$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda\Omega_F(x)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 1.15), we deduce that Ω_F has a fixed point $x \in \overline{U}$, which is a solution of the problem (3.37). This completes the proof. □

Example 3.6 Consider the problem

$$\begin{cases} D^{3/2}x(t) \in F(t, x(t)), & 1 < t < e, \\ x(1) = 0, \quad I^{1/2}x(2) + x(e) = 4. \end{cases} \tag{3.38}$$

Here $\alpha = 3/2, \gamma = 1/2, \ \eta = 2, A = 1, B = 1$ and $c = 4$. With the given values, we find that $\Omega \approx 2.228571, \ \omega \approx 1.197596$. Let $F : [1, e] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ be a multivalued map given by

$$x \to F(t, x) = \left[\frac{|x|^5}{|x|^5 + 3} + t^3 + t^2 + 4, \ \frac{|x|^3}{|x|^3 + 1} + t + 2 \right].$$

For $f \in F$, we have

$$|f| \le \max\left(\frac{|x|^5}{|x|^5 + 3} + t^3 + t^2 + 4, \ \frac{|x|^3}{|x|^3 + 1} + t + 2 \right) \le 7, \ \ x \in \mathbb{R}.$$

Thus,

$$\|F(t, x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t, x)\} \le 7 = p(t)\psi(\|x\|), \ \ x \in \mathbb{R},$$

with $p(t) = 1, \ \psi(\|x\|) = 7$. In this case by the condition (3.7.3), we find that $M > 10.178044$. Hence, by Theorem 3.7, the problem (3.38) has a solution on $[1, e]$.

3.4.2 The Lower Semicontinuous Case

Theorem 3.8 *Assume that (3.7.1), (3.7.2) and the following condition hold:*

(3.8.1) $F : [1, e] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*

- (a) $(t, x) \longmapsto F(t, x)$ *is* $\mathscr{L} \otimes \mathscr{B}$ *measurable,*
- (b) $x \longmapsto F(t, x)$ *is lower semicontinuous for each* $t \in [1, e]$.

Then the boundary value problem (3.37) has at least one solution on $[1, e]$.

Proof It follows from (3.7.2) and (3.8.1) that F is of l.s.c. type [82]. Then, from Lemma 1.3, there exists a continuous function $f : \mathscr{E} \to L^1([1, e], \mathbb{R})$ such that $f(x) \in \mathscr{F}(x)$ for all $x \in \mathscr{E}$.

Consider the problem

$$\begin{cases} D^{\alpha}x(t) = f(x(t)), & 1 < t < e, \ \ 1 < \alpha \le 2, \\ x(1) = 0, \quad AI^{\gamma}x(\eta) + Bx(e) = c, & 1 < \eta < e. \end{cases} \tag{3.39}$$

Observe that if $x \in \mathscr{C}^2([1,e],\mathbb{R})$ is a solution of (3.39), then x is a solution to the problem (3.37). In order to transform the problem (3.39) into a fixed point problem, we define the operator $\overline{\Omega}_F$ as

$$\overline{\Omega}_F x(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f(x(s))}{s}ds \\ +\frac{(\log t)^{\alpha-1}}{\Omega}\left\{c - \frac{A}{\Gamma(\gamma+\alpha)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{f(x(s))}{s}ds \right. \\ \left. -\frac{B}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{f(x(s))}{s}ds\right\}.$$

It can easily be shown that $\overline{\Omega}_F$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.7. So, we omit it. This completes the proof. □

3.4.3 The Lipschitz Case

Theorem 3.9 *Assume that:*

(3.9.1) $F : [1,e]\times\mathbb{R} \to \mathscr{P}_{cp}(\mathbb{R})$ *is such that* $F(\cdot,x) : [1,e] \to \mathscr{P}_{cp}(\mathbb{R})$ *is measurable for each* $x \in \mathbb{R}$;

(3.9.2) $H_d(F(t,x),F(t,\bar{x})) \le m(t)|x-\bar{x}|$ *for almost all* $t \in [1,e]$ *and* $x,\bar{x} \in \mathbb{R}$ *with* $m \in L^1([1,e],\mathbb{R}^+)$ *and* $d(0,F(t,0)) \le m(t)$ *for almost all* $t \in [1,e]$.

Then the boundary value problem (3.37) has at least one solution on $[1,e]$ *if*

$$\|m\|\omega + |c|/|\Omega| < 1,$$

where Ω *and* ω *are defined by (3.23) and (3.25) respectively.*

Proof Observe that the set $S_{F,x}$ is nonempty for each $x \in \mathscr{E}$ by the assumption (3.9.1), so F has a measurable selection (see Theorem III.6 [57]). Now, we show that the operator Ω_F, defined in the beginning of proof of Theorem 3.7, satisfies the assumptions of Theorem 1.18. To show that $\Omega_F(x) \in \mathscr{P}_{cl}(\mathscr{E})$ for each $x \in \mathscr{E}$, let $\{u_n\}_{n\ge 0} \in \Omega_F(x)$ be such that $u_n \to u$ $(n\to\infty)$ in $\mathscr{E}$. Then $u \in \mathscr{E}$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [1,e]$,

$$u_n(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v_n(s)}{s}ds \\ +\frac{(\log t)^{\alpha-1}}{\Omega}\left\{c - \frac{A}{\Gamma(\gamma+\alpha)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{v_n(s)}{s}ds\right.$$

$$-\frac{B}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v_n(s)}{s}ds\Bigg\}.$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([1,e],\mathbb{R})$. Thus, $v\in S_{F,x}$ and for each $t\in[1,e]$, we have

$$\begin{aligned}v_n(t)\to v(t) = &\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds\\&+\frac{(\log t)^{\alpha-1}}{\Omega}\Bigg\{c-\frac{A}{\Gamma(\gamma+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{v(s)}{s}ds\\&-\frac{B}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds\Bigg\}.\end{aligned}$$

Hence, $u\in\Omega_F(x)$.

Next, we show that there exists $\delta<1$ ($\delta:=\|m\|\omega+|c|/|\Omega|$) such that

$$H_d(\Omega_F(x),\Omega_F(\bar{x}))\le\delta\|x-\bar{x}\|\ \text{ for each }\ x,\bar{x}\in\mathscr{E}.$$

Let $x,\bar{x}\in\mathscr{E}$ and $h_1\in\Omega_F(x)$. Then there exists $v_1(t)\in F(t,x(t))$ such that, for each $t\in[1,e]$,

$$\begin{aligned}h_1(t) = &\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v_1(s)}{s}ds\\&+\frac{(\log t)^{\alpha-1}}{\Omega}\Bigg\{c-\frac{A}{\Gamma(\gamma+\alpha)}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\gamma+\alpha-1}\frac{v_1(s)}{s}ds\\&-\frac{B}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v_1(s)}{s}ds\Bigg\}.\end{aligned}$$

By (3.9.2), we have

$$H_d(F(t,x),F(t,\bar{x}))\le m(t)|x(t)-\bar{x}(t)|.$$

So, there exists $w\in F(t,\bar{x}(t))$ such that

$$|v_1(t)-w|\le m(t)|x(t)-\bar{x}(t)|,\quad t\in[1,e].$$

Define $U:[1,e]\to\mathscr{P}(\mathbb{R})$ by

$$U(t)=\{w\in\mathbb{R}:|v_1(t)-w|\le m(t)|x(t)-\bar{x}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{x}(t))$ is measurable (Proposition III.4 [57]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [1, e]$, we have $|v_1(t) - v_2(t)| \leq m(t)|x(t) - \bar{x}(t)|$.

For each $t \in [1, e]$, let us define

$$\begin{aligned} h_2(t) = & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_2(s)}{s} ds \\ & + \frac{(\log t)^{\alpha-1}}{\Omega} \left\{ c - \frac{A}{\Gamma(\gamma+\alpha)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{v_2(s)}{s} ds \right. \\ & \left. - \frac{B}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v_2(s)}{s} ds \right\}. \end{aligned}$$

Thus,

$$\begin{aligned} |h_1(t) - h_2(t)| \leq & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{|v_1(s) - v_2(s)|}{s} ds \\ & + \frac{(\log t)^{\alpha-1}}{|\Omega|} \left\{ |c| + \frac{|A|}{\Gamma(\gamma+\alpha)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{|v_1(s) - v_2(s)|}{s} ds \right. \\ & \left. + \frac{|B|}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{|v_1(s) - v_2(s)|}{s} ds \right\} \\ \leq & \frac{\|m\|}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{1}{s} ds \\ & + \frac{(\log t)^{\alpha-1}}{|\Omega|} \left[|c| + \frac{|A|}{\Gamma(\gamma+\alpha)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{1}{s} ds \right. \\ & \left. + \frac{|B|}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{1}{s} ds \right] \|x - \bar{x}\| \\ \leq & \Big(\|m\|\omega + |c|/|\Omega|\Big) \|x - \bar{x}\|. \end{aligned}$$

Hence,

$$\|h_1 - h_2\| \leq \Big(\|m\|\omega + |c|/|\Omega|\Big) \|x - \bar{x}\|.$$

Analogously, interchanging the roles of x and $\bar{x}$, we obtain

$$H_d(\Omega_F(x), \Omega_F(\bar{x})) \leq \Big(\|m\|\omega + |c|/|\Omega|\Big) \|x - \bar{x}\|.$$

Since Ω_F is a contraction, it follows by Theorem 1.18, that Ω_F has a fixed point x, which is a solution of (3.37). This completes the proof. □

3.5 Boundary Value Problems of Hadamard-Type Fractional Differential Equations and Inclusions with Nonlocal Conditions

In this section, we study boundary value problems of Hadamard type fractional differential equations and inclusions with nonlocal boundary conditions. Firstly, we discuss the existence and uniqueness of solutions for the following boundary value problem

$$\begin{cases} D^{\alpha}x(t) = f(t, x(t)), \quad 1 < t < e, \quad 1 < \alpha \leq 2, \\ x(1) = 0, \quad x(\eta) = g(x), \; 1 < \eta < e, \end{cases} \tag{3.40}$$

where D^{α} is the Hadamard fractional derivative of order α, $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ and $g : C([0, 1], \mathbb{R}) \to \mathbb{R}$ are given continuous functions.

In passing, we remark that the nonlocal conditions are more plausible than the standard initial conditions for the formulation of certain physical phenomena involving interior points of the given domain. In (3.40), $g(x)$ may be regarded as $g(x) = \sum_{j=1}^{p} \alpha_j x(t_j)$ where $\alpha_j, j = 1, \ldots, p$, are given constants and $0 < t_1 < \ldots < t_p \leq 1$. Further details can be found in the work by Byszewski [54, 55].

The main results for the problem (3.40) rely on Banach's contraction principle and a fixed point theorem due to O'Regan (Theorem 1.6).

Secondly, we extend our study to the case of inclusions by considering the following boundary value problem:

$$\begin{cases} D^{\alpha}x(t) \in F(t, x(t)), \quad 1 < t < e, \quad 1 < \alpha \leq 2, \\ x(1) = 0, \quad x(\eta) = g(x), \; 1 < \eta < e, \end{cases} \tag{3.41}$$

where $F : [1, e] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map, $\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$. We show the existence of solutions for the problem (3.41) by using the nonlinear alternative for contractive maps, when the multivalued map $F(t, x)$ is convex valued.

Lemma 3.4 *Given $y \in \mathscr{E}$, the boundary value problem*

$$\begin{cases} D^{\alpha}x(t) = y(t), \quad 1 < t < e, \quad 1 < \alpha \leq 2, \\ x(1) = 0, \quad x(\eta) = y_0 \in \mathbb{R}, \end{cases} \tag{3.42}$$

is equivalent to the integral equation

$$x(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds \\ -\frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds + \frac{(\log t)^{\alpha-1}}{(\log\eta)^{\alpha-1}}y_0,\ t\in[1,e]. \tag{3.43}$$

Proof We omit the proof as it is similar to that of Lemma 3.3. □

3.6 Existence Results: The Single-Valued Case

In view of Lemma 3.4, we define an operator $\mathscr{Q}:\mathscr{E}\to\mathscr{E}$ by

$$\mathscr{Q}x(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f(s,x(s))}{s}ds \\ -\frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{f(s,x(s))}{s}ds \\ +\frac{(\log t)^{\alpha-1}}{(\log\eta)^{\alpha-1}}g(x),\quad t\in[1,e]. \tag{3.44}$$

Observe that the existence of a fixed point for the operator $\mathscr{Q}$ defined by (3.44) implies the existence of a solution for the problem (3.40).

Theorem 3.10 *Let* $f:[1,e]\times\mathbb{R}\to\mathbb{R}$ *and* $g:C([1,e],\mathbb{R})\to\mathbb{R}$ *be continuous functions. Assume that:*

(3.10.1) $|f(t,x)-f(t,y)|\le L|x-y|,\ \forall t\in[1,e],\ L>0,\ x,y\in\mathbb{R}$;
(3.10.2) $|g(u)-g(v)|\le \ell\|u-v\|,\ 0<\ell<(\log\eta)^{\alpha-1}$, *for all* $u,v\in\mathscr{E}$;
(3.10.3) $\gamma := \dfrac{L}{\Gamma(\alpha+1)}(1+\log\eta)+\dfrac{\ell}{(\log\eta)^{\alpha-1}}<1.$

Then the boundary value problem (3.40) has a unique solution on $[1,e]$.

Proof For $x,y\in\mathscr{E}$ and for each $t\in[1,e]$, from the definition of $\mathscr{Q}$ and assumptions (3.10.1) and (3.10.2), we obtain

$$|(\mathscr{Q}x)(t)-(\mathscr{Q}y)(t)| \\ \le \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|f(s,x(s))-f(s,y(s))|}{s}ds \\ +\frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{|f(s,x(s))-f(s,y(s))|}{s}ds$$

$$
\begin{aligned}
&+\frac{(\log t)^{\alpha-1}}{(\log \eta)^{\alpha-1}}|g(x)-g(y)| \\
&\leq L\|x-y\|\left[\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{s}ds\right. \\
&\left.+\frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log \eta)^{\alpha-1}}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{1}{s}ds\right]+\frac{\ell}{(\log \eta)^{\alpha-1}}\|x-y\| \\
&\leq \|x-y\|\left\{\frac{L}{\Gamma(\alpha+1)}(1+\log\eta)+\frac{\ell}{(\log \eta)^{\alpha-1}}\right\}.
\end{aligned}
$$

Hence

$$\|\mathscr{Q}x-\mathscr{Q}y\| \leq \gamma\|x-y\|.$$

As $\gamma < 1$ by (3.10.3), $\mathscr{Q}$ is a contraction map from the Banach space $\mathscr{E}$ into itself. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Example 3.7 Consider the following fractional boundary value problem

$$
\begin{cases}
D^{3/2}x(t)=\dfrac{L}{2}\left(x+\dfrac{|\sin x|}{1+|\sin x|}+\cos t\right), \quad 1<t<e, \\
x(1)=0, \quad x\left(\dfrac{5}{4}\right)=\dfrac{1}{7}x\left(\dfrac{3}{2}\right)+\dfrac{1}{9}x\left(2\right)+\dfrac{1}{11}x\left(\dfrac{5}{2}\right),
\end{cases}
\tag{3.45}
$$

where L will be fixed later. Clearly $\eta=\frac{5}{4}$, $g(x)=\frac{1}{7}x\left(\frac{3}{2}\right)+\frac{1}{9}x\left(2\right)+\frac{1}{11}x\left(\frac{5}{2}\right)$. With the given values, it is found that $\ell \simeq 0.344877$ and the assumption (3.10.3) is satisfied for $L < 0.293351$. Thus all the conditions of Theorem 3.10 are satisfied. Hence the boundary value problem (3.45) has a unique solution on $[1, e]$.

In the sequel, we will use Theorem 1.6, by taking C to be E. For more details of such fixed point theorems, we refer a paper by Petryshyn [140].

Theorem 3.11 *Let $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that (3.10.2) holds. In addition, we assume that:*

(3.11.1) $g(0) = 0$;
(3.11.2) there exists a nonnegative function $p \in C([1, e], \mathbb{R}^+)$ and a nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that

$$|f(t,u)| \leq p(t)\psi(|u|) \ \textit{for any} \ (t,u) \in [1,e]\times\mathbb{R};$$

(3.11.3) $\sup_{r\in(0,\infty)} \frac{r}{p_0\psi(r)} > \frac{1}{1-\ell(\log\eta)^{1-\alpha}}$, *where*

$$p_0 = \frac{\|p\|}{\Gamma(\alpha+1)}(1+\log\eta). \tag{3.46}$$

Then the boundary value problem (3.40) has at least one solution on $[1, e]$.

Proof Let us decompose the operator $\mathcal{Q} : \mathcal{E} \to \mathcal{E}$ defined by (3.44) into a sum of two operators as

$$(\mathcal{Q}x)(t) = (\mathcal{Q}_1x)(t) + (\mathcal{Q}_2x)(t), \quad t \in [1, e], \tag{3.47}$$

where

$$\begin{aligned}(\mathcal{Q}_1x)(t) = &\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f(s,x(s))}{s}ds\\ &-\frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{f(s,x(s))}{s}ds, \quad t\in[1,e],\end{aligned} \tag{3.48}$$

and

$$(\mathcal{Q}_2x)(t) = \frac{(\log t)^{\alpha-1}}{(\log\eta)^{\alpha-1}}g(x), \quad t\in[1,e]. \tag{3.49}$$

Let

$$\Omega_r = \{x \in \mathcal{E} : \|x\| < r\}.$$

From (3.11.3), there exists a number $r_0 > 0$ such that

$$\frac{r_0}{p_0\psi(r_0)} > \frac{1}{1-\ell(\log\eta)^{\alpha-1}}. \tag{3.50}$$

We shall show that the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ satisfy all the conditions of Theorem 1.6.

Step 1. *The operator* $\mathcal{Q}_2 : \bar{\Omega}_{r_0} \to \mathcal{E}$ *is contractive*. Indeed, we have:

$$\begin{aligned}|(\mathcal{Q}_2x)(t) - (\mathcal{Q}_2y)(t)| &= \frac{(\log t)^{\alpha-1}}{(\log\eta)^{\alpha-1}}|g(x)-g(y)|\\ &\le \frac{\ell}{(\log\eta)^{\alpha-1}}\|x-y\|,\end{aligned}$$

and hence by (3.10.2), $\mathcal{Q}_2$ is contractive.

Step 2. *The operator $\mathscr{Q}_1$ is continuous and completely continuous*. We first show that $\mathscr{Q}_1(\bar{\Omega}_{r_0})$ is bounded. For any $x \in \bar{\Omega}_{r_0}$, we have

$$\begin{aligned}\|\mathscr{Q}_1 x\| \leq & \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|f(s,x(s))|}{s}ds \\ & + \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{|f(s,x(s))|}{s}ds \\ \leq & \|p\|\psi(r_0)\left[\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{s}ds\right. \\ & \left. + \frac{1}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{1}{s}ds\right] \\ \leq & \|p\|\psi(r_0)\frac{1}{\Gamma(\alpha+1)}(1+\log\eta).\end{aligned}$$

This proves that $\mathscr{Q}_1(\bar{\Omega}_{r_0})$ is uniformly bounded.
In addition, for any $\tau_1, \tau_2 \in [1, e]$, $\tau_1 < \tau_2$, we have:

$$\begin{aligned} & |(\mathscr{Q}_1 x)(\tau_2) - (\mathscr{Q}_1 x)(\tau_1)| \\ \leq & \frac{\psi(r_0)\|p\|}{\Gamma(\alpha)}\left|\int_1^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{1}{s}ds - \int_1^{\tau_1}\left(\log\frac{\tau_1}{s}\right)^{\alpha-1}\frac{1}{s}ds\right| \\ & + \frac{\psi(r_0)\|p\||(\log\tau_2)^{\alpha-1}-(\log\tau_1)^{\alpha-1}|}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta\left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{1}{s}ds \\ \leq & \frac{\psi(r_0)\|p\|}{\Gamma(\alpha+1)}[|(\log\tau_2)^\alpha - (\log\tau_1)^\alpha| + 2(\log(\tau_2/\tau_1))^\alpha] \\ & + \frac{\psi(r_0)\|p\||(\log\tau_2)^{\alpha-1}-(\log\tau_1)^{\alpha-1}|}{\Gamma(\alpha+1)}(\log\eta),\end{aligned}$$

which is independent of x, and tends to zero as $\tau_2 - \tau_1 \to 0$. Thus, $\mathscr{Q}_1$ is equicontinuous. Hence, by the Arzelá-Ascoli Theorem, $\mathscr{Q}_1(\bar{\Omega}_{r_0})$ is a relatively compact set. Now, let $x_n, x \subset \bar{\Omega}_{r_0}$ with $\|x_n - x\| \to 0$. Then the limit $|x_n(t) - x(t)| \to 0$ as $n \to \infty$ uniformly on $[1, e]$. From the uniform continuity of $f(t, x)$ on the compact set $[1, e] \times [-r_0, r_0]$ it follows that $\|f(t, x_n(t)) - f(t, x(t))\| \to 0$ is uniformly valid on $[1, e]$. Hence $\|\mathscr{Q}_1 x_n - \mathscr{Q}_1 x\| \to 0$ as $n \to \infty$, which proves the continuity of $\mathscr{Q}_1$. Hence Step 2 is completely established.

Step 3. *The set $F(\bar{\Omega}_{r_0})$ is bounded.* By (3.10.2) and (3.11.2), we get

$$\|\mathscr{Q}_2(x)\| \leq \frac{1}{(\log\eta)^{\alpha-1}}\ell r_0,$$

for any $x \in \bar{\Omega}_{r_0}$. This, with the boundedness of the set $\mathscr{Q}_1(\bar{\Omega}_{r_0})$, implies that the set $\mathscr{Q}(\bar{\Omega}_{r_0})$ is bounded.

Step 4. *Finally, it will be shown that the case (C2) in Theorem 1.6 does not occur.* To this end, we suppose that (C2) holds. Then, there exist $\lambda \in (0,1)$ and $x \in \partial\Omega_{r_0}$ such that $x = \lambda \mathscr{Q}x$. So, we have $\|x\| = r_0$ and

$$x(t) = \lambda \Bigg\{ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s,x(s))}{s} ds - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log \eta)^{\alpha-1}} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{f(s,x(s))}{s} ds + \frac{(\log t)^{\alpha-1}}{(\log \eta)^{\alpha-1}} g(x) \Bigg\}, \ t \in [1,e].$$

With hypotheses (3.11.1)–(3.11.3), we have

$$|x(t)| \le \|p\|\psi(\|x\|)\frac{1+\log \eta}{\Gamma(\alpha+1)} + \frac{1}{(\log \eta)^{\alpha-1}}\ell\|x\|.$$

Taking supremum over $t \in [1,e]$, we get $r_0 \le p_0\psi(r_0) + (\log \eta)^{1-\alpha}\ell r_0$. Thus,

$$\frac{r_0}{p_0\psi(r_0)} \le \frac{1}{1-\ell(\log \eta)^{1-\alpha}},$$

which contradicts (3.50). Thus it follows that the operators $\mathscr{Q}_1$ and $\mathscr{Q}_2$ satisfy all the conditions of Theorem 1.6. Hence, the operator $\mathscr{Q}$ has at least one fixed point $x \in \bar{\Omega}_{r_0}$, which is the solution of the boundary value problem (3.40). □

Example 3.8 Consider the following fractional boundary value problem

$$\begin{cases} D^{3/2}x(t) = \dfrac{1}{\sqrt{63+t^2}} 2^{1+|\sin x|}, \quad 1 < t < e, \\ x(1) = 0, \quad x\left(\dfrac{5}{4}\right) = \dfrac{1}{7}x\left(\dfrac{3}{2}\right) + \dfrac{1}{9}x\left(2\right) + \dfrac{1}{11}x\left(\dfrac{5}{2}\right), \end{cases} \tag{3.51}$$

Since $\left|\dfrac{1}{\sqrt{63+t^2}} 2^{1+|\sin x|}\right| \le \dfrac{1}{2}$, we take $\|p\| = \dfrac{1}{2}$ and $\psi(|u|) = 1$. By the condition (3.11.3), we find that $r_0 > 1.70445$. Obviously all the conditions of Theorem 3.11 are satisfied. Therefore, the conclusion of Theorem 3.11 applies to the problem (3.51).

3.7 Existence Result: The Multivalued Case

Definition 3.2 A function $x \in \mathscr{C}^2([1,e],\mathbb{R})$ is called a solution of problem (3.41) if there exists a function $f \in L^1([1,e],\mathbb{R})$ with $f(t) \in F(t,x(t))$, a.e. on $[1,e]$ such that $D^\alpha x(t) = f(t)$, a.e. on $[1,e]$ and $x(1) = 0, \ x(\eta) = g(x)$.

Theorem 3.12 *Assume that (3.10.2) holds. In addition, we suppose that:*

(3.12.1) *$F : [1,e] \times \mathbb{R} \to \mathscr{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory multivalued map;*
(3.12.2) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([1,e],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t,x)\} \leq p(t)\psi(\|x\|) \ \textit{for each} \ (t,x) \in [1,e] \times \mathbb{R};$$

(3.12.3) *there exists a number $M > 0$ such that*

$$\frac{(1-\ell(\log\eta)^{1-\alpha})M}{\|p\|\psi(M)\dfrac{(1+\log\eta)}{\Gamma(\alpha+1)}} > 1. \tag{3.52}$$

Then the boundary value problem (3.41) has at least one solution on $[1,e]$.

Proof To transform the problem (3.41) into a fixed point problem, we define an operator $\mathscr{F} : \mathscr{E} \longrightarrow \mathscr{P}(\mathscr{E})$ as

$$\mathscr{F}(x) = \left\{ \begin{array}{l} h \in \mathscr{E} : \\ h(t) = \left\{ \begin{array}{l} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds \\ -\dfrac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}} \displaystyle\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds \\ +\dfrac{(\log t)^{\alpha-1}}{(\log\eta)^{\alpha-1}} g(x), \quad t \in [1,e], \end{array} \right. \end{array} \right\} \tag{3.53}$$

for $f \in S_{F,x}$.

Next, we introduce two operators $\mathscr{A} : \mathscr{E} \longrightarrow \mathscr{E}$ and $\mathscr{B} : \mathscr{E} \longrightarrow \mathscr{P}(\mathscr{E})$ as follows:

$$\mathscr{A}x(t) = \frac{(\log t)^{\alpha-1}}{(\log\eta)^{\alpha-1}} g(x), \quad t \in [1,e], \tag{3.54}$$

and

$$\mathscr{B}(x) = \left\{ \begin{array}{l} h \in \mathscr{E} : \\ h(t) = \left\{ \begin{array}{l} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds \\ -\dfrac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}} \displaystyle\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds, \ t \in [1,e]. \end{array} \right. \end{array} \right\} \tag{3.55}$$

Observe that $\mathscr{F} = \mathscr{A} + \mathscr{B}$. We shall show that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 1.17 on $[1, e]$. First, we show that the operators $\mathscr{A}$ and $\mathscr{B}$ define the multivalued operators $\mathscr{A}, \mathscr{B} : B_r \to \mathscr{P}_{cp,c}(\mathscr{E})$ where $B_r = \{x \in \mathscr{E} : \|x\| \leq r\}$ is a bounded set in $\mathscr{E}$. We prove that $\mathscr{B}$ is compact-valued on B_r. Note that the operator $\mathscr{B}$ is equivalent to the composition $\mathscr{L} \circ S_F$, where $\mathscr{L}$ is the continuous linear operator on $L^1([1, e], \mathbb{R})$ into $\mathscr{E}$, defined by

$$\mathscr{L}(v)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log \eta)^{\alpha-1}} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds.$$

Suppose that $x \in B_r$ is arbitrary and let $\{v_n\}$ be a sequence in $S_{F,x}$. Then, by definition of $S_{F,x}$, we have $v_n(t) \in F(t, x(t))$ for almost all $t \in [1, e]$. Since $F(t, x(t))$ is compact for all $t \in J$, there is a convergent subsequence of $\{v_n(t)\}$, (we denote it by $\{v_n(t)\}$ again) that converges in measure to some $v(t) \in S_{F,x}$ for almost all $t \in J$. On the other hand, $\mathscr{L}$ is continuous, so $\mathscr{L}(v_n)(t) \to \mathscr{L}(v)(t)$ point-wise on $[1, e]$.

In order to show that the convergence is uniform, we have to show that $\{\mathscr{L}(v_n)\}$ is an equi-continuous sequence. Let $\tau_1, \tau_2 \in [1, e]$ with $\tau_1 < \tau_2$. Then, we have

$$\begin{aligned}
&|\mathscr{L}(v_n)(\tau_2) - \mathscr{L}(v_n)(\tau_1)| \\
&\leq \frac{\psi(r)\|p\|}{\Gamma(\alpha)} \left| \int_1^{\tau_2} \left(\log \frac{\tau_2}{s}\right)^{\alpha-1} \frac{1}{s} ds - \int_1^{\tau_1} \left(\log \frac{\tau_1}{s}\right)^{\alpha-1} \frac{1}{s} ds \right| \\
&\quad + \frac{\psi(\rho)\|p\| |(\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1}|}{\Gamma(\alpha)(\log \eta)^{\alpha-1}} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{1}{s} ds \\
&\leq \frac{\psi(\rho)\|p\|}{\Gamma(\alpha+1)} [|(\log \tau_2)^\alpha - (\log \tau_1)^\alpha| + 2(\log(\tau_2/\tau_1))^\alpha] \\
&\quad + \frac{\psi(r)\|p\| |(\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1}|}{\Gamma(\alpha+1)} (\log \eta).
\end{aligned}$$

We see that the right hand of the above inequality tends to zero as $\tau_2 \to \tau_1$. Thus, the sequence $\{\mathscr{L}(v_n)\}$ is equi-continuous by the Arzelá-Ascoli Theorem, and hence there exists a uniformly convergent subsequence. So, there is a subsequence of $\{v_n\}$, (we denote it again by $\{v_n\}$) such that $\mathscr{L}(v_n) \to \mathscr{L}(v)$. Note that $\mathscr{L}(v) \in \mathscr{L}(S_{F,x})$. Hence, $\mathscr{B}(x) = \mathscr{L}(S_{F,x})$ is compact for all $x \in B_r$. So $\mathscr{B}(x)$ is compact.

Now, we show that $\mathscr{B}(x)$ is convex for all $x \in \mathscr{E}$. Let $z_1, z_2 \in \mathscr{B}(x)$. We select $f_1, f_2 \in S_{F,x}$ such that

$$z_i(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f_i(s)}{s} ds - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log \eta)^{\alpha-1}} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{f_i(s)}{s} ds,$$

$i = 1, 2$, for almost all $t \in [1, e]$. Let $0 \le \lambda \le 1$. Then, we have

$$\begin{aligned}
&[\lambda z_1 + (1-\lambda)z_2](t) \\
&= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{[\lambda f_1(s) + (1-\lambda)f_2(s)]}{s} ds \\
&\quad - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log \eta)^{\alpha-1}} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{[\lambda f_1(s) + (1-\lambda)f_2(s)]}{s} ds.
\end{aligned}$$

Since F has convex values, so $S_{F,x}$ is convex and $\lambda f_1(s) + (1-\lambda)f_2(s) \in S_{F,x}$. Thus

$$\lambda z_1 + (1-\lambda)z_2 \in \mathscr{B}(x).$$

Consequently, $\mathscr{B}$ is convex-valued. Obviously, $\mathscr{A}$ is compact and convex-valued.

For the sake of clarity, we split the rest of the proof into a number of steps and claims.

Step 1. *$\mathscr{A}$ is a contraction on $\mathscr{E}$*. This is a consequence of (3.10.2).

Step 2. *$\mathscr{B}$ is compact, convex valued and completely continuous*. This will be established in several claims.

CLAIM I. *$\mathscr{B}$ maps bounded sets into bounded sets in $\mathscr{E}$*. For that, let $B_\rho = \{x \in \mathscr{E} : \|x\| \le \rho\}$ be a bounded set in $\mathscr{E}$. Then, for each $h \in \mathscr{B}(x), x \in B_\rho$, we have

$$\begin{aligned}
|h(t)| &\le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{|f(s)|}{s} ds \\
&\quad + \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log \eta)^{\alpha-1}} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{|f(s)|}{s} ds \\
&\le \|p\|\psi(\rho)\left[\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{1}{s} ds \right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)(\log \eta)^{\alpha-1}} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{1}{s} ds\right] \\
&\le \|p\|\psi(\rho)\frac{1}{\Gamma(\alpha+1)}(1 + \log \eta),
\end{aligned}$$

and consequently, for each $h \in \mathscr{B}(B_\rho)$, we have

$$\|h\| \le \|p\|\psi(r)\frac{1}{\Gamma(\alpha+1)}(1 + \log \eta).$$

CLAIM II. *$\mathscr{B}$ maps bounded sets into equicontinuous sets*. As before, let B_ρ be a bounded set and let $h \in \mathscr{B}(x)$ for $x \in B_\rho$. Let $\tau_1, \tau_2 \in [1, e]$ with $\tau_1 < \tau_2$ and

$x \in B_\rho$. For each $h \in \mathscr{B}(x)$, as before, we obtain

$$|h(\tau_2) - h(\tau_1)| \le \frac{\psi(\rho)\|p\|}{\Gamma(\alpha)}[|(\log\tau_2)^\alpha - (\log\tau_1)^\alpha| + 2(\log(\tau_2/\tau_1))^\alpha] + \frac{\psi(r)\|p\||\log\tau_2)^{\alpha-1} - (\log\tau_1)^{\alpha-1}|}{\Gamma(\alpha+1)}(\log\eta),$$

which is independent of x and tends to zero as $\tau_2 - \tau_1 \to 0$. Therefore, it follows by the Arzelá-Ascoli Theorem, that $\mathscr{B} : \mathscr{E} \to \mathscr{P}(\mathscr{E})$ is completely continuous.

CLAIM III. *$\mathscr{B}$ has a closed graph.* Let $x_n \to x_*, h_n \in \mathscr{B}(x_n)$ and $h_n \to h_*$. Then, we need to show that $h_* \in \mathscr{B}(x_*)$. Associated with $h_n \in \mathscr{B}(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in [1, e]$,

$$h_n(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f_n(s)}{s}ds - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{f_n(s)}{s}ds.$$

Then, we have to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [1, e]$,

$$h_*(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f_*(s)}{s}ds - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{f_*(s)}{s}ds.$$

Let us consider the continuous linear operator $\Theta : L^1([1, e], \mathbb{R}) \to \mathscr{C}$ given by

$$f \mapsto \Theta(f)(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f(s)}{s}ds - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{f(s)}{s}ds.$$

Observe that

$$\|h_n(t) - h_*(t)\| = \left\| \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{(f_n(s) - f_*(s))}{s}ds - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{(f_n(s) - f_*(s))}{s}ds \right\| \to 0,$$

as $n \to \infty$. Thus, it follows by Lemma 1.2 that $\Theta \circ S_{F,x}$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{f_*(s)}{s}ds - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha-1}\frac{f_*(s)}{s}ds,$$

for some $f_* \in S_{F,x_*}$. Hence $\mathscr{B}$ has a closed graph (and therefore has closed values). In consequence, the operator $\mathscr{B}$ is compact valued.

Thus, the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy hypotheses of Theorem 1.17 and therefore, by its application, it follows either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible. If $x \in \lambda\mathscr{A}(x) + \lambda\mathscr{B}(x)$ for $\lambda \in (0, 1)$, then there exists $f \in S_{F,x}$ such that

$$x(t) = \lambda\Bigg\{ \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} \frac{f(s)}{s} ds \\ - \frac{(\log t)^{\alpha-1}}{\Gamma(\alpha)(\log\eta)^{\alpha-1}} \int_1^\eta \Big(\log\frac{\eta}{s}\Big)^{\alpha-1} \frac{f(s)}{s} ds + \frac{(\log t)^{\alpha-1}}{(\log\eta)^{\alpha-1}} g(x)\Bigg\}, t \in [1, e].$$

Consequently, we have

$$|x(t)| \le \|p\|\psi(\|x\|)\frac{1}{\Gamma(\alpha+1)}(1 + \log\eta) + \frac{1}{(\log\eta)^{\alpha-1}}\ell\|x\|.$$

If condition (ii) of Theorem 1.17 holds, then there exists $\lambda \in (0, 1)$ and $x \in \partial B_M$ with $x = \lambda\mathscr{F}(x)$. Then, x is a solution of (3.44) with $\|x\| = M$. Now, by the last inequality, we have

$$\frac{(1 - \ell(\log\eta)^{1-\alpha})M}{\|p\|\psi(M)\dfrac{(1 + \log\eta)}{\Gamma(\alpha+1)}} \le 1,$$

which contradicts (3.52). Hence, $\mathscr{F}$ has a fixed point in $[1, e]$ by Theorem 1.17, and consequently the problem (3.41) has a solution. This completes the proof. □

Example 3.9 Consider the following fractional boundary value problem

$$\begin{cases} D^{3/2}x(t) \in F(t, x), \quad 1 < t < e, \\ x(1) = 0, \quad x\Big(\dfrac{5}{4}\Big) = \dfrac{1}{7}x\Big(\dfrac{3}{2}\Big) + \dfrac{1}{9}x\Big(2\Big) + \dfrac{1}{11}x\Big(\dfrac{5}{2}\Big), \end{cases} \tag{3.56}$$

where

$$F(t, x) = \Big[\frac{1}{(t+2)}\frac{|\sin x|^3}{8(|\sin x|^3 + 3)} + \frac{1}{10}, \ \frac{t}{3\sqrt{(t+1)}}\frac{|x|}{|x|+1}\Big].$$

Clearly $\|F(t, x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t, x)\} \le p(t)\psi(\|x\|)$ for each $(t, x) \in [1, e] \times \mathbb{R}$ with $p(t) = \frac{t}{3\sqrt{(t+1)}}$, $\psi(\|x\|) = 1$. By the condition (3.12.3), it is found that $M > 1.601815$. Thus all the conditions of Theorem 3.12 are satisfied and in consequence, there exists a solution for the problem (3.56) on $[1, e]$.

3.8 Notes and Remarks

In Sect. 3.2, we studied a three-point boundary value problem of Hadamard type fractional differential equations via Krasnoselskii-Zabreiko fixed point theorem.

We have investigated in Sect. 3.3 the existence and uniqueness of solutions for a semi-linear Hadamard type fractional differential equation supplemented with nonlocal non-conserved boundary conditions involving Hadamard integral. The uniqueness result is proved by applying Banach's fixed point theorem while the three existence results are established by means of Krasnoselskii's fixed point theorem, Leray-Schauder's nonlinear alternative, and Leray-Schauder degree respectively. We have also discussed a companion problem (3.34) by replacing the condition $AI^{\gamma}x(\eta) + Bx(e) = c$ with $AI^{\gamma}x(e) + Bx(\eta) = c$ in problem (3.18). The results presented in this section are more general and correspond to several known and new results corresponding to specific values of the parameters involved in the problem (3.18). For instance, we have:

- By taking $A = 0, c = 0, B \neq 0$, our results correspond to the ones for Hadamard type fractional differential equations with Dirichlet boundary conditions.
- Letting $A = 1, B = -1, c = 0$ and $\eta \to e$ in the results of this paper, we obtain the ones presented in [22].
- With $A \neq 0, B = 0, c = 0$, our problem becomes an "average type" nonlocal boundary value problem in the sense of Hadamard integral ($\gamma = 1$ in classical sense). This reduced integral condition can also be termed as a "conserved" condition in the sense of Hadamard. In this case, the operator $Q : \mathscr{E} \to \mathscr{E}$ takes the form:

$$Qx(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s, x(s))}{s} ds - \frac{A(\log t)^{\alpha-1}}{\Delta\Gamma(\gamma+\alpha)} \int_1^{\eta} \left(\log \frac{\eta}{s}\right)^{\gamma+\alpha-1} \frac{f(s, x(s))}{s} ds, \quad t \in [1, e].$$

In relation to problem (3.34), we can make similar observations.

The multivalued case of the problems studied in Sect. 3.3 is considered in Sect. 3.4. We have established some existence results when the right hand side is convex as well as non-convex valued. The first result relies on the nonlinear alternative of Leray-Schauder type. In the second result, we combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values, while in the third result, a fixed point theorem for contractive multivalued maps due to Covitz and Nadler is used. Note that the special case $A = -1, B = 1, c = 0$ was studied in [12].

In Sect. 3.5, we studied boundary value problems of Hadamard type fractional differential equations and inclusions with nonlocal boundary conditions. The results for single-valued case are proved via Banach's contraction principle and a fixed point theorem due to O'Regan, while the result for multivalued case is obtained by means of the nonlinear alternative for contractive maps when the multivalued map $F(t,x)$ is convex valued.

The content of this chapter is based on the papers [15, 24, 30] and [23].

Chapter 4
Existence Results for Mixed Hadamard and Riemann-Liouville Fractional Integro-Differential Equations and Inclusions

4.1 Introduction

We introduce a new class of mixed initial value problems involving Hadamard derivative and Riemann-Liouville fractional integrals. Existence as well existence and uniqueness results are proved for mixed initial value problems involving Hadamard and Riemann-Liouville type integro-differential equations and inclusions via appropriate fixed point theorems. We also obtain an existence result for the inclusion case by following a relatively new approach known as "endpoint theory".

4.2 Existence Results for Mixed Hadamard and Riemann-Liouville Fractional Integro-Differential Equations

In this section, we discuss the existence and uniqueness results for a new class of mixed initial value problems involving Hadamard derivative and Riemann-Liouville fractional integrals given by

$$\begin{cases} D^{\alpha}\left(x(t)-\displaystyle\sum_{i=1}^{m} I^{\beta_i}h_i(t,x(t))\right)=g(t,x(t),Kx(t)), \quad t\in J:=[1,T], \\ x(1)=0, \end{cases} \tag{4.1}$$

where D^{α} denotes the Hadamard fractional derivative of order α, $0<\alpha\leq 1$, I^{ϕ} is the Riemann-Liouville fractional integral of order $\phi>0$, $\phi\in\{\beta_1,\beta_2,\ldots,\beta_m\}$, $g\in C(J\times\mathbb{R}^2,\mathbb{R})$, $h_i\in C(J\times\mathbb{R},\mathbb{R})$ with $h(1,0)=0$, $i=1,2,\ldots,m$, and $Kx(t)=\int_1^t \varphi(t,s)x(s)ds$, $\varphi(t,s)\in C(J^2,\mathbb{R})$.

B. Ahmad et al., *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, DOI 10.1007/978-3-319-52141-1_4

The main tools of our study for (4.1) include Krasnoselskii's fixed point theorem, Banach's fixed point theorem and Leray-Schauder's nonlinear alternative.

In order to define the solution for problem (4.1), we need the following lemma. The proof of this lemma is obvious [96] in view of the fact that Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral.

Lemma 4.1 *Let* $0 < \alpha \leq 1$*, and the functions* g, h_i*,* $i = 1, 2, \ldots, m$ *satisfy problem* (4.1)*. Then the unique solution of the problem* (4.1) *is*

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g(s, x(s), Kx(s)) \frac{ds}{s} + \sum_{i=1}^{m} I^{\beta_i} h_i(t, x(t)), \quad t \in J. \tag{4.2}$$

Let $E = C(J, \mathbb{R})$ be the space of continuous real-valued functions defined on $J = [1, T]$ endowed with the norm $\|x\| = \sup_{t \in J} |x(t)|$.

Theorem 4.1 *Assume that:*

(4.1.1) there exists a constant $L_0 > 0$*, such that*

$$|h_i(t, x) - h_i(t, y)| \leq L_0 |x - y|, \tag{4.3}$$

for $t \in J$ *and* $x, y \in \mathbb{R}$*,* $i = 1, 2, \ldots, m$*;*

(4.1.2) there exist functions $\theta_i \in C(J, \mathbb{R}^+)$*,* $i = 1, 2, \ldots, m$*, such that*

$$|h_i(t, x)| \leq \theta_i(t), \quad \forall (t, x) \in J \times \mathbb{R};$$

(4.1.3) there exist functions $\nu, \mu \in C(J, \mathbb{R}^+)$*, such that*

$$|g(t, x, y)| \leq \nu(t) + \mu(t)|y|, \forall (t, x, y) \in J \times \mathbb{R}^2.$$

Then the problem (4.1) *has at least one solution on* J*, provided that*

$$L_0 \sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i + 1)} < 1. \tag{4.4}$$

Proof Setting $\sup_{t \in J} |\nu(t)| = \|\nu\|$, $\sup_{t \in J} |\mu(t)| = \|\mu\|$, $\sup_{t \in J} |\theta_i(t)| = \|\theta_i\|$, $i = 1, 2, \ldots, m$, we consider $B_R = \{x \in E : \|x\| \leq R\}$, where

$$R \geq \left(\sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i + 1)} \|\theta_i\| + \frac{(\log T)^{\alpha}}{\Gamma(\alpha + 1)} \|\nu\| \right) \Big/ \left(1 - \varphi_0 \|\mu\| \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha + 1)} \right] \right),$$

$$\gamma = T \int_0^{\log T} u^{\alpha-1} e^{-u} du, \; \varphi_0 = \sup\{|\varphi(t, s)| \; : \; (t, s) \in J \times J\} \text{ and}$$

$$\varphi_0 \|\mu\| \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha + 1)} \right] < 1.$$

Next we define the operators $\mathscr{Q} : B_R \to E$ and $\mathscr{T} : B_R \to E$ by

$$\mathscr{Q}x(t) = \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i - 1} h_i(s, x(s))ds, \quad t \in J, \tag{4.5}$$

$$\mathscr{T}x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g(s, x(s), Kx(s))\frac{ds}{s}, \quad t \in J. \tag{4.6}$$

For any $x, y \in B_R$, we have

$$\begin{aligned}
|\mathscr{Q}x(t) + \mathscr{T}y(t)| &\le \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i-1}|h_i(s, x(s))|ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |g(s, y(s), Ky(s))| \frac{ds}{s} \\
&\le \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i-1}|\theta_i(s)|ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} (|\nu(s)| + |\mu(s)||Ky(s)|) \frac{ds}{s} \\
&\le \sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)}\|\theta_i\| + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\|\nu\| \\
&\quad + \varphi_0\|\mu\|R\left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right] \\
&\le R.
\end{aligned}$$

Now we show that $\mathscr{T}$ is continuous and compact. The operator $\mathscr{T}$ is obviously continuous. Also, $\mathscr{T}$ is uniformly bounded on B_R as

$$\|\mathscr{T}x\| \le \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\|\nu\| + \|\mu\|\varphi_0 R\left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right].$$

Let $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$ and $x \in B_R$. We define $\sup_{(t,x,y)\in J\times B_R\times B_R} |g(t, x, y)| = \bar{g} < \infty$. Then, we have

$$\begin{aligned}
|\mathscr{T}x(\tau_2) - \mathscr{T}x(\tau_1)| &= \left| \frac{1}{\Gamma(\alpha)} \int_1^{\tau_2} \left(\log \frac{\tau_2}{s}\right)^{\alpha-1} g(s, x(s), Kx(s))\frac{ds}{s} \right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha)} \int_1^{\tau_1} \left(\log \frac{\tau_1}{s}\right)^{\alpha-1} g(s, x(s), Kx(s))\frac{ds}{s} \right| \\
&\le \frac{\bar{g}}{\Gamma(\alpha+1)}|(\log \tau_2)^{\alpha} - (\log \tau_1)^{\alpha}| + 2(\log(\tau_2/\tau_1))^{\alpha},
\end{aligned}$$

which is independent of x, and tends to zero as $\tau_2 - \tau_1 \rightarrow 0$. Thus, $\mathscr{T}$ is equicontinuous. So $\mathscr{T}$ is relatively compact on B_R. Hence, by the Arzelá-Ascoli Theorem, $\mathscr{T}$ is compact on B_R.

Now, we show that $\mathscr{Q}$ is a contraction. Let $x, y \in B_R$. Then, for $t \in J$, we have

$$
\begin{aligned}
|\mathscr{Q}x(t) - \mathscr{Q}y(t)| &\leq \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i - 1} |h_i(s, x(s)) - h_i(s, y(s))| ds \\
&\leq L_0 \|x - y\| \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i - 1} ds \\
&\leq L_0 \|x - y\| \sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i + 1)}.
\end{aligned}
$$

Hence, by the given assumption (4.4), $\mathscr{T}$ is a contraction.

Thus all the assumptions of Krasnoselskii fixed point theorem (Theorem 1.2) are satisfied, which implies that the problem (4.1) has at least one solution on J. □

Example 4.1 Consider the following mixed Hadamard and Riemann-Liouville fractional integro-differential initial value problem:

$$
\begin{cases}
D^{1/4}\left(x(t) - \sum_{i=1}^{3} I^{\beta_i} h_i(t, x(t))\right) = \dfrac{1}{2} + \dfrac{(\sqrt{t} + \log t)|x(t)|}{|x(t)| + 3} \\
\qquad\qquad + (1 + \log t) \displaystyle\int_1^t \frac{\sin(\pi \log t) x(s)}{3(s^2 + 1)} ds, \quad t \in \left[1, e^{1/2}\right], \\
x(1) = 0.
\end{cases}
\tag{4.7}
$$

Here $\alpha = 1/4$, $\beta_1 = 2/5$, $\beta_2 = 3/5$, $\beta_3 = 4/5$, $m = 3$, $T = e^{1/2}$, and

$$
h_1(t, x) = \frac{e^{-t}}{\sqrt{2}} \left(\frac{|x|}{1 + |x|}\right), \; h_2(t, x) = \frac{1}{4t}\left(\frac{1}{|x| + 3} + 1\right)|x|, \; h_3(t, x) = \frac{2(\log t)}{3} \sin |x|,
$$

$$
g(t, x, y) = \frac{1}{2} + \frac{(\sqrt{t} + \log t)|x|}{|x| + 3} + y(1 + \log t), \qquad \varphi(t, s) = \frac{\sin(\pi \log t)}{3(s^2 + 1)}.
$$

Using the given values, we have $\varphi_0 = 1/6$, $|h_i(t, x) - h_i(t, y)| \leq (1/3)|x - y|$, $i = 1, 2, 3$, $|g(t, x, y)| \leq (1/2) + 2\sqrt{t} + (1 + \log t)|y|$ which satisfy (4.1.1)–(4.1.3) with $L_0 = (1/3)$, $\nu(t) = (1/2) + 2\sqrt{t}$ and $\mu(t) = 1 + \log t$. We can show that $\gamma = 5.059974208$, $L_0 \sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i + 1)} = 0.856880869 < 1$ and

$$
\varphi_0 \|\mu\| \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha + 1)}\right] = 0.580837503 < 1.
$$

Therefore, by Theorem 4.1, the problem (4.7) has at least one solution on $[1, e^{1/2}]$.

Theorem 4.2 *Assume that $h_i : J \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are continuous functions satisfying the condition (4.1.1). In addition, we assume that:*

(4.2.1) $|g(t,x,y) - g(t,\bar{x},\bar{y})| \le L_1|x-\bar{x}| + L_2|y-\bar{y}|$, $\forall t \in J$, $L_1, L_2 > 0$, $x, \bar{x}, y, \bar{y} \in \mathbb{R}$.

Then the problem (4.1) has a unique solution if

$$\Lambda := L_0 \sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)} + L_1 \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} + L_2\varphi_0 \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} \right] < 1.$$

Proof Let us fix $\sup_{t\in[1,T]} |g(t,0,0)| = N$, $\sup_{t\in[1,T]} |h_i(t,0)| = K_i$, $i = 1, 2, \ldots, m$ and choose $r \ge \frac{M}{1-\Lambda}$, where $M = \sum_{i=1}^{m} K_i \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)} + N \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}$. Then, we show that $FB_r \subset B_r$, where $B_r = \{x \in E : \|x\| \le r\}$ with the operator $F : E \to E$ defined by

$$(Fx)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g(s, x(s), Kx(s)) \frac{ds}{s} + \sum_{i=1}^{m} I^{\beta_i} h_i(t, x(t)), \quad t \in J. \tag{4.8}$$

For $x \in B_r$, we have

$$\begin{aligned}
&|(Fx)(t)| \\
&\le \sup_{t\in[1,T]} \Bigg\{ \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i-1} |h_i(s,x(s))| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |g(s,x(s),Kx(s))| \frac{ds}{s} \Bigg\} \\
&\le \sup_{t\in[1,T]} \Bigg\{ \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i-1} (|h_i(s,x(s)) - h_i(s,0)| + |h_i(s,0)|) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} (|g(s,x(s),Kx(s)) - g(s,0,0)| + |g(s,0,0)|) \frac{ds}{s} \Bigg\} \\
&\le \sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)} (L_0 r + K_i) + (L_1 r + N) \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} + L_2\varphi_0 r \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} \right] \\
&= \Lambda r + M \le r,
\end{aligned}$$

and hence $\|Fx\| \le r$, which implies that $FB_r \subset B_r$.

Now, for $x, y \in E$ and for each $t \in J$, we obtain

$$
\begin{aligned}
&|(Fx)(t) - (Fy)(t)| \\
&\le \sup_{t\in[1,T]} \left\{ \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i - 1} |h_i(s, x(s)) - h_i(s, y(s))| ds \right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |g(s, x(s), Kx(s)) - g(s, y(s), Ky(s))| \frac{ds}{s} \right\} \\
&\le \left\{ L_0 \sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i + 1)} + L_1 \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} + L_2 \varphi_0 \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} \right] \right\} \|x - y\|.
\end{aligned}
$$

Therefore $\|Fx - Fy\| \le \Lambda \|x - y\|$. As $\Lambda < 1$, F is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Example 4.2 Consider the following mixed Hadamard and Riemann-Liouville fractional integro-differential initial value problem:

$$
\begin{cases}
D^{1/2}\left(x(t) - \sum_{i=1}^{4} I^{\beta_i} h_i(t, x(t))\right) = \dfrac{1}{10(1+t^2)} \left(\dfrac{x^2(t) + 10|x(t)|}{5 + |x(t)|} \right) \\
\qquad + \dfrac{1}{6} \displaystyle\int_1^t \dfrac{e^{1-st} x(s)}{(1+t)(1 + |\cos(\pi s/2)|)} ds + \dfrac{3}{4}, \ t \in [1, e], \\
x(1) = 0.
\end{cases}
\tag{4.9}
$$

Here $\alpha = 1/2$, $\beta_1 = 1/2$, $\beta_2 = 3/2$, $\beta_3 = 5/2$, $\beta_4 = 7/2$, $m = 4$, $T = e$, and

$$
\begin{aligned}
h_1(t, x) &= \frac{1}{20t}\left(\frac{1}{1+|x|} + 1\right)|x|, \quad h_2(t, x) = \frac{2e^{-2t}}{3} \sin|x|, \\
h_3(t, x) &= \frac{1}{5(2 + \log t)} \tan^{-1}|x|, \quad h_4(t, x) = \frac{1}{10}\left(\frac{|x|}{t + |x|}\right), \\
g(t, x, y) &= \frac{1}{10(1+t^2)} \left(\frac{x^2 + 10|x|}{5 + |x|}\right) + \frac{1}{6} y + \frac{3}{4}, \\
\varphi(t, s) &= \frac{e^{1-st}}{(1+t)(1 + |\cos(\pi s/2)|)}.
\end{aligned}
$$

With the given data, we find that $\varphi_0 = 1/2$, $|h_i(t, x) - h_i(t, y)| \le (1/10)|x - y|$, $i = 1, 2, 3, 4$, $|g(t, x, y) - g(t, \bar{x}, \bar{y})| \le (1/10)|x - \bar{x}| + (1/6)|y - \bar{y}|$ which satisfy (4.1.1) and (4.2.1) with $L_0 = (1/10)$, $L_1 = (1/10)$ and $L_2 = (1/6)$. Since $\int_0^1 u^{-1/2} e^{-u} du = \sqrt{\pi}\ \mathrm{erf}(1)$, where $\mathrm{erf}(\cdot)$ is the Gauss error function, we have $\gamma = 4.06015694$. Hence, we obtain $\Lambda = 0.88873633 < 1$. Therefore, by Theorem 4.2, the problem (4.9) has a unique solution on $[1, e]$.

Our final existence result is based on Leray-Schauder nonlinear alternative.

Theorem 4.3 *Assume that $g : J \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and the following conditions hold:*

(4.3.1) there exist functions $p_1, p_2 \in C(J, \mathbb{R}^+)$, and a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|g(t,x,y)| \leq p_1(t)\psi(|x|) + p_2(t)|y|$ for each $(t,x,y) \in J \times \mathbb{R}^2$;

(4.3.2) there exist functions $q_i \in C(J, \mathbb{R}^+)$, and nondecreasing functions $\Omega_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|h_i(t,x)| \leq q_i(t)\Omega_i(|x|)$ for each $(t,x) \in J \times \mathbb{R}$, $i = 1, 2, \ldots, m$;

(4.3.3) there exists a number $M_0 > 0$ such that

$$\frac{\left(1 - \|p_2\|\varphi_0\left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right]\right) M_0}{\sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)}\|q_i\|\Omega_i(M_0) + \|p_1\|\psi(M_0)\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}} > 1,$$

$$\text{with } \gamma = T\int_0^{\log T} u^{\alpha-1}e^{-u}du, \text{ and } \|p_2\|\varphi_0\left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right] < 1.$$

Then the problem (4.1) has at least one solution on J.

Proof Consider the operator F defined by (4.8). It is easy to prove that F is continuous. Next, we show that F maps bounded sets into bounded sets in E. For a positive number ρ, let $B_\rho = \{x \in E : \|x\| \leq \rho\}$ be a bounded set in E. Then, for each $x \in B_\rho$, we have

$$\begin{aligned}
|(Fx)(t)| &\leq \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)}\int_1^t (t-s)^{\beta_i-1}|h_i(s,x(s))|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}|g(s,x(s),Kx(s))|\frac{ds}{s} \\
&\leq \sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)}\|q_i\|\Omega_i(\rho) + \|p_1\|\psi(\rho)\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} \\
&\quad + \|p_2\|\varphi_0\rho\left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\|Fx\| &\leq \sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)}\|q_i\|\Omega_i(r) + \|p_1\|\psi(\rho)\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} \\
&\quad + \|p_2\|\varphi_0\rho\left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right].
\end{aligned}$$

Now, we show that F maps bounded sets into equicontinuous sets of E. Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in B_\rho$, where B_ρ is a bounded set of E. Then, we have

$$\begin{aligned}
&|(Fx)(t_2) - (Fx)(t_1)| \\
&\leq \left| \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^{t_2} (t_2 - s)^{\beta_1 - 1} h_i(s, x(s)) ds - \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^{t_1} (t_1 - s)^{\beta_i - 1} h_i(s, x(s)) ds \right| \\
&+ \left| \frac{1}{\Gamma(\alpha)} \int_1^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha - 1} g(s, x(s), Kx(s)) \frac{ds}{s} \right. \\
&\left. - \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{\alpha - 1} g(s, x(s), Kx(s)) \frac{ds}{s} \right| \\
&\leq \sum_{i=1}^{m} \frac{\|q_i\| \Omega_i(\rho)}{\Gamma(\beta_i + 1)} [2(t_2 - t_1)^{\beta_i} + |(t_2 - 1)^{\beta_i} - (t_1 - 1)^{\beta_i}|] \\
&+ \frac{\|p_1\| \psi(\rho) + \|p_2\| \varphi_0 \rho (T - 1)}{\Gamma(\alpha + 1)} [|(\log t_2)^\alpha - (\log t_1)^\alpha| + 2(\log(t_2/t_1))^\alpha].
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero, independently of $x \in B_\rho$ as $t_2 - t_1 \to 0$. Therefore it follows by the Arzelá-Ascoli Theorem that $F : E \to E$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative once we have shown the boundedness of the set of all solutions to equations $x = \lambda F x$ for $\lambda \in [0, 1]$.

Let x be a solution. Then, for $t \in J$, following the computations used in proving that F is bounded, we obtain

$$\begin{aligned}
|x(t)| \leq &\sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i + 1)} \|q_i\| \Omega_i(\|x\|) + \|p_1\| \psi(\|x\|) \frac{(\log T)^\alpha}{\Gamma(\alpha + 1)} \\
&+ \|p_2\| \varphi_0 \|x\| \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha + 1)} \right].
\end{aligned}$$

Consequently, we get

$$\frac{\left(1 - \|p_2\| \varphi_0 \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha + 1)} \right]\right) \|x\|}{\sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i + 1)} \|q_i\| \Omega_i(\|x\|) + \|p_1\| \psi(\|x\|) \frac{(\log T)^\alpha}{\Gamma(\alpha + 1)}} \leq 1.$$

In view of (4.3.3), there exists M_0 such that $\|x\| \neq M_0$. Let us set

$$U = \{x \in E : \|x\| < M_0 + 1\}.$$

Note that the operator $F : \overline{U} \to E$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda F x$ for some $\lambda \in (0, 1)$. Hence, by the Leray-Schauder alternative (Theorem 1.4), we deduce that F has a fixed point $x \in \overline{U}$ which is a solution of the problem (4.1). □

Example 4.3 Consider the following mixed Hadamard and Riemann-Liouville fractional integro-differential initial value problem

$$\begin{cases} D^{3/4}\left(x(t) - \sum_{i=1}^{5} I^{\beta_i} h_i(t, x(t))\right) = \frac{1}{1+2t^2}\left(\frac{\log t}{3}x(t) + \frac{1}{2}\right) \\ \qquad + \frac{e^{1-t}}{3}\int_1^t \frac{1 + |\sin \pi st|}{7+3st}x(s)ds, \ t \in [1, e^{3/2}], \\ x(1) = 0. \end{cases} \tag{4.10}$$

Here $\alpha = 3/4$, $\beta_1 = 1/2$, $\beta_2 = 3/4$, $\beta_3 = 5/4$, $\beta_4 = 3/2$, $\beta_5 = 7/4$, $m = 5$, $T = e^{3/2}$ and

$$h_i(t, x) = \left(\frac{1}{i + 2\log t}\right)\left(\frac{x}{24 + i}\right), \ i = 1, 2, 3, 4, 5,$$

$$g(t, x, y) = \frac{1}{1+2t^2}\left(\frac{\log t}{3}x + \frac{1}{2}\right) + \frac{ye^{1-t}}{3}, \quad \varphi(t, s) = \frac{1 + |\sin \pi st|}{7 + 3st}.$$

With the given data, we find that $\varphi_0 = 1/5$, $|g(t, x, y)| \le (1/(1 + 2t^2))((1/2)|x| + (1/2)) + (e^{1-t}/3)|y|$, $|h_i(t, x)| \le (1/(i + 2\log t))(|x|/(24 + i))$, $i = 1, 2, 3, 4, 5$, which satisfy (4.3.1)–(4.3.3) with $p_1(t) = 1/(1 + 2t^2)$, $\psi(|x|) = (1/2)(|x| + 1)$, $p_2(t) = e^{1-t}/3$, $q_i(t) = 1/(i + 2\log t)$, $\Omega_i(|x|) = |x|/(24 + i)$, $i = 1, 2, 3, 4, 5$. Further, we find that $\gamma = 4.681329240$, $\|p_1\| = 1/3$, $\|p_2\| = 1/3$, $\|q_i\| = 1/i$, $i = 1, 2, 3, 4, 5$, and $\|p_2\|\varphi_0\left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right] \approx 0.3529974 < 1$. Hence there exists a positive number $M_0 > 1.888596954$. Therefore, by Theorem 4.3, the problem (4.10) has at least one solution on $[1, e^{3/2}]$.

4.3 Existence Results for Mixed Hadamard and Riemann-Liouville Fractional Integro-Differential Inclusions

In this section, we consider the following mixed Hadamard and Riemann-Liouville fractional integro-differential inclusions:

$$\begin{cases} D^{\alpha}\left(x(t) - \sum_{i=1}^{m} I^{\beta_i} h_i(t, x(t))\right) \in F(t, x(t), Kx(t)), \quad t \in J := [1, T], \\ x(1) = 0, \end{cases} \tag{4.11}$$

where α, I^{β_i}, $h_i, i = 1, 2, \ldots, m$, Kx are as in problem (4.1), and $F : J \times \mathbb{R}^2 \to \mathscr{P}(\mathbb{R})$ a multivalued map, ($\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$).

Definition 4.1 A function $x \in \mathscr{C}^1(J, \mathbb{R})$ is called a solution of problem (4.11) if there exists a function $v \in L^1(J, \mathbb{R})$ with $v(t) \in F(t, x(t), Kx(t))$, a.e. on J such that $x(1) = 0$ and

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v(s) \frac{ds}{s} + \sum_{i=1}^m I^{\beta_i} h_i(t, x(t)), \quad t \in J.$$

4.3.1 The Upper Semicontinuous Case

Our first existence result, for the initial value problem (4.11) deals with the convex valued right-hand side of the inclusion and is based on Krasnoselskii's fixed point theorem for multivalued maps (Theorem 1.16).

Theorem 4.4 *Assume that (4.1.1) and (4.1.2) hold. In addition, we suppose that:*

(4.4.1) there exist functions $\nu, \mu \in C(J, \mathbb{R}^+)$ *such that*

$$\|F(t, x, y)\| := \sup\{|v| : v \in F(t, x, y)|\} \le \nu(t) + \mu(t)|y|,$$

$\forall (t, x, y) \in J \times \mathbb{R}^2$ *with* $\varphi_0 \|\mu\| \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right] < 1$, *where* $\sup_{t \in J} |\mu(t)| = \|\mu\|$ *and*

$$\gamma = T \int_0^{\log T} u^{\alpha-1} e^{-u} du. \tag{4.12}$$

Then the problem (4.11) *has at least one solution on* J, *provided that*

$$L_0 \sum_{i=1}^m \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i + 1)} < 1. \tag{4.13}$$

Proof Define an operator $\Omega_F : E \to \mathscr{P}(E)$ by

$$\Omega_F(x) = \left\{ \begin{array}{l} h \in E : \\ h(t) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v(s) \frac{ds}{s} + \sum_{i=1}^m I^{\beta_i} h_i(t, x(t)) \end{array} \right\} \tag{4.14}$$

for $v \in S_{F,x}$.

Setting $\sup_{t\in J}|\nu(t)| = \|\nu\|$, $\sup_{t\in J}|\theta_i(t)| = \|\theta_i\|$, $i = 1, 2, \ldots, m$, we consider $B_R = \{x \in C(J, \mathbb{R}) : \|x\| \le R\}$, where

$$R \ge \left(\sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)}\|\theta_i\| + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\|\nu\|\right) \Big/ \left(1 - \varphi_0\|\mu\|\left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right]\right).$$

We define the operators $\mathscr{Q} : B_R \to E$ by

$$\mathscr{Q}x(t) = \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i - 1} h_i(s, x(s))ds, \quad t \in J,$$

and a multivalued operator $\mathscr{T} : B_R \to \mathscr{P}(E)$ by

$$\mathscr{T}x(t) = \left\{h \in E : h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v(s)\frac{ds}{s},\ v \in S_{F,x}\right\}.$$

In this way, the problem (4.11) is equivalent to the inclusion problem $u \in \mathscr{Q}u + \mathscr{T}u$. We show that the operators $\mathscr{Q}$ and $\mathscr{T}$ satisfy the conditions of Theorem 1.16 on B_R.

First, we show that the operators $\mathscr{Q}$ and $\mathscr{T}$ define the multivalued operators $\mathscr{Q}, \mathscr{T} : B_R \to \mathscr{P}_{cp,c}(E)$. We prove that $\mathscr{T}$ is compact-valued on B_R. Note that the operator $\mathscr{T}$ is equivalent to the composition $\mathscr{L} \circ S_F$, where $\mathscr{L}$ is the continuous linear operator on $L^1(J, \mathbb{R})$ into E, defined by

$$\mathscr{L}(v)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v(s)\frac{ds}{s}.$$

Suppose that $x \in B_R$ is arbitrary and let $\{v_n\}$ be a sequence in $S_{F,x}$. Then, by definition of $S_{F,x}$, we have $v_n(t) \in F(t, x(t), Kx(t))$ for almost all $t \in J$. Since $F(t, x(t), Kx(t))$ is compact for all $t \in J$, there is a convergent subsequence of $\{v_n(t)\}$, (we denote it by $\{v_n(t)\}$ again) that converges in measure to some $v(t) \in S_{F,x}$ for almost all $t \in J$. On the other hand, $\mathscr{L}$ is continuous, so $\mathscr{L}(v_n)(t) \to \mathscr{L}(v)(t)$ pointwise on J.

In order to show that the convergence is uniform, we need to establish that $\{\mathscr{L}(v_n)\}$ is an equi-continuous sequence. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then, we have

$$\begin{aligned}
&|\mathscr{L}(v_n)(t_2) - \mathscr{L}(v_n)(t_1)| \\
&= \left|\frac{1}{\Gamma(\alpha)} \int_1^{t_2} \left(\log\frac{t_2}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s} - \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left(\log\frac{t_1}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s}\right| \\
&\le \frac{\|\nu\| + \|\mu\|\varphi_0 R(T-1)}{\Gamma(\alpha+1)} |[(\log t_2)^{\alpha} - (\log t_1)^{\alpha}| + 2(\log(t_2/t_1))^{\alpha}].
\end{aligned}$$

We see that the right hand of the above inequality tends to zero as $t_2 \to t_1$. Thus, the sequence $\{\mathscr{L}(v_n)\}$ is equi-continuous and by using the Arzelá-Ascoli Theorem, we get that there is a uniformly convergent subsequence. So, there is a subsequence of $\{v_n\}$, (we denote it again by $\{v_n\}$) such that $\mathscr{L}(v_n) \to \mathscr{L}(v)$. Note that, $\mathscr{L}(v) \in \mathscr{L}(S_{F,x})$. Hence, $\mathscr{T}(x) = \mathscr{L}(S_{F,x})$ is compact for all $x \in B_R$. So $\mathscr{T}(x)$ is compact.

Now, we show that $\mathscr{T}(x)$ is convex for all $x \in E$. Let $z_1, z_2 \in \mathscr{T}(x)$. We select $f_1, f_2 \in S_{F,x}$ such that

$$z_i(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f_i(s) \frac{ds}{s}, \quad i = 1, 2,$$

for almost all $t \in J$. Let $0 \leq \lambda \leq 1$. Then, we have

$$[\lambda z_1 + (1-\lambda) z_2](t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} [\lambda f_1(s) + (1-\lambda) f_2(s)] \frac{ds}{s}.$$

Since F has convex values, so $S_{F,x}$ is convex and $\lambda f_1(s) + (1-\lambda) f_2(s) \in S_{F,x}$. Thus

$$\lambda z_1 + (1-\lambda) z_2 \in \mathscr{T}(x).$$

Consequently, $\mathscr{T}$ is convex-valued. Obviously, $\mathscr{Q}$ is compact and convex-valued.

Next, we show that $\mathscr{Q}(x) + \mathscr{T}(x) \subset B_R$ for all $x \in B_R$. Suppose $x \in B_R$ and $h \in \mathscr{Q}$ are arbitrary elements. Choose $v \in S_{F,x}$ such that

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v(s) \frac{ds}{s} + \sum_{i=1}^m I^{\beta_i} h_i(t, x(t)),$$

for almost all $t \in J$. Hence, we get

$$\begin{aligned}
|h(t)| &\leq \sum_{i=1}^m \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i - 1} |h_i(s, x(s))| ds + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |v(s)| \frac{ds}{s} \\
&\leq \sum_{i=1}^m \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i - 1} |\theta_i(s)| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} (|\nu(s)| + |\mu(s)||Kx(s)|) \frac{ds}{s} \\
&\leq \sum_{i=1}^m \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i + 1)} \|\theta_i\| + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} \|\nu\| + \varphi_0 \|\mu\| R \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right] \\
&\leq R.
\end{aligned}$$

Thus $\|h\| \leq R$, which means that $\mathscr{Q}(x) + \mathscr{T}(x) \subset B_R$ for all $x \in B_R$.

The rest of the proof consists of several steps and claims.

Step 1: We show that $\mathscr{Q}$ *is a contraction on* E. This is a consequence of (4.1.1). Indeed, for $x, y \in E$, we have

$$|\mathscr{Q}x(t) - \mathscr{Q}y(t)| \leq \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i - 1} |h_i(s, x(s)) - h_i(s, y(s))| ds$$

$$\leq L_0 \|x - y\| \sum_{i=1}^{m} \frac{1}{\Gamma(\beta_i)} \int_1^t (t-s)^{\beta_i - 1} ds$$

$$\leq L_0 \|x - y\| \sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i + 1)}.$$

Hence, by the given assumption (4.13), $\mathscr{Q}$ is a contraction.

Step 2: $\mathscr{T}$ *is compact and upper semicontinuous.*

This will be established in several claims.

CLAIM I: $\mathscr{T}$ *maps bounded sets into bounded sets in* E. For each $h \in \mathscr{T}(x),\ x \in B_R$, there exists $v \in S_{F,x}$ such that

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha - 1} v(s) \frac{ds}{s}.$$

Then, we have

$$\|h\| \leq \frac{(\log T)^{\alpha}}{\Gamma(\alpha + 1)} \|\nu\| + \|\mu\| \varphi_0 R \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha + 1)}\right]$$

and thus the operator $\mathscr{T}(B_R)$ is uniformly bounded.

CLAIM II: $\mathscr{T}$ *maps bounded sets into equi-continuous sets*. Let $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$ and $x \in B_R$. Then, we have

$$\begin{aligned} &|\mathscr{T}x(\tau_2) - \mathscr{T}x(\tau_1)| \\ &= \left| \frac{1}{\Gamma(\alpha)} \int_1^{\tau_2} \left(\log \frac{\tau_2}{s}\right)^{\alpha - 1} v(s) \frac{ds}{s} - \frac{1}{\Gamma(\alpha)} \int_1^{\tau_1} \left(\log \frac{\tau_1}{s}\right)^{\alpha - 1} v(s) \frac{ds}{s} \right| \\ &\leq \frac{\|\nu\| + \|\mu\| \varphi_0 R (T-1)}{\Gamma(\alpha + 1)} [|(\log \tau_2)^{\alpha} - (\log \tau_1)^{\alpha}| + 2(\log(\tau_2/\tau_1))^{\alpha}], \end{aligned}$$

which is independent of x, and tends to zero as $\tau_2 - \tau_1 \to 0$. Thus, $\mathscr{T}$ is equicontinuous. So $\mathscr{T}$ is relatively compact on B_R. Hence, by the Arzelá-Ascoli Theorem, $\mathscr{T}$ is compact on B_R.

In our next step, we show that $\mathscr{T}$ is upper semicontinuous. By Lemma 1.1, $\mathscr{T}$ will be upper semicontinuous if we establish that it has a closed graph, since $\mathscr{T}$ is already shown to be completely continuous.

CLAIM III: *$\mathscr{T}$ has a closed graph.* Let $x_n \to x_*, h_n \in \mathscr{T}(x_n)$ and $h_n \to h_*$. Then, we need to show that $h_* \in \mathscr{T}(x_*)$. Associated with $h_n \in \mathscr{T}(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in J$,

$$h_n(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_n(s) \frac{ds}{s}.$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in J$,

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_*(s) \frac{ds}{s}.$$

Let us consider the linear operator $\Theta : L^1(J, \mathbb{R}) \to E$ given by

$$f \mapsto \Theta(f)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v(s) \frac{ds}{s}.$$

Observe that

$$\|h_n(t) - h_*(t)\| = \left\| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} (v_n(s) - v_*(s)) \frac{ds}{s} \right\| \to 0,$$

as $n \to \infty$.

Thus, it follows by Lemma 1.2 that $\Theta \circ S_{F,x}$ is a closed graph operator. Further, we have that $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, we have

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_*(s) \frac{ds}{s},$$

for some $v_* \in S_{F,x_*}$. Hence $\mathscr{T}$ has a closed graph (and therefore has closed values). In consequence, the operator $\mathscr{T}$ is upper semicontinuous.

Thus, the operators $\mathscr{Q}$ and $\mathscr{T}$ satisfy all the conditions of Theorem 1.16 and hence its conclusion implies that $x \in \mathscr{Q}(x) + \mathscr{T}(x)$ is a solution in B_r. Therefore the problem (4.11) has a solution in B_r and the proof is completed. □

4.3.2 The Lipschitz Case

Now, we prove the existence of solutions for the problem (4.11) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued maps due to Covitz and Nadler (Theorem 1.18).

Theorem 4.5 *Let (4.1.1) and the following conditions hold:*

(4.5.1) $F : J \times \mathbb{R}^2 \to \mathscr{P}_{cp}(\mathbb{R})$ *is such that* $F(\cdot, x, y) : J \to \mathscr{P}_{cp}(\mathbb{R})$ *is measurable for each* $x, y \in \mathbb{R}$*;*

(4.5.2) $H_d(F(t,x,y), F(t,\bar{x},\bar{y})) \le m(t)(|x-\bar{x}| + |y-\bar{y}|)$ *for almost all* $t \in J$ *and* $x, \bar{x}, y, \bar{y} \in \mathbb{R}$ *with* $m \in C(J, \mathbb{R}^+)$ *and* $d(0, F(t,0,0)) \le m(t)$ *for almost all* $t \in J$*.*

Then the problem (4.11) has at least one solution on J if

$$\|m\| \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}(1 + \varphi_0(T-1)) + L_0 \sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_1+1)} < 1.$$

Proof Observe that the set $S_{F,x}$ is nonempty for each $x \in E$ by the assumption (4.5.1), so F has a measurable selection (see Theorem III.6 [57]). Now, we show that the operator Ω_F, defined by (4.14), satisfies the assumptions of Theorem 1.18. To show that $\Omega_F(x) \in \mathscr{P}_{cl}(E)$ for each $x \in E$, let $\{u_n\}_{n\ge 0} \in \Omega_F(x)$ be such that $u_n \to u$ $(n \to \infty)$ in E. Then $u \in E$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in J$,

$$u_n(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_n(s) \frac{ds}{s} + \sum_{i=1}^{m} I^{\beta_i} h_i(t, x_n(t)).$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1(J, \mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in J$, we have

$$v_n(t) \to v(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v(s) \frac{ds}{s} + \sum_{i=1}^{m} I^{\beta_i} h_i(t, x(t)).$$

Hence, $u \in \Omega_F(x)$.

Next. we show that there exists $\delta < 1$ $(\delta := \|m\| \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}(1 + \varphi_0(T-1)) + L_0 \sum_{i=1}^{m} \frac{(T-1)^{\beta_i}}{\Gamma(\beta_1+1)})$ such that

$$H_d(\Omega_F(x), \Omega_F(\bar{x})) \le \delta \|x - \bar{x}\| \text{ for each } x, \bar{x} \in E.$$

Let $x, \bar{x} \in E$ and $h_1 \in \Omega_F(x)$. Then there exists $v_1(t) \in F(t, x(t), Kx(t))$ such that, for each $t \in J$,

$$h_1(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_1(s) \frac{ds}{s} + \sum_{i=1}^{m} I^{\beta_i} h_i(t, x(t)).$$

By (4.5.2), we have

$$H_d(F(t,x,Kx), F(t,\bar{x},K\bar{x})) \le m(t)(|x(t) - \bar{x}(t)| + |Kx(t) - K\bar{x}(t)|).$$

So, there exists $w \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \leq m(t)(|x(t) - \bar{x}(t)| + |Kx(t) - K\bar{x}(t)|), \quad t \in J.$$

Define $U : J \to \mathscr{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t)(|x(t) - \bar{x}(t)| + |Kx(t) - K\bar{x}(t)|)\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{x}(t), K\bar{x}(t))$ is measurable (Proposition III.4 [57]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t, \bar{x}(t), K\bar{x}(t))$ and for each $t \in J$, we have $|v_1(t) - v_2(t)| \leq m(t)(|x(t) - \bar{x}(t)| + |Kx(t) - K\bar{x}(t)|)$.

For each $t \in J$, let us define

$$h_2(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} v_2(s) \frac{ds}{s} + \sum_{i=1}^m I^{\beta_i} h_i(t, \bar{x}(t)).$$

Thus,

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |v_1(s) - v_2(s)| \frac{ds}{s} \\ &\quad + \sum_{i=1}^m I^{\beta_i} |h_i(t, x(s)) - h_i(t, \bar{x}(s))| \\ &\leq \|m\| \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} (1 + \varphi_0(T-1)) \|x - \bar{x}\| \\ &\quad + L_0 \sum_{i=1}^m \frac{(T-1)^{\beta_i}}{\Gamma(\beta_1+1)} \|x - \bar{x}\|. \end{aligned}$$

Hence,

$$\begin{aligned} \|h_1 - h_2\| \leq \Bigg\{ &\|m\| \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} (1 + \phi_0(T-1)) \\ &+ L_0 \sum_{i=1}^m \frac{(T-1)^{\beta_i}}{\Gamma(\beta_1+1)} \Bigg\} \|x - \bar{x}\|. \end{aligned}$$

Analogously, interchanging the roles of x and $\bar{x}$, we obtain

$$\begin{aligned} H_d(\Omega_F(x), \Omega_F(\bar{x})) \leq \Bigg\{ &\|m\| \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} (1 + \phi_0(T-1)) \\ &+ L_0 \sum_{i=1}^m \frac{(T-1)^{\beta_i}}{\Gamma(\beta_i+1)} \Bigg\} \|x - \bar{x}\|. \end{aligned}$$

Since Ω_F is a contraction, it follows by Theorem 1.18 that Ω_F has a fixed point x which is a solution of (4.11). This completes the proof. □

4.3.3 Examples

Example 4.4 Consider the following mixed Hadamard and Riemann-Liouville fractional integro-differential initial value problem

$$\begin{cases} D^{1/2}\left(x(t) - \sum_{i=1}^{3} I^{(2i+1)/2} h_i(t, x(t))\right) \in F(t, x(t), Kx(t)), \quad t \in [1, e], \\ x(1) = 0, \end{cases} \tag{4.15}$$

where

$$h_1(t,x) = \frac{\log t}{25}\frac{|x|}{1+|x|}, \quad h_2(t,x) = \frac{\tan^{-1}|x|}{29(1+\log t)}, \quad h_3(t,x) = \frac{2e^{-t}}{21}\sin|x|.$$

(i) Consider the multivalued map $F : [1,e]\times\mathbb{R}\times\mathbb{R} \to \mathscr{P}(\mathbb{R})$ given by

$$\begin{aligned} x \to F(t,x,Kx) = \Bigg[&(t^2+1)\frac{|x|}{3+|x|} + \frac{e^{-t}}{4}\int_1^t \frac{\cos^2(t-s)}{2}x(s)ds, \\ &\left(\sqrt{t}+\frac{1}{2}\right)e^{-x^2} + \frac{1}{2+\log t}\int_1^t \frac{\cos^2(t-s)}{2}x(s)ds\Bigg]. \end{aligned} \tag{4.16}$$

Here $\alpha = 1/2$, $\beta_1 = 3/2$, $\beta_2 = 5/2$, $\beta_3 = 7/2$, $m = 3$, $T = e$. With the given data, we find that $\varphi_0 = 1/2$, $|h_i(t,x) - h_i(t,y)| \le (1/25)|x-y|$, $i = 1,2,3$, which satisfies (4.1.1) with $L_0 = 1/25$. Since $\int_0^1 u^{-1/2}e^{-u}du = \sqrt{\pi}\,\mathrm{erf}(1)$, where erf(·) is the Gauss error function, we have $\gamma = 4.06015694$. For $f \in F$, we have

$$\begin{aligned} |f| &\le \max\left((t^2+1)\frac{|x|}{3+|x|} + \frac{e^{-t}}{4}K|x|, \left(\sqrt{t}+\frac{1}{2}\right)e^{-x^2} + \frac{1}{2+\log t}K|x|\right) \\ &\le t^2 + 1 + \frac{|Kx|}{2+\log t}. \end{aligned}$$

Thus

$$\|F(t,x,y)\| := \sup\{|f|, f \in F(t,x,y)\} \le t^2 + 1 + \frac{|y|}{2+\log t},$$

$\forall (t, x, y) \in [1, e] \times \mathbb{R}^2$ with $\nu(t) = t^2 + 1$, $\mu(t) = 1/(2 + \log t)$. Then, we have

$$\varphi_0 \|\mu\| \left[\frac{\gamma}{\Gamma(\alpha)} + \frac{(\log T)^\alpha}{\Gamma(\alpha + 1)} \right] = 0.8547693548 < 1.$$

Hence, (4.4.1) is satisfied. It is easy to verify that $|h_1(t, x)| \leq (\log t)/25$, $|h_2(t, x)| \leq \pi/(58(1 + \log t))$ and $|h_3(t, x)| \leq 2e^{-t}/21$. In addition, we can show that

$$L_0 \sum_{i=1}^{3} \frac{(T - 1)^{\beta_i}}{\Gamma(\beta_i + 1)} = 0.1372254755 < 1.$$

Thus all conditions of Theorem 4.4 are satisfied. Therefore, by the conclusion of Theorem 4.4, the problem (5.21) with $F(t, x, Kx)$ given by (4.16) has at least one solution on $[1, e]$

(ii) Let $F : [1, e] \times \mathbb{R}^2 \to \mathscr{P}(\mathbb{R})$ be a multivalued map given by

$$x \to F(t, x, Kx) = \Bigg[0, \frac{|x|}{(\sqrt{2} + \log t)^2 (3 + |x|)} + \frac{1}{(\sqrt{2} + \log t)^2} \sin \left| \int_0^t e^{-\sqrt{t-s}} x(s) ds \right| + \frac{1}{9} \Bigg]. \quad (4.17)$$

Then, we have

$$\sup\{|x| : x \in F(t, x, Kx)\} \leq \frac{2}{(\sqrt{2} + \log t)^2} + \frac{1}{9},$$

and

$$H_d(F(t, x, Kx), F(t, \bar{x}, K\bar{x})) \leq \frac{1}{(\sqrt{2} + \log t)^2} \left(|x - \bar{x}| + |Kx - K\bar{x}| \right).$$

Let $m(t) = 1/(\sqrt{2} + \log t)^2$. Then, we have $H_d(F(t, x, Kx), F(t, \bar{x}, K\bar{x})) \leq m(t)|x - \bar{x}|$ with $d(0, F(t, 0, 0)) = 1/9 \leq m(t)$ and $\|m\| = 1/2$. Also

$$\|m\| \frac{(\log T)^\alpha}{\Gamma(\alpha + 1)} (1 + \varphi_0 (T - 1)) + L_0 \sum_{i=1}^{3} \frac{(T - 1)^{\beta_i}}{\Gamma(\beta_1 + 1)} = 0.9437742360 < 1.$$

Thus all the conditions of Theorem 4.5 are satisfied. Therefore, by the conclusion of Theorem 4.5, the problem (5.21) with $F(t, x, Kx)$ given by (4.17) has at least one solution on $[1, e]$.

4.4 Existence Result via Endpoint Theory

In this section, we consider the following mixed initial value problem involving Hadamard derivative and Riemann-Liouville fractional integrals:

$$\begin{cases} D^{\alpha}\left(x(t)-\sum_{i=1}^{m} I^{\beta_i} h_i(t,x(t))\right) \in F(t,x(t)), \quad t \in J := [1,T], \\ x(1)=0, \end{cases} \tag{4.18}$$

where D^{α} denotes the Hadamard fractional derivative of order α, $0 < \alpha \le 1$, I^{ϕ} is the Riemann-Liouville fractional integral of order $\phi > 0$, $\phi \in \{\beta_1, \beta_2, \ldots, \beta_m\}$, $F : J \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$, ($\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$), $h_i \in C(J \times \mathbb{R}, \mathbb{R})$ with $h_i(1,0)=0$, $i=1,2,\ldots,m$.

Definition 4.2 A function $x \in \mathscr{C}^1(J,\mathbb{R})$ is called a solution of problem (4.18) if there exists a function $v \in L^1(J,\mathbb{R})$ with $v(t) \in F(t,x(t))$, a.e. on J such that $x(1)=0$ and

$$x(t)=\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v(s)\frac{ds}{s}+\sum_{i=1}^{m} I^{\beta_i} h_i(t,x(t)), \quad t \in J.$$

Define an operator $N : E \to \mathscr{P}(E)$ by

$$N(u)=\left\{\begin{array}{l} h \in E: \\ h(t)=\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v(s)\frac{ds}{s}+\sum_{i=1}^{m} I^{\beta_i} h_i(t,u(t)) \end{array}\right\} \tag{4.19}$$

for $v \in S_{F,u}$, where $S_{F,u}$ denote the set of selections of F defined by

$$S_{F,u} := \{v \in L^1(J,\mathbb{R}) : v(t) \in F(t,u(t)) \text{ for a.e. } t \in J\}.$$

Theorem 4.6 *Suppose that $\psi : [0,\infty) \to [0,\infty)$ is a nondecreasing upper semi-continuous mapping such that $\liminf_{t\to\infty}(t-\psi(t)) > 0$ and $\psi(t) < t$ for all $t > 0$. Also, let $F : J \times \mathbb{R} \to \mathscr{P}_{cp}(\mathbb{R})$ be an integrable bounded multifunction such that $F(\cdot,u) : J \to \mathscr{P}_{cp}(\mathbb{R})$ is measurable for all $u \in \mathbb{R}$. Assume that there exist functions $\eta, \widehat{\eta} \in C(J,[0,\infty))$ such that*

$$H_d(F(t,u(t))-F(t,v(t))) \le \varepsilon_1\left(\|\eta\|\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\right)^{-1}\eta(t)\psi(|u(t)-v(t)|),$$

$$|h_i(t,u)-h_i(t,v)| \le \varepsilon_2\left(\|\widehat{\eta}\|\sum_{i=1}^{m}\frac{(T-1)^{\beta_i}}{\Gamma(\beta_1+1)}\right)^{-1}\|\widehat{\eta}\|\psi(|u(t)-v(t)|),$$

where $\sup_{t\in J}|\kappa(t)| = \|\kappa\|$ *with* $\kappa = \eta, \widehat{\eta}$, *and* $\varepsilon_1, \varepsilon_2$ *are positive constants such that* $\varepsilon_1 + \varepsilon_2 \le 1$. *If the multifunction N has the approximate endpoint property, then the inclusion problem (4.18) has a solution.*

Proof We show that the multifunction $N : E \to \mathscr{P}(E)$, defined by (4.19), has an endpoint. For this, we prove that $N(u)$ is a closed subset of $\mathscr{P}(E)$ for all $u \in E$. Since the multivalued map $t \mapsto F(t, u(t))$ is measurable and has closed values for all $u \in E$, so it has measurable selection and thus, $S_{F,u}$ is nonempty for all $u \in E$. Assume that $u \in E$ and $\{z_n\}_{n\ge 1}$ be a sequence in $N(u)$ with $z_n \to z$. For every $n \in \mathbb{N}$, choose $v_n \in S_{F,u_n}$ such that

$$z_n(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v_n(s)\frac{ds}{s} + \sum_{i=1}^m I^{\beta_i}h_i(t, u_n(t)), \quad t \in J.$$

By compactness of F, the sequence $\{v_n\}_{n\ge1}$ has a subsequence which converges to some $v \in L^1(J)$. We denote this subsequence again by $\{v_n\}_{n\ge1}$. It is clear that $v \in S_{F,u}$ and for all $t \in J$, we have

$$z_n(t) \to z(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v(s)\frac{ds}{s} + \sum_{i=1}^m I^{\beta_i}h_i(t, u(t)).$$

This shows that $z \in N(u)$ and so N is closed-valued. On the other hand, $N(u)$ is a bounded set for all $u \in E$ as F is a compact multivalued map.

Finally, we show that $H_d(N(u), N(w)) \le \psi(\|u - w\|)$. Let $u, w \in E$ and $h_1 \in N(w)$. Choose $v_1 \in S_{F,w}$ such that

$$h_1(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v_1(s)\frac{ds}{s} + \sum_{i=1}^m I^{\beta_i}h_i(t, w(t)),$$

for almost all $t \in J$. Since

$$H_d(F(t,u(t)) - F(t,w(t))) \le \varepsilon_1\Big(\|\eta\|\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\Big)^{-1}\eta(t)\psi(|u(t) - w(t)|)$$

for all $t \in J$, there exists $z \in F(t, u(t))$ provided that

$$|v_1(t) - z| \le \varepsilon_1\Big(\|\eta\|\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\Big)^{-1}\eta(t)\psi(|u(t) - w(t)|)$$

for all $t \in J$. Now, we consider the multivalued map $U : J \to \mathscr{P}(\mathbb{R})$ as follows:

$$U(t) = \left\{z \in \mathbb{R} : \ |v_1(t) - z| \le \varepsilon_1\Big(\|\eta\|\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\Big)^{-1}\eta(t)\psi(|u(t) - w(t)|)\right\}.$$

Since v_1 and $\varphi = \varepsilon_1\Big(\|\eta\|\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\Big)^{-1}\eta\psi(|u-w|)$ are measurable, the multifunction $U(\cdot) \cap F(\cdot, u(\cdot))$ is measurable. Choose $v_2(t) \in F(t, u(t))$ such that

$$|v_1(t) - v_2(t)| \le \frac{\varepsilon_1}{\|\eta\|\dfrac{(\log T)^\alpha}{\Gamma(\alpha+1)}}\eta(t)\psi(|u(t) - w(t)|)$$

for all $t \in J$. We define the element $h_2 \in N(u)$ by

$$h_2(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} v_2(s)\frac{ds}{s} + \sum_{i=1}^m I^{\beta_i}h_i(t, u(t)),$$

for all $t \in J$. Thus, one can get

$$\begin{aligned}
|h_1(t) - h_2(t)| &\le \frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} |v_1(s) - v_2(s)|\frac{ds}{s} \\
&\quad + \sum_{i=1}^m I^{\beta_i}|h_i(t, u(t)) - h_i(t, w(t))| \\
&\le \varepsilon_1\Big(\|\eta\|\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\Big)^{-1}\|\eta\|\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\psi(\|u-w\|) \\
&\quad +\varepsilon_2\Big(\|\widehat{\eta}\|\sum_{i=1}^m\frac{(T-1)^{\beta_i}}{\Gamma(\beta_1+1)}\Big)^{-1}\|\widehat{\eta}\|\sum_{i=1}^m\frac{(T-1)^{\beta_i}}{\Gamma(\beta_1+1)}\psi(\|u-v\|) \\
&= \psi(\|u-w\|).
\end{aligned}$$

Hence,

$$\|h_1 - h_2\| \le \psi(\|u-w\|).$$

Therefore $H_d(N(u), N(w)) \le \psi(\|u-w\|)$ for all $u, w \in E$. By hypothesis, since the multifunction N has approximate endpoint property, by Theorem 1.19, there exists $u^* \in X$ such that $N(u^*) = \{u^*\}$. Consequently, the problem (4.18) has the solution u^* and the proof is completed. □

Example 4.5 Consider the following mixed Hadamard and Riemann-Liouville fractional integro-differential problem

$$\begin{cases} D^{1/2}\Big(x(t) - \displaystyle\sum_{i=1}^4 I^{\frac{2i+1}{2}}h_i(t, x(t))\Big) \in F(t, x(t)), \quad t \in [1, e], \\ x(1) = 0, \end{cases} \tag{4.20}$$

where

$$h_i(t, x(t)) = \left(\frac{1}{i + \sqrt{3}\log t}\right)\left(\frac{x(t)}{25 + i}\right).$$

Clearly $\widehat{\eta}(t) = 1/(26(1 + \sqrt{3}\log t))$ with $\|\eta\| = 1/26$. Let $F : [1, e] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ be a multivalued map given by

$$x \to F(t, x) = \left[0, \frac{3\log t}{2}\frac{|x|}{1 + |x|} + \frac{2}{3}\right]. \tag{4.21}$$

Setting $\eta(t) = (3\log t)/2$, $t \in [1, e]$, we have $\|\eta\| = 3/2$. Choosing $\psi(y) = y/2$, it is clear the function ψ is nondecreasing upper semi-continuous on $[1, e]$ such that $\liminf_{y\to\infty}(y - \psi(y)) > 0$ and $\psi(y) < y$ for all $y > 0$. Also we have

$$H_d(F(t, x) - F(t, \bar{x})) \leq \frac{3\log t}{2}|x - \bar{x}| < \left(\frac{\|\eta\|(\log T)^{\alpha}}{\Gamma(\alpha + 1)}\right)^{-1}\frac{3\log t}{2}\psi(|x - \bar{x}|),$$

for $x, \bar{x} \in \mathbb{R}$. Let $X = C([1, e], \mathbb{R})$. Let $N : X \to \mathscr{P}(X)$ be an operator defined by

$$N(u) = \{z \in X \ : \ \text{there exists } v \in S_{F,u} \text{ such that } z(t) = w(t) \text{ for all } t \in [1, e]\},$$

where

$$w(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} v(s)\frac{ds}{s} + \sum_{i=1}^{m} I^{\beta_i} h_i(t, w(t)), \quad t \in [1, e].$$

Since $\sup_{u\in N(0)}\|u\| = 0$, thus $\inf_{u\in X}\sup_{s\in N(u)}\|u - s\| = 0$. Consequently, the operator N has the approximate endpoint property. Thus all the conditions of Theorem 4.6 are satisfied. Therefore, by the conclusion of Theorem 4.6, the problem (4.20) with $F(t, x)$ given by (4.21) has at least one solution on $[1, e]$.

4.5 Notes and Remarks

Section 4.2 contains the existence and uniqueness results for mixed initial value problems for fractional differential equations involving Hadamard derivative and Riemann-Liouville fractional integrals, while the inclusions (multivalued) analog of the problem considered in Sect. 4.2 is studied in Sect. 4.3. Section 4.4 contains an existence result for a mixed initial value problem involving Hadamard derivative and Riemann-Liouville fractional integrals, via endpoint theory. The papers [25, 26] and [27] are the sources of the work presented in this chapter.

Chapter 5
Nonlocal Hadamard Fractional Integral Conditions and Nonlinear Riemann-Liouville Fractional Differential Equations and Inclusions

5.1 Introduction

In this chapter, we develop the existence theory for nonlocal boundary value problems of nonlinear Riemann-Liouville fractional differential equations and inclusions supplemented with the Hadamard fractional integral boundary conditions. The key tool for the present study is the Property 2.25 from [96, p. 113] (see Lemma 1.6).

5.2 Nonlocal Hadamard Fractional Integral Conditions and Nonlinear Riemann-Liouville Fractional Differential Equations

In this section, we study existence and uniqueness of solutions for the following nonlinear Riemann-Liouville fractional differential equation with nonlocal Hadamard fractional integral boundary conditions:

$$ {}_{RL}D^q x(t) = f(t, x(t)), \quad t \in [0, T], \tag{5.1} $$

$$ x(0) = 0, \qquad x(T) = \sum_{i=1}^{n} \alpha_{i\,H} I^{p_i} x(\eta_i), \tag{5.2} $$

where $1 < q \leq 2$, ${}_{RL}D^q$ is the standard Riemann-Liouville fractional derivative of order q, ${}_H I^{p_i}$ is the Hadamard fractional integral of order $p_i > 0$, $\eta_i \in (0, T)$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\alpha_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ are real constants such that $\sum_{i=1}^{n} \frac{\alpha_i \eta_i^{q-1}}{(q-1)^{p_i}} \neq T^{q-1}$.

B. Ahmad et al., *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, DOI 10.1007/978-3-319-52141-1_5

The uniqueness results are obtained via Banach's fixed point theorem, Banach's fixed point theorem combined with Hölder's inequality and nonlinear contractions. Existence results are established by means of Krasnoselskii's fixed point theorem, Leray-Schauder's nonlinear alternative and Leray-Schauder's degree theory. All the results are illustrated by examples.

Lemma 5.1 *Let* $\Lambda_1 := T^{q-1} - \sum_{i=1}^{n} \frac{\alpha_i \eta_i^{q-1}}{(q-1)^{p_i}} \neq 0$, $1 < q \leq 2$, $p_i > 0$, $\alpha_i \in \mathbb{R}$, $\eta_i \in (0, T)$, $i = 1, 2, 3, \ldots, n$ *and* $h \in C([0, T], \mathbb{R})$. *Then, the nonlocal Hadamard fractional boundary value problem of linear Riemann-Liouville fractional differential equation*

$$ {}_{RL}D^q x(t) = h(t), \qquad 0 < t < T, \tag{5.3}$$

subject to the boundary conditions

$$x(0) = 0, \qquad x(T) = \sum_{i=1}^{n} \alpha_{i\,H} I^{p_i} x(\eta_i), \tag{5.4}$$

is equivalent to the following fractional integral equation

$$x(t) = {}_{RL}I^q h(t) - \frac{t^{q-1}}{\Lambda_1} \left({}_{RL}I^q h(T) - \sum_{i=1}^{n} \alpha_i ({}_H I^{p_i}\, {}_{RL}I^q h)(\eta_i) \right). \tag{5.5}$$

Proof Using Lemmas 1.4 and 1.5, the equation (5.3) can be transformed into an equivalent integral equation

$$x(t) = {}_{RL}I^q h(t) - c_1 t^{q-1} - c_2 t^{q-2}, \tag{5.6}$$

for $c_1, c_2 \in \mathbb{R}$. The first condition in (5.4) implies that $c_2 = 0$. Applying the Hadamard fractional integral operator of order $p_i > 0$ on (5.6) and using property: $({}_H I^{p_i} s^{q-1})(t) = (q-1)^{-p_i} t^{q-1}$ (see Lemma 1.6), we get

$${}_H I^{p_i} x(t) = ({}_H I^{p_i}\, {}_{RL}I^q h)(t) - c_1 ({}_H I^{p_i} s^{q-1})(t) = ({}_H I^{p_i}\, {}_{RL}I^q h)(t) - c_1 \frac{t^{q-1}}{(q-1)^{p_i}},$$

which, together with the second condition of (5.4), implies that

$${}_{RL}I^q h(T) - c_1 T^{q-1} = \sum_{i=1}^{n} \alpha_i ({}_H I^{p_i}\, {}_{RL}I^q h)(\eta_i) - c_1 \sum_{i=1}^{n} \frac{\alpha_i \eta_i^{q-1}}{(q-1)^{p_i}}.$$

Thus,

$$c_1 = \frac{1}{\Lambda_1} \left({}_{RL}I^q h(T) - \sum_{i=1}^{n} \alpha_i ({}_H I^{p_i}\, {}_{RL}I^q h)(\eta_i) \right).$$

Substituting the values of c_1 and c_2 in (5.6), we obtain the solution (5.5).

Conversely, it can easily be shown by direct computation, that the integral equation (5.5) satisfies the problem (5.3) and (5.4). This completes the proof. □

Throughout this chapter, for convenience, we use the following notations:

$$_{RL}I^{\alpha}f(s,x(s))(z) = \frac{1}{\Gamma(\alpha)}\int_0^z (z-s)^{\alpha-1}f(s,x(s))ds, \quad z\in\{t,T\},$$

for $t \in [0,T]$ and

$$_{H}I^{p_i}{}_{RL}I^{\alpha}f(s,x(s))(\eta_i) = \frac{1}{\Gamma(p_i)\Gamma(\alpha)}\int_0^{\eta_i}\int_0^r \left(\log\frac{\eta_i}{r}\right)^{p_i-1}(r-s)^{\alpha-1}\frac{f(s,x(s))}{r}ds\,dr,$$

where $\eta_i \in (0,T)$ for $i = 1,2,\ldots,n$.

Let $\mathscr{E}_0 = C([0,T],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup_{t\in[0,T]}|x(t)|$. By Lemma 5.1, we define an operator $\mathscr{A} : \mathscr{E}_0 \to \mathscr{E}_0$ associated with the problem (5.1)–(5.2) as follows:

$$\begin{aligned}(\mathscr{A}x)(t) &= {}_{RL}I^{q}f(s,x(s))(t)\\ &\quad - \frac{t^{q-1}}{\Lambda_1}\left({}_{RL}I^{q}f(s,x(s))(T) - \sum_{i=1}^{n}\alpha_i({}_{H}I^{p_i}{}_{RL}I^{q}f(s,x(s)))(\eta_i)\right).\end{aligned} \tag{5.7}$$

Observe that the problem (5.1)–(5.2) has solutions if and only if the operator $\mathscr{A}$ has fixed points.

In the sequel, we set a constant

$$\Phi_1 := \frac{T^q}{\Gamma(q+1)} + \frac{T^{2q-1}}{|\Lambda_1|\Gamma(q+1)} + \frac{T^{q-1}}{|\Lambda_1|\Gamma(q+1)}\sum_{i=1}^{n}|\alpha_i|q^{-p_i}\eta_i^q. \tag{5.8}$$

In the following subsections, we present existence, as well as existence and uniqueness results, for the problem (5.1)–(5.2).

5.2.1 *Existence and Uniqueness Result via Banach's Fixed Point Theorem*

Theorem 5.1 *Assume that:*

(5.1.1) there exists a constant $L > 0$ such that $|f(t,x) - f(t,y)| \le L|x-y|$, for each $t \in [0,T]$ and $x,y \in \mathbb{R}$.

If

$$L\Phi_1 < 1, \tag{5.9}$$

where Φ_1 is defined by (5.8), then the problem (5.1)–(5.2) has a unique solution on $[0,T]$.

Proof We transform the problem (5.1)–(5.2) into a fixed point problem, $x = \mathscr{A}x$, where the operator $\mathscr{A}$ is defined by (5.7). Observe that the fixed points of the operator $\mathscr{A}$ are solutions of the problem (5.1)–(5.2). Applying the Banach's contraction mapping principle, we shall show that $\mathscr{A}$ has a unique fixed point.

We let $\sup_{t\in[0,T]}|f(t,0)| = M < \infty$, and choose $r \geq \dfrac{M\Phi_1}{1-L\Phi_1}$ to show that $\mathscr{A}B_r \subset B_r$, where $B_r = \{x \in \mathscr{E}_0 : \|x\| \leq r\}$. For any $x \in B_r$, we have

$$
\begin{aligned}
|(\mathscr{A}x)(t)| \leq \sup_{t\in[0,T]} \Bigg\{ & {}_{RL}I^q|f(s,x(s))|(t) + \frac{t^{q-1}}{|\Lambda_1|}{}_{RL}I^q|f(s,x(s))|(T) \\
& + \frac{t^{q-1}}{|\Lambda_1|}\sum_{i=1}^n |\alpha_i|{}_HI^{p_i}{}_{RL}I^q|f(s,x(s))|(\eta_i)\Bigg\} \\
\leq {} & {}_{RL}I^q(|f(s,x(s)) - f(s,0)| + |f(s,0)|)(T) \\
& + \frac{T^{q-1}}{|\Lambda_1|}{}_{RL}I^q(|f(s,x(s)) - f(s,0)| + |f(s,0)|)(T) \\
& + \frac{T^{q-1}}{|\Lambda_1|}\sum_{i=1}^n |\alpha_i|{}_HI^{p_i}{}_{RL}I^q(|f(s,x(s)) - f(s,0)| + |f(s,0)|)(\eta_i) \\
\leq {} & (L\|x\| + M){}_{RL}I^q(1)(T) + (L\|x\| + M)\frac{T^{q-1}}{|\Lambda_1|}{}_{RL}I^q(1)(T) \\
& + (L\|x\| + M)\frac{T^{q-1}}{|\Lambda_1|}\sum_{i=1}^n |\alpha_i|{}_HI^{p_i}{}_{RL}I^q(1)(\eta_i) \\
= {} & (Lr + M)\left(\frac{T^q}{\Gamma(q+1)} + \frac{T^{2q-1}}{|\Lambda_1|\Gamma(q+1)} + \frac{T^{q-1}}{|\Lambda_1|\Gamma(q+1)}\sum_{i=1}^n |\alpha_i|q^{-p_i}\eta_i^q\right) \\
= {} & (Lr + M)\Phi_1 \leq r.
\end{aligned}
$$

Thus we get $\mathscr{A}B_r \subset B_r$.

Next, we let $x, y \in \mathscr{E}_0$. Then, for $t \in [0,T]$, we have

$$
\begin{aligned}
& |(\mathscr{A}x)(t) - (\mathscr{A}y)(t)| \\
\leq {} & {}_{RL}I^q|f(s,x(s)) - f(s,y(s))|(t) + \frac{T^{q-1}}{|\Lambda_1|}{}_{RL}I^q|f(s,x(s)) - f(s,y(s))|(T) \\
& + \frac{T^{q-1}}{|\Lambda_1|}\sum_{i=1}^n |\alpha_i|{}_HI^{p_i}{}_{RL}I^q|f(s,x(s)) - f(s,y(s))|(\eta_i) \\
\leq {} & L\left(\frac{T^q}{\Gamma(q+1)} + \frac{T^{2q-1}}{|\Lambda_1|\Gamma(q+1)} + \frac{T^{q-1}}{|\Lambda_1|\Gamma(q+1)}\sum_{i=1}^n |\alpha_i|q^{-p_i}\eta_i^q\right)\|x - y\| \\
= {} & L\Phi_1\|x - y\|,
\end{aligned}
$$

which implies that $\|\mathscr{A}x - \mathscr{A}y\| \leq L\Phi_1\|x-y\|$. As $L\Phi_1 < 1$, $\mathscr{A}$ is a contraction. Therefore, by the Banach's contraction mapping principle, we deduce that $\mathscr{A}$ has a unique fixed point which corresponds to the unique solution of the problem (5.1)–(5.2). The proof is completed. □

Example 5.1 Consider the following nonlocal boundary value problem for a nonlinear Riemann-Liouville fractional differential equation with Hadamard fractional integral boundary conditions:

$$\begin{cases} {}_{RL}D^{3/2}x(t) = \dfrac{\sin^2(\pi t)}{(e^t+3)^2}\cdot\dfrac{|x(t)|}{|x(t)|+1} + \dfrac{\sqrt{3}}{2}, & t \in [0,3], \\ x(0) = 0, \quad x(3) + \sqrt{5}{}_HI^{1/2}x\left(\dfrac{9}{4}\right) = \dfrac{4}{5}{}_HI^{\sqrt{2}}x\left(\dfrac{3}{4}\right) + \dfrac{\sqrt{3}}{2}{}_HI^{\pi}x\left(\dfrac{3}{2}\right). \end{cases} \tag{5.10}$$

Here $q = 3/2$, $n = 3$, $T = 3$, $\alpha_1 = 4/5$, $\alpha_2 = \sqrt{3}/2$, $\alpha_3 = -\sqrt{5}$, $p_1 = \sqrt{2}$, $p_2 = \pi$, $p_3 = 1/2$, $\eta_1 = 3/4$, $\eta_2 = 3/2$, $\eta_3 = 9/4$ and $f(t,x) = (\sin^2(\pi t)/(e^t+3)^2)(|x|/(1+|x|)) + (\sqrt{3}/2)$. Since $|f(t,x) - f(t,y)| \leq (1/16)|x-y|$, the condition (5.1.1) is satisfied with $L = 1/16$. Further, it is found that $\Phi_1 \approx 7.239901027$, and that $L\Phi_1 \approx 0.4524938142 < 1$. Hence, by Theorem 5.1, the problem (5.10) has a unique solution on $[0,3]$.

5.2.2 Existence and Uniqueness Result via Banach's Fixed Point Theorem and Hölder's Inequality

Theorem 5.2 *Suppose that:* $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ *is a continuous function satisfying the following assumption:*

(5.2.1) $|f(t,x) - f(t,y)| \leq \delta(t)|x-y|$, *for* $t \in [0,T]$, $x,y \in \mathbb{R}$ *and* $\delta \in L^{\frac{1}{\sigma}}([0,T],\mathbb{R}^+)$, $\sigma \in (0,1)$.

Denote $\|\delta\| = \left(\int_0^T |\delta(s)|^{\frac{1}{\sigma}}ds\right)^{\sigma}$. *If*

$$\gamma_0 := \|\delta\|\left\{\frac{T^{q-\sigma}}{\Gamma(q)}\left(\frac{1-\sigma}{q-\sigma}\right)^{1-\sigma} + \frac{T^{2q-\sigma-1}}{|\Lambda_1|\Gamma(q)}\left(\frac{1-\sigma}{q-\sigma}\right)^{1-\sigma} + \frac{T^{q-1}}{|\Lambda_1|\Gamma(q)}\left(\frac{1-\sigma}{q-\sigma}\right)^{1-\sigma}\sum_{i=1}^{n}|\alpha_i|(q-\sigma)^{p_i}\eta_i^{q-\sigma}\right\} < 1, \tag{5.11}$$

then the problem (5.1)–(5.2) has a unique solution on $[0,T]$.

Proof For $x, y \in \mathscr{E}_0$ and for each $t \in [0, T]$, by Hölder's inequality, we have

$$
\begin{aligned}
&|(\mathscr{A}x)(t) - (\mathscr{A}y)(t)| \\
\leq\ & {}_{RL}I^q|f(s,x(s)) - f(s,y(s))|(t) + \frac{T^{q-1}}{|\Lambda_1|}{}_{RL}I^q|f(s,x(s)) - f(s,y(s))|(T) \\
&+\frac{T^{q-1}}{|\Lambda_1|}\sum_{i=1}^{n}|\alpha_i|_H I^{p_i}{}_{RL}I^q|f(s,x(s)) - f(s,y(s))|(\eta_i) \\
=\ & \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\delta(s)|x(s)-y(s)|ds \\
&+\frac{T^{q-1}}{|\Lambda_1|\Gamma(q)}\int_0^T (T-s)^{q-1}\delta(s)|x(s)-y(s)|ds \\
&+\frac{T^{q-1}}{|\Lambda_1|}\sum_{i=1}^{n}\frac{|\alpha_i|}{\Gamma(p_i)\Gamma(q)}\int_0^{\eta_i}\int_0^s\left(\log\frac{\eta_i}{s}\right)^{p_i-1}(s-r)^{q-1}\delta(r)|x(r)-y(r)|dr\frac{ds}{s} \\
\leq\ & \frac{1}{\Gamma(q)}\left(\int_0^t ((t-s)^{q-1})^{\frac{1}{1-\sigma}}ds\right)^{1-\sigma}\left(\int_0^t (\delta(s))^{\frac{1}{\sigma}}ds\right)^{\sigma}\|x-y\| \\
&+\frac{T^{q-1}}{|\Lambda_1|\Gamma(q)}\left(\int_0^T ((T-s)^{q-1})^{\frac{1}{1-\sigma}}ds\right)^{1-\sigma}\left(\int_0^T (\delta(s))^{\frac{1}{\sigma}}ds\right)^{\sigma}\|x-y\| \\
&+\frac{T^{q-1}}{|\Lambda_1|}\sum_{i=1}^{n}\frac{|\alpha_i|}{\Gamma(q)\Gamma(p_i)}\int_0^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{p_i-1}\left(\int_0^s ((s-r)^{q-1})^{\frac{1}{1-\sigma}}dr\right)^{1-\sigma} \\
&\times\left(\int_0^s (\delta(r))^{\frac{1}{\sigma}}dr\right)^{\sigma}\frac{ds}{s}\|x-y\| \\
\leq\ & \|\delta\|\frac{T^{q-\sigma}}{\Gamma(q)}\left(\frac{1-\sigma}{q-\sigma}\right)^{1-\sigma}\|x-y\| + \|\delta\|\frac{T^{2q-\sigma-1}}{|\Lambda_1|\Gamma(q)}\left(\frac{1-\sigma}{q-\sigma}\right)^{1-\sigma}\|x-y\| \\
&+\|\delta\|\frac{T^{q-1}}{|\Lambda_1|\Gamma(q)}\left(\frac{1-\sigma}{q-\sigma}\right)^{1-\sigma}\sum_{i=1}^{n}\frac{|\alpha_i|}{\Gamma(p_i)}\int_0^{\eta_i}\left(\log\frac{\eta_i}{s}\right)^{p_i-1}s^{q-\sigma}\frac{ds}{s}\|x-y\| \\
\leq\ & \|\delta\|\Bigg[\frac{T^{q-\sigma}}{\Gamma(q)}\left(\frac{1-\sigma}{q-\sigma}\right)^{1-\sigma} + \frac{T^{2q-\sigma-1}}{|\Lambda_1|\Gamma(q)}\left(\frac{1-\sigma}{q-\sigma}\right)^{1-\sigma} + \frac{T^{q-1}}{|\Lambda_1|\Gamma(q)}\left(\frac{1-\sigma}{q-\sigma}\right)^{1-\sigma} \\
&\times\sum_{i=1}^{n}|\alpha_i|(q-\sigma)^{p_i}\eta_i^{q-\sigma}\Bigg]\|x-y\|.
\end{aligned}
$$

In view of the condition (5.11), it follows that $\mathscr{A}$ is a contraction. Hence, Banach's fixed point theorem implies that $\mathscr{A}$ has a unique fixed point, which is the unique solution of the problem (5.1)–(5.2). The proof is completed. □

Example 5.2 Consider a nonlocal boundary value problem of a nonlinear Riemann-Liouville fractional differential equation with Hadamard fractional integral boundary conditions of the form:

$$\begin{cases} {}_{RL}D^{4/3}x(t) = \dfrac{e^t}{e^t+8}\cdot\dfrac{|x(t)|}{|x(t)|+2}+1, \qquad t\in\left[0,\dfrac{3}{2}\right], \\ x(0)=0, \quad x\left(\dfrac{3}{2}\right)+\dfrac{2}{3}{}_HI^{\sqrt{2}/2}x\left(\dfrac{3}{5}\right)+\pi\,{}_HI^{\sqrt{3}}x\left(\dfrac{6}{5}\right) \\ \qquad\qquad = \dfrac{1}{5}{}_HI^{1/4}x\left(\dfrac{3}{10}\right)+\dfrac{1}{\sqrt{3}}{}_HI^{6/5}x\left(\dfrac{9}{10}\right). \end{cases} \tag{5.12}$$

Here $q = 4/3$, $n = 4$, $T = 3/2$, $\alpha_1 = 1/5$, $\alpha_2 = -2/3$, $\alpha_3 = 1/\sqrt{3}$, $\alpha_4 = -\pi/2$, $p_1 = 1/4$, $p_2 = \sqrt{2}/2$, $p_3 = 6/5$, $p_4 = \sqrt{3}$, $\eta_1 = 3/10$, $\eta_2 = 3/5$, $\eta_3 = 9/10$ and $\eta_4 = 6/5$. Since $|f(t,x)-f(t,y)| \le (2e^t/(e^t+8))|x-y|$, (5.2.1) is satisfied with $\delta(t) = 2e^t/(e^t+8)$ and $\sigma = 1/2$. Using the given values, we find that $\gamma_0 \approx 0.9380422264 < 1$. Hence, by Theorem 5.2, the problem (5.12) has a unique solution on $[0, 3/2]$.

5.2.3 Existence and Uniqueness Result via Nonlinear Contractions

Theorem 5.3 *Let $f : [0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the assumption:*

(5.3.1) $|f(t,x)-f(t,y)| \le h(t)\dfrac{|x-y|}{H^*+|x-y|}$ *for* $t\in[0,T], x,y\ge 0$, *where* $h : [0,T]\to\mathbb{R}^+$ *is continuous and* H^* *the constant defined by*

$$H^* := {}_{RL}I^q h(T) + \frac{T^{q-1}}{|\Lambda_1|}{}_{RL}I^q h(T) + \frac{T^{q-1}}{|\Lambda_1|}\sum_{i=1}^{n}|\alpha_i|{}_HI^{p_i}{}_{RL}I^q h(\eta_i).$$

Then the problem (5.1)–(5.2) has a unique solution on $[0,T]$.

Proof We define a continuous nondecreasing function $\Psi : \mathbb{R}^+\to\mathbb{R}^+$ by $\Psi(\varepsilon) = \dfrac{H^*\varepsilon}{H^*+\varepsilon}$, $\forall\varepsilon\ge 0$, such that $\Psi(0)=0$ and $\Psi(\varepsilon)<\varepsilon$ for all $\varepsilon>0$.

For any $x, y\in\mathscr{E}_0$ and for each $t\in[0,T]$, by (5.7), we have

$$\begin{aligned} &|(\mathscr{A}x)(t)-(\mathscr{A}y)(t)| \\ &\le {}_{RL}I^q|f(s,x(s))-f(s,y(s))|(t) + \frac{T^{q-1}}{|\Lambda_1|}{}_{RL}I^q|f(s,x(s))-f(s,y(s))|(T) \\ &\quad + \frac{T^{q-1}}{|\Lambda_1|}\sum_{i=1}^{n}|\alpha_i|{}_HI^{p_i}{}_{RL}I^q|f(s,x(s))-f(s,y(s))|(\eta_i) \end{aligned}$$

$$
\begin{aligned}
&\le {}_{RL}I^q\left(h(s)\frac{|x-y|}{H^*+|x-y|}\right)(T)+\frac{T^{q-1}}{|\Lambda_1|}{}_{RL}I^q\left(h(s)\frac{|x-y|}{H^*+|x-y|}\right)(T)\\
&\quad+\frac{T^{q-1}}{|\Lambda_1|}\sum_{i=1}^{n}|\alpha_i|{}_HI^{p_i}{}_{RL}I^q\left(h(s)\frac{|x-y|}{H^*+|x-y|}\right)(\eta_i)\\
&\le \frac{\Psi(\|x-y\|)}{H^*}\left({}_{RL}I^q h(T)+\frac{T^{q-1}}{|\Lambda_1|}{}_{RL}I^q h(T)+\frac{T^{q-1}}{|\Lambda_1|}\sum_{i=1}^{n}|\alpha_i|{}_HI^{p_i}{}_{RL}I^q h(\eta_i)\right)\\
&=\Psi(\|x-y\|).
\end{aligned}
$$

This implies that $\|\mathscr{A}x-\mathscr{A}y\|\le\Psi(\|x-y\|)$. Therefore $\mathscr{A}$ is a nonlinear contraction. Hence, by Theorem 1.11, the operator $\mathscr{A}$ has a unique fixed point, which is the unique solution of the problem (5.1)–(5.2). This completes the proof. ☐

Example 5.3 Consider the following nonlocal boundary value problem:

$$
\begin{cases}
{}_{RL}D^{7/6}x(t)=\dfrac{t^2}{(t+2)^2}\cdot\dfrac{|x(t)|}{|x(t)|+1}+3t+\dfrac{4}{5}, & t\in[0,2],\\
x(0)=0,\quad x(2)=2{}_HI^{\sqrt{\pi}}x\left(\dfrac{2}{5}\right)+\dfrac{2}{3}{}_HI^{5/4}x\left(\dfrac{4}{3}\right)+\sqrt{3}{}_HI^{3/7}x\left(\dfrac{3}{2}\right).
\end{cases}
\tag{5.13}
$$

Here $q=7/6$, $n=3$, $T=2$, $\alpha_1=2$, $\alpha_2=2/3$, $\alpha_3=\sqrt{3}$, $p_1=\sqrt{\pi}$, $p_2=5/4$, $p_3=3/7$, $\eta_1=2/5$, $\eta_2=4/3$, $\eta_3=3/2$ and $f(t,x)=(t^2|x|/((t+2)^2)(|x|+1))+3t+(4/5)$. We choose $h(t)=t^2/4$. Then, we find $H^*\approx 0.6432886158$. Clearly,

$$
|f(t,x)-f(t,y)|=\frac{t^2}{(t+2)^2}\left|\frac{|x|-|y|}{1+|x|+|y|+|x||y|}\right|\le\frac{t^2}{4}\left(\frac{|x-y|}{0.6432886158+|x-y|}\right).
$$

Hence, by Theorem 5.3, the problem (5.13) has a unique solution on $[0,2]$.

5.2.4 Existence Result via Krasnoselskii's Fixed Point Theorem

Theorem 5.4 *Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying (5.1.1). In addition, we assume that:*

(5.4.1) $|f(t,x)|\le\varphi(t)$, $\forall(t,x)\in[0,T]\times\mathbb{R}$, *and* $\varphi\in C([0,T],\mathbb{R}^+)$.

Then the problem (5.1)–(5.2) has at least one solution on $[0,T]$, *provided that*

$$
\gamma_1:=L\left(\frac{T^{2q-1}}{|\Lambda_1|\Gamma(q+1)}+\frac{T^{q-1}}{|\Lambda_1|\Gamma(q+1)}\sum_{i=1}^{n}|\alpha_i|q^{-p_i}\eta_i^q\right)<1. \tag{5.14}
$$

Proof Setting $\sup_{t\in[0,T]}\varphi(t)=\|\varphi\|$ and $\rho\ge\|\varphi\|\Phi_1$, where Φ_1 is defined by (5.8), we consider $B_\rho=\{x\in\mathscr{E}_0:\|x\|\le\rho\}$ and introduce the operators $\mathscr{A}_1$ and $\mathscr{A}_2$ on B_ρ by

$$\mathscr{A}_1x(t) = {}_{RL}I^q f(s,x(s))(t), \quad t \in [0,T],$$
$$\mathscr{A}_2x(t) = -\frac{t^{q-1}}{\Lambda_1}\left({}_{RL}I^q f(s,x(s))(T) - \sum_{i=1}^{n}\alpha_i({}_HI^{p_i}{}_{RL}I^q f(s,x(s)))(\eta_i)\right), \quad t \in [0,T].$$

For any $x, y \in B_\rho$, we have

$$\begin{aligned}
&|(\mathscr{A}_1x)(t) + (\mathscr{A}_2y)(t)| \\
&\leq \sup_{t\in[0,T]}\left\{{}_{RL}I^q|f(s,x(s))|(t) + \frac{t^{q-1}}{|\Lambda_1|}{}_{RL}I^q|f(s,y(s))|(T)\right. \\
&\quad \left. + \frac{t^{q-1}}{|\Lambda_1|}\sum_{i=1}^{n}|\alpha_i|{}_HI^{p_i}{}_{RL}I^q|f(s,y(s))|(\eta_i)\right\} \\
&\leq \|\varphi\|\left(\frac{T^q}{\Gamma(q+1)} + \frac{T^{2q-1}}{|\Lambda_1|\Gamma(q+1)} + \frac{T^{q-1}}{|\Lambda_1|\Gamma(q+1)}\sum_{i=1}^{n}|\alpha_i|q^{-p_i}\eta_i^q\right) \\
&= \|\varphi\|\Phi_1 \leq \rho.
\end{aligned}$$

This shows that $\mathscr{A}_1x + \mathscr{A}_2y \in B_\rho$. It is easy to check that $\mathscr{A}_2$ is a contraction by using (5.14).

Continuity of f implies that the operator $\mathscr{A}_1$ is continuous. Also, $\mathscr{A}_1$ is uniformly bounded on B_ρ as

$$\|\mathscr{A}_1x\| \leq \frac{T^q}{\Gamma(q+1)}\|\varphi\|.$$

Now, we prove the compactness of the operator $\mathscr{A}_1$.

We define $\sup_{(t,x)\in[0,T]\times B_\rho}|f(t,x)| = \bar{f} < \infty$, and consequently, for $0 < t_1, t_2 < T$, we have

$$\begin{aligned}
|(\mathscr{A}_1x)(t_2) - (\mathscr{A}_1x)(t_1)| &= \frac{1}{\Gamma(q)}\left|\int_0^{t_1}[(t_2-s)^{q-1} - (t_1-s)^{q-1}]f(s,x(s))ds\right. \\
&\quad \left. + \int_{t_1}^{t_2}(t_2-s)^{q-1}f(s,x(s))ds\right| \\
&\leq \frac{\bar{f}}{\Gamma(q+1)}[2|t_2-t_1|^q + |t_1^q - t_2^q|],
\end{aligned}$$

which is independent of x, and tends to zero as $t_2 - t_1 \to 0$. Thus, $\mathscr{A}_1$ is equicontinuous. So $\mathscr{A}_1$ is relatively compact on B_ρ. Hence, by the Arzelá-Ascoli Theorem, $\mathscr{A}_1$ is compact on B_ρ. Thus, all the assumptions of Theorem 1.2 are satisfied. So the conclusion of Theorem 1.2 implies that the problem (5.1)–(5.2) has at least one solution on $[0,T]$. □

Example 5.4 Consider the nonlocal problem for a nonlinear Riemann-Liouville fractional differential equation with Hadamard fractional integral boundary conditions given by

$$\begin{cases} {}_{RL}D^{5/4}x(t) = \dfrac{e^{-t^2}\sin^2(2t)}{(t+3)^2}\cdot\dfrac{|x(t)|}{|x(t)|+1} + \dfrac{t-1}{t+1}, \qquad t \in [0, 2\pi], \\ x(0) = 0, \\ x(2\pi) + \sqrt{3}{}_HI^{1/2}x\left(\dfrac{\pi}{3}\right) + \dfrac{3}{4}{}_HI^{3/4}x\left(\dfrac{2\pi}{3}\right) = \\ \qquad {}_HI^{4/5}x(\pi) + \dfrac{1}{9}{}_HI^{4/3}x\left(\dfrac{4\pi}{3}\right) + 2{}_HI^{2/3}x\left(\dfrac{5\pi}{3}\right). \end{cases} \tag{5.15}$$

Here $q = 5/4$, $n = 5$, $T = 2\pi$, $\alpha_1 = -\sqrt{3}$, $\alpha_2 = -3/4$, $\alpha_3 = 1$, $\alpha_4 = 1/9$, $\alpha_5 = 2$, $p_1 = 1/2$, $p_2 = 3/4$, $p_3 = 4/5$, $p_4 = 4/3$, $p_5 = 2/3$, $\eta_1 = \pi/3$, $\eta_2 = 2\pi/3$, $\eta_3 = \pi$, $\eta_4 = 4\pi/3$, $\eta_5 = 5\pi/3$, and $f(t,x) = (e^{-t^2}\sin^2(2t)|x|)/(((t+3)^2)(|x|+1)) + (t-1)/(t+1)$. Since $|f(t,x) - f(t,y)| \le (1/9)|x-y|$, (5.2.1) is satisfied with $L = 1/36$. Further, we have that $\gamma_1 \approx 0.9518560542 < 1$. Clearly,

$$|f(t,x)| = \left|\frac{e^{-t^2}\sin^2(2t)}{(t+3)^2}\cdot\frac{|x(t)|}{|x(t)|+1} + \frac{t-1}{t+1}\right| \le \frac{e^{-t^2}}{9} + \frac{|t-1|}{t+1}.$$

Hence, by Theorem 5.4, the problem (5.15) has at least one solution on $[0, 2\pi]$.

5.2.5 Existence Result via Leray-Schauder's Nonlinear Alternative

Theorem 5.5 *Assume that:*

(5.5.1) there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in C([0,T],\mathbb{R}^+)$ *such that*

$$|f(t,u)| \le p(t)\psi(\|x\|) \ \textit{for each}\ (t,x) \in [0,T]\times\mathbb{R};$$

(5.5.2) there exists a constant $M > 0$ *such that*

$$\frac{M}{\psi(M)\|p\|\Phi_1} > 1,$$

where Φ_1 *is defined by (5.8).*

Then the problem (5.1)–(5.2) has at least one solution on $[0,T]$.

Proof Firstly, we shall show that the operator $\mathscr{A}$ *defined by (5.7) maps bounded sets (balls) into bounded sets in* $\mathscr{E}_0$. For a number $r > 0$, let $B_r = \{x \in \mathscr{E}_0 : \|x\| \le r\}$ be a bounded ball in $\mathscr{E}_0$. Then, for $t \in [0, T]$, we have

$$
\begin{aligned}
&|(\mathscr{A}x)(t)| \\
&\le \sup_{t\in[0,T]} \Bigg\{ {}_{RL}I^q|f(s,x(s))|(t) + \frac{t^{q-1}}{|\Lambda_1|}{}_{RL}I^q|f(s,x(s))|(T) \\
&\quad + \frac{t^{q-1}}{|\Lambda_1|}\sum_{i=1}^{n}|\alpha_i|{}_HI^{p_i}{}_{RL}I^q|f(s,x(s))|(\eta_i)\Bigg\} \\
&\le \psi(\|x\|)_{RL}I^q p(s)(T) + \psi(\|x\|)\frac{T^{q-1}}{|\Lambda_1|}{}_{RL}I^q p(s)(T) \\
&\quad + \psi(\|x\|)\frac{T^{q-1}}{|\Lambda_1|}\sum_{i=1}^{n}|\alpha_i|{}_HI^{p_i}{}_{RL}I^q p(s)(\eta_i) \\
&\le \psi(\|x\|)\|p\|\left(\frac{T^q}{\Gamma(q+1)} + \frac{T^{2q-1}}{|\Lambda_1|\Gamma(q+1)} + \frac{T^{q-1}}{|\Lambda_1|\Gamma(q+1)}\sum_{i=1}^{n}|\alpha_i|q^{-p_i}\eta_i^q\right),
\end{aligned}
$$

and consequently,

$$\|\mathscr{A}x\| \le \psi(r)||p||\Phi_1.$$

Next, we will show that $\mathscr{A}$ *maps bounded sets into equicontinuous sets of* $\mathscr{E}_0$. Let $\tau_1, \tau_2 \in [0, T]$ with $\tau_1 < \tau_2$ and $x \in B_r$. Then, we have

$$
\begin{aligned}
&|(\mathscr{A}x)(\tau_2) - (\mathscr{A}x)(\tau_1)| \\
&\le \frac{1}{\Gamma(q)}\left|\int_0^{\tau_1}[(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}]f(s,x(s))ds + \int_{\tau_1}^{\tau_2}(\tau_2 - s)^{q-1}f(s,x(s))ds\right| \\
&\quad + \frac{(\tau_2^{q-1} - \tau_1^{q-1})}{|\Lambda_1|}{}_{RL}I^q|f(s,x(s))|(T) + \frac{(\tau_2^{q-1} - \tau_1^{q-1})}{|\Lambda_1|}\sum_{i=1}^{n}|\alpha_i|{}_HI^{p_i}{}_{RL}I^q|f(s,x(s))|(\eta_i) \\
&\le \frac{\psi(r)}{\Gamma(q+1)}[2(\tau_2 - \tau_1)^q + |\tau_2^q - \tau_1^q|] \\
&\quad + \frac{(\tau_2^{q-1} - \tau_1^{q-1})\psi(r)}{|\Lambda_1|}\Big[{}_{RL}I^q p(s)(T) + \sum_{i=1}^{n}|\alpha_i|{}_HI^{p_i}{}_{RL}I^q p(s)(\eta_i)\Big].
\end{aligned}
$$

As $\tau_2 - \tau_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $x \in B_r$. Therefore, by the Arzelá-Ascoli Theorem, the operator $\mathscr{A} : \mathscr{E}_0 \to \mathscr{E}_0$ is completely continuous.

Finally, we show that there exists an open set $U \subset \mathscr{E}_0$ with $x \ne \nu\mathscr{A}x$ for $\nu \in (0, 1)$ and $x \in \partial U$.

Let x be a solution. Then, as in the first step, we have

$$|x(t)| \le \psi(\|x\|)||p||\Phi_1, \ t \in [0, T],$$

which leads to

$$\frac{\|x\|}{\psi(\|x\|)\|p\|\Phi_1} \leq 1.$$

In view of (5.5.1), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in \mathscr{E}_0 \,:\, \|x\| < M\}.$$

Notice that the operator $\mathscr{A} : \overline{U} \to \mathscr{E}_0$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \nu \mathscr{A} x$ for some $\nu \in (0, 1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 1.4), we deduce that $\mathscr{A}$ has a fixed point $x \in \overline{U}$, which is a solution of the problem (5.1)–(5.2). This completes the proof. □

Example 5.5 Consider the following nonlocal boundary value problem:

$$\begin{cases} {}_{RL}D^{6/5}x(t) = \dfrac{1}{64}(1+t^2)\left(\dfrac{x^2}{|x|+1} + \dfrac{\sqrt{|x|}}{2(1+\sqrt{|x|})} + \dfrac{1}{2}\right), & t \in [0, e], \\ x(0) = 0, \qquad x(e) = \dfrac{1}{2}{}_HI^{\sqrt{2}}x\left(\dfrac{1}{2}\right) - 5{}_HI^{\sqrt{3}}x\left(\dfrac{2}{3}\right) + \sqrt{3}{}_HI^{\sqrt{5}}x(1). \end{cases} \tag{5.16}$$

Here $q = 6/5$, $n = 3$, $T = e$, $\alpha_1 = 1/2$, $\alpha_2 = -5$, $\alpha_3 = \sqrt{3}$, $p_1 = \sqrt{2}$, $p_2 = \sqrt{3}$, $p_3 = \sqrt{5}$, $\eta_1 = 1/2$, $\eta_2 = 2/3$, $\eta_3 = 1$, and $f(t, x) = (1/64)(1+t^2)((x^2/(|x|+1)) + (\sqrt{x})/(2(1+\sqrt{x})) + (1/2))$. It is easy to find that $\Phi_1 \approx 3.905177250$. Clearly,

$$|f(t,x)| = \left|\frac{1}{64}(1+t^2)\left(\frac{x^2}{|x|+1} + \frac{\sqrt{|x|}}{2(1+\sqrt{|x|})} + \frac{1}{2}\right)\right| \leq \frac{1}{64}(1+t^2)(|x|+1).$$

Choosing $p(t) = (1/64)(1+t^2)$ and $\psi(|x|) = |x| + 1$, we can show that (5.5.2) is satisfied for $M > 1.048704821$. Hence, by Theorem 5.5, the problem (5.16) has at least one solution on $[0, e]$.

5.2.6 *Existence Result via Leray-Schauder's Degree Theory*

Theorem 5.6 *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that:*

(5.6.1) there exist constants $0 \leq \kappa < \Phi_1^{-1}$, and $K > 0$ such that

$$|f(t,x) \leq \kappa|x| + K \ \textit{for all} \ \ (t,x) \in [0,T] \times \mathbb{R},$$

where Φ_1 is defined by (5.8).

Then the problem (5.1)–(5.2) has at least one solution on $[0, T]$.

Proof Consider the fixed point problem

$$x = \mathscr{A}x, \tag{5.17}$$

where the operator $\mathscr{A}$ is given by (5.7). We shall prove the existence of at least one solution $x \in \mathscr{E}_0$ satisfying (5.17). Define a ball $B_R = \{x \in \mathscr{E}_0 : |x(t)| < R\}$, with a constant radius $R > 0$, and show that $\mathscr{A} : \overline{B}_R \to \mathscr{E}_0$ satisfies a condition

$$x \neq \theta\mathscr{A}x, \qquad \forall x \in \partial B_R, \qquad \forall \theta \in [0, 1]. \tag{5.18}$$

We set

$$H(\theta, x) = \theta\mathscr{A}x, \qquad x \in \mathscr{E}_0, \qquad \theta \in [0, 1].$$

As shown in Theorem 5.5, the operator $\mathscr{A}$ is continuous, uniformly bounded and equicontinuous. Then, by the Arzelá-Ascoli Theorem, a continuous map h_θ defined by $h_\theta(x) = x - H(\theta, x) = x - \theta\mathscr{A}x$ is completely continuous. If (5.18) holds, then Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, we have

$$\begin{aligned}\deg(h_\theta, B_R, 0) &= \deg(I - \theta\mathscr{A}, B_R, 0) = \deg(h_1, B_R, 0)\\ &= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \qquad 0 \in B_R,\end{aligned}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, $h_1(x) = x - \mathscr{A}x = 0$ for at least one $x \in B_R$. Let us assume that $x = \theta\mathscr{A}x$ for some $\theta \in [0, 1]$ and for all $t \in [0, T]$ so that

$$\begin{aligned}|x(t)| &= |\theta(\mathscr{A}x)(t)|\\
&\leq {}_{RL}I^q|f(s, x(s))|(t) + \frac{t^{q-1}}{|\Lambda_1|}{}_{RL}I^q|f(s, x(s))|(T)\\
&\quad + \frac{t^{q-1}}{|\Lambda_1|}\sum_{i=1}^{n}|\alpha_i|{}_HI^{p_i}{}_{RL}I^q|f(s, x(s))|(\eta_i)\\
&\leq (\kappa|x| + K){}_{RL}I^q(1)(T) + (\kappa|x| + M)\frac{T^{q-1}}{|\Lambda_1|}{}_{RL}I^q(1)(T)\\
&\quad + (\kappa|x| + K)\frac{T^{q-1}}{|\Lambda_1|}\sum_{i=1}^{n}|\alpha_i|{}_HI^{p_i}{}_{RL}I^q p(s)(\eta_i)\\
&\leq (\kappa|x| + K)\left(\frac{T^q}{\Gamma(q+1)} + \frac{T^{2q-1}}{|\Lambda_1|\Gamma(q+1)} + \frac{T^{q-1}}{|\Lambda_1|\Gamma(q+1)}\sum_{i=1}^{n}|\alpha_i|q^{-p_i}\eta_i^q\right)\\
&= (\kappa|x| + K)\Phi_1,\end{aligned}$$

which, on taking norm ($\sup_{t\in[0,T]} |x(t)| = \|x\|$) and solving for $\|x\|$, yields

$$\|x\| \le \frac{K\Phi_1}{1-\kappa\Phi_1}.$$

If $R = \dfrac{K\Phi_1}{1-\kappa\Phi_1} + 1$, the inequality (5.18) holds. This completes the proof. □

Example 5.6 Consider a nonlinear Riemann-Liouville fractional differential equation with Hadamard fractional integral boundary conditions of the form:

$$\begin{cases} {}_{RL}D^{7/4}x(t) = \dfrac{1}{2\pi}\sin\left(\dfrac{\pi}{2}x\right)\cdot\dfrac{|x|}{|x|+1} + 1, \qquad t\in[0,1], \\ x(0) = 0, \qquad x(1) = 3{}_HI^{1/2}x\left(\dfrac{1}{2}\right) - 2{}_HI^{3/2}x\left(\dfrac{3}{4}\right). \end{cases} \tag{5.19}$$

Here $q = 7/4$, $n = 2$, $T = 1$, $\alpha_1 = 3$, $\alpha_2 = -2$, $p_1 = 1/2$, $p_2 = 3/2$, $\eta_1 = 1/2$, $\eta_2 = 3/4$, and $f(t,x) = (1/2\pi)(\sin(\pi x/2))(|x|/(|x|+1)) + 1$. Using the given values $\Phi_1 \approx 1.582207843$. Since

$$|f(t,x)| = \left|\frac{1}{2\pi}\sin\left(\frac{\pi}{2}x\right)\cdot\frac{|x|}{|x|+1} + 1\right| \le \frac{1}{4}|x| + 1,$$

(5.6.1) is satisfied with $\kappa = 1/4$ and $M = 1$. Note that $\kappa = \dfrac{1}{4} < \dfrac{1}{\Phi_1} \approx 0.6320282158$. Hence, by Theorem 5.6, the problem (5.19) has at least one solution on $[0,1]$.

5.3 Nonlocal Hadamard Fractional Integral Conditions and Nonlinear Riemann-Liouville Fractional Differential Inclusions

In this section, we study the multivalued variant of the problem (5.1)–(5.2) given by

$$\begin{cases} {}_{RL}D^q x(t) \in F(t, x(t)), \quad 0 < t < T, \quad 1 < q \le 2, \\ x(0) = 0, \qquad x(T) = \displaystyle\sum_{i=1}^{n} \alpha_{i\,H}I^{p_i}x(\eta_i), \end{cases} \tag{5.20}$$

where $F : [0,T]\times\mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map, $\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

Definition 5.1 A function $x \in \mathscr{C}^2([0,T],\mathbb{R})$ is called a solution of problem (5.20) if there exists a function $v \in L^1([0,T],\mathbb{R})$ with $v(t) \in F(t,x(t))$, a.e. on $[0,T]$ such that $D^q x(t) = v(t)$, $1 < q \le 2$, a.e. on $[0,T]$ and $x(0) = 0$, $\quad x(T) = \sum_{i=1}^{n} \alpha_{i\,H}I^{p_i}x(\eta_i)$.

5.3.1 The Lipschitz Case

In this subsection, we prove the existence of solutions for the problem (5.20) with a not necessary non-convex valued right hand side, by applying a fixed point theorem for multivalued maps due to Covitz and Nadler (Theorem 1.18).

Theorem 5.7 *Assume that:*

(5.7.1) $F : [0,T] \times \mathbb{R} \to \mathscr{P}_{cp}(\mathbb{R})$ *is such that* $F(\cdot, x) : [0,T] \to \mathscr{P}_{cp}(\mathbb{R})$ *is measurable for each* $x \in \mathbb{R}$;

(5.7.2) $H_d(F(t,x), F(t,\bar{x})) \leq m(t)|x-\bar{x}|$ *for almost all* $t \in [0,T]$ *and* $x, \bar{x} \in \mathbb{R}$ *with* $m \in C([0,T], \mathbb{R}^+)$ *and* $d(0, F(t,0)) \leq m(t)$ *for almost all* $t \in [0,T]$.

Then the problem (5.20) has at least one solution on $[0,T]$ *if*

$$\|m\|\Phi_1 < 1,$$

where Φ_1 *is defined by (5.8).*

Proof Define an operator $\mathscr{B} : \mathscr{E}_0 \to \mathscr{P}(\mathscr{E}_0)$ by

$$\mathscr{B}(x) = \left\{ \begin{array}{l} h \in \mathscr{E}_0 : \\ h(t) = \left\{ \begin{array}{l} \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} v(s)ds + \dfrac{t^{q-1}}{\Lambda_1 \Gamma(q)} \displaystyle\int_0^T (T-s)^{q-1} v(s)ds \\ -\dfrac{t^{q-1}}{\Lambda_1 \Gamma(q)} \displaystyle\sum_{i=1}^n \dfrac{\alpha_i}{\Gamma(p_i)} \displaystyle\int_0^{\eta_i} \int_0^s \left(\log \dfrac{\eta_i}{s}\right)^{p_i - 1} (s-r)^{q-1} \dfrac{v(r)}{s} drds \end{array} \right. \end{array} \right\}$$

for $v \in S_{F,x}$.

Observe that the set $S_{F,x}$ is nonempty for each $x \in \mathscr{E}_0$ by the assumption (5.7.1), so F has a measurable selection (see [57, Theorem III.6]). Now, we show that the operator $\mathscr{B}$ satisfies the assumptions of Theorem 1.18. To show that $\mathscr{B}(x) \in \mathscr{P}_{cl}(\mathscr{E}_0)$ for each $x \in \mathscr{E}_0$, let $\{u_n\}_{n \geq 0} \in \mathscr{B}(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $\mathscr{E}_0$. Then $u \in \mathscr{E}_0$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [0,T]$, we have

$$u_n(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v_n(s)ds + \frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \int_0^T (T-s)^{q-1} v_n(s)ds$$

$$-\frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \sum_{i=1}^n \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i - 1} (s-r)^{q-1} \frac{v_n(r)}{s} drds.$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([0,T], \mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0,T]$, we have

$$v_n(t) \to v(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(s)ds + \frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \int_0^T (T-s)^{q-1} v(s)ds$$

$$- \frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \sum_{i=1}^{n} \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-r)^{q-1} \frac{v(r)}{s} drds.$$

Hence, $u \in \mathscr{B}(x)$. Next, we show that there exists $\hat{\delta} < 1$ ($\hat{\delta} = \|m\|\Phi_1$) such that

$$H_d(\mathscr{B}(x), \mathscr{B}(\bar{x})) \leq \hat{\delta}\|x - \bar{x}\| \text{ for each } x, \bar{x} \in \mathscr{E}.$$

Let $x, \bar{x} \in \mathscr{E}_0$ and $h_1 \in \mathscr{B}(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in [0, T]$,

$$h_1(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v_1(s)ds + \frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \int_0^T (T-s)^{q-1} v_1(s)ds$$

$$- \frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \sum_{i=1}^{n} \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-r)^{q-1} \frac{v_1(r)}{s} drds.$$

By (5.7.2), we have

$$H_d(F(t,x), F(t,\bar{x})) \leq m(t)|x(t) - \bar{x}(t)|.$$

So, there exists $w \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \leq m(t)|x(t) - \bar{x}(t)|, \quad t \in [0, T].$$

Define $U : [0, T] \to \mathscr{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t)|x(t) - \bar{x}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{x}(t))$ is measurable [57, Proposition III.4], there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [0, T]$, we have $|v_1(t) - v_2(t)| \leq m(t)|x(t) - \bar{x}(t)|$.

For each $t \in [0, T]$, let us define

$$h_2(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v_2(s)ds + \frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \int_0^T (T-s)^{q-1} v_2(s)ds$$

$$- \frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \sum_{i=1}^{n} \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-r)^{q-1} \frac{v_2(r)}{s} drds.$$

Then

$$
\begin{aligned}
&|h_1(t) - h_2(t)| \\
&= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}|v_1(s) - v_2(s)|ds + \frac{t^{q-1}}{|\Lambda_1|\Gamma(q)} \int_0^T (T-s)^{q-1}|v_1(s) - v_2(s)|ds \\
&\quad + \frac{t^{q-1}}{|\Lambda_1|\Gamma(q)} \sum_{i=1}^{n} \frac{|\alpha_i|}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-r)^{q-1} \frac{|v_1(r) - v_2(r)|}{s} drds \\
&\le \|m\| \left(\frac{T^q}{\Gamma(q+1)} + \frac{T^{2q-1}}{|\Lambda_1|\Gamma(q+1)} + \frac{T^{q-1}}{|\Lambda_1|\Gamma(q+1)} \sum_{i=1}^{n} \frac{|\alpha_i|\eta_i^q}{q^{p_i}} \right) \|x - \bar{x}\|.
\end{aligned}
$$

Hence,

$$
\|h_1 - h_2\| \le \|m\|\Phi_1\|x - \bar{x}\|.
$$

Analogously, interchanging the roles of x and $\bar{x}$, we obtain

$$
H_d(\mathscr{B}(x), \mathscr{B}(\bar{x})) \le \|m\|\Phi_1\|x - \bar{x}\|.
$$

Since $\mathscr{B}$ is a contraction by the given assumption, it follows by Theorem 1.18 that $\mathscr{B}$ has a fixed point x which is a solution of (5.20). This completes the proof. □

Example 5.7 Consider the following boundary value problem for Riemann-Liouville fractional differential inclusions with nonlocal Hadamard fractional integral boundary conditions:

$$
\begin{cases}
{}_{RL}D^{3/2}x(t) \in F(t, x(t)), \quad t \in (0, 5/2), \\
x(0) = 0, \; x(5/2) + 3/2_H I^{\sqrt{2}/2} x(3/2) = \pi_H I^{\sqrt{3}/2} x(1/2) + \sqrt{2}_H I^{1/2} x(2),
\end{cases}
\tag{5.21}
$$

where $q = 3/2, n = 3, T = 5/2, \alpha_1 = \pi, \alpha_2 = -3/2, \alpha_3 = \sqrt{2}, \rho_1 = \sqrt{3}/2,\ \rho_2 = \sqrt{2}/2, \rho_3 = 1/2, \eta_1 = 1/2, \eta_2 = 3/2, \eta_3 = 2$. By using computer program, we find that $\Lambda_1 \approx -1.56277153 \neq 0$.

Let the multivalued map $F : [0, 5/2] \to \mathscr{P}(\mathbb{R})$ be given by

$$
x \to F(t, x) = \left[0, \frac{1 + \sin^2 x}{16(1+t)^2} + \frac{1}{3}\right]. \tag{5.22}
$$

Then, we have

$$
\sup\{|x| : x \in F(t, x)\} \le \frac{1}{8(1+t)^2} + \frac{1}{3}, \text{ and } H_d(F(t, x), F(t, \bar{x})) \le \frac{1}{8(1+t)^2}|x - \bar{x}|.
$$

Let $m(t) = 1/(8(1+t)^2)$. Then $H_d(F(t,x), F(t,\bar{x})) \le m(t)|x-\bar{x}|$ and $\|m\|_{L^1} = 5/56$. Using the given data, we find that $\hat{\delta} \approx 0.95635768 < 1$. Thus all the conditions of Theorem 5.7 are satisfied. Therefore, by the conclusion of Theorem 5.7, the problem (5.21) with $F(t,x)$ given by (5.22) has at least one solution on $[0, 5/2]$.

5.3.2 *The Carathéodory Case*

In this subsection, we consider the case when F has convex values and prove an existence result based on nonlinear alternative of Leray-Schauder type, assuming that F is Carathéodory.

Theorem 5.8 *Assume that:*

(5.8.1) $F : [0,T] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ *is Carathéodory and has nonempty compact and convex values;*

(5.8.2) there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in C([0,T], \mathbb{R}^+)$ *such that*

$$\|F(t,x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|) \text{ for each } (t,x) \in [0,T] \times \mathbb{R};$$

(5.8.3) there exists a constant $M > 0$ *such that*

$$\frac{M}{\psi(M)\|p\|\Phi_1} > 1,$$

where Φ_1 *is defined by (5.8).*

Then the problem (5.20) has at least one solution on $[0,T]$.

Proof Consider the operator $\mathscr{B} : \mathscr{E}_0 \to \mathscr{P}(\mathscr{E}_0)$ defined in the beginning of the proof of Theorem 5.7. We will show that $\mathscr{B}$ satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps. As a first step, we show that $\mathscr{B}$ *is convex for each* $x \in \mathscr{E}_0$. This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore, we omit the proof.

In the second step, we show that $\mathscr{B}$ *maps bounded sets (balls) into bounded sets in* $\mathscr{E}_0$. For a positive number ρ, let $B_\rho = \{x \in \mathscr{E}_0 : \|x\| \le \rho\}$ be a bounded ball in $\mathscr{E}_0$. Then, for each $h \in \mathscr{B}(x), x \in B_\rho$, there exists $v \in S_{F,x}$ such that

$$h(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(s)ds + \frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \int_0^T (T-s)^{q-1} v(s)ds$$

$$-\frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \sum_{i=1}^n \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-r)^{q-1} \frac{v(r)}{s} drds.$$

Then, for $t \in [0, T]$, we have

$$|h(t)| \le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |v(s)| ds + \frac{t^{q-1}}{|\Lambda_1|\Gamma(q)} \int_0^T (T-s)^{q-1} |v(s)| ds$$

$$+ \frac{t^{q-1}}{|\Lambda_1|\Gamma(q)} \sum_{i=1}^n \frac{|\alpha_i|}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-r)^{q-1} \frac{|v(r)|}{s} drds$$

$$\le \psi(\|x\|)\|p\| \left(\frac{T^q}{\Gamma(q+1)} + \frac{T^{2q-1}}{|\Lambda_1|\Gamma(q+1)} + \frac{T^{q-1}}{|\Lambda_1|\Gamma(q+1)} \sum_{i=1}^n \frac{|\alpha_i|\eta_i^q}{q^{p_i}} \right),$$

and consequently,

$$\|\mathscr{B}x\| \le \psi(\rho)||p||\Phi_1.$$

Now, we show that the operator $\mathscr{B}$ *maps bounded sets into equicontinuous sets of* $\mathscr{E}_0$. Let $\tau_1, \tau_2 \in [0, T]$ with $\tau_1 < \tau_2$ and $x \in B_\rho$. For each $h \in \mathscr{B}(x)$, we obtain

$$|h(\tau_2) - h(\tau_1)|$$

$$\le \frac{1}{\Gamma(q)} \left| \int_0^{\tau_1} [(\tau_2-s)^{q-1} - (\tau_1-s)^{q-1}] f(s, x(s)) ds + \int_{\tau_1}^{\tau_2} (\tau_2-s)^{q-1} f(s, x(s)) ds \right|$$

$$+ \frac{(\tau_2^{q-1} - \tau_1^{q-1})}{|\Lambda_1|\Gamma(q)} \int_0^T (T-s)^{q-1} |v(s)| ds$$

$$+ \frac{(\tau_2^{q-1} - \tau_1^{q-1})}{|\Lambda_1|\Gamma(q)} \sum_{i=1}^n \frac{|\alpha_i|}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-r)^{q-1} \frac{|v(r)|}{s} drds$$

$$\le \frac{\|p\|\psi(\rho)}{\Gamma(q+1)} [2(\tau_2 - \tau_1)^q + |\tau_2^q - \tau_1^q|]$$

$$+ \frac{(\tau_2^{q-1} - \tau_1^{q-1}) T^{q-1} \|p\|\psi(\rho)}{|\Lambda_1|\Gamma(q+1)} \left(T + \sum_{i=1}^n \frac{|\alpha_i|\eta_i^q}{q^{p_i}} \right).$$

Obviously the right hand side of the above inequality tends to zero, independently of $x \in B_\rho$ as $\tau_2 - \tau_1 \to 0$. As $\mathscr{B}$ satisfies the above three assumptions, therefore it follows by the Arzelá-Ascoli Theorem that $\mathscr{B} : \mathscr{E}_0 \to \mathscr{P}(\mathscr{E}_0)$ is completely continuous.

By Lemma 1.1, $\mathscr{B}$ will be upper semi-continuous (u.s.c.) if we prove that it has a closed graph, since $\mathscr{B}$ is already shown to be completely continuous.

Thus, in our next step, we show that $\mathscr{B}$ *has a closed graph.* Let $x_n \to x_*, h_n \in \mathscr{B}(x_n)$ and $h_n \to h_*$. Then, we need to show that $h_* \in \mathscr{B}(x_*)$. Associated with $h_n \in \mathscr{B}(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in [0, T]$,

$$h_n(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v_n(s) ds + \frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \int_0^T (T-s)^{q-1} v_n(s) ds$$

$$-\frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \sum_{i=1}^{n} \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-r)^{q-1} \frac{v_n(r)}{s} drds.$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [0, T]$,

$$h_*(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v_*(s) ds + \frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \int_0^T (T-s)^{q-1} v_*(s) ds$$

$$-\frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \sum_{i=1}^{n} \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-r)^{q-1} \frac{v_*(r)}{s} drds.$$

Let us consider the linear operator $\Theta : L^1([0, T], \mathbb{R}) \to \mathscr{E}_0$ given by

$$f \mapsto \Theta(v)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(s) ds + \frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \int_0^T (T-s)^{q-1} v(s) ds$$

$$-\frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \sum_{i=1}^{n} \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-r)^{q-1} \frac{v(r)}{s} drds.$$

Observe that

$$\begin{aligned}
&\|h_n(t) - h_*(t)\| \\
&= \Bigg\| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} (v_n(s) - v_*(s)) ds + \frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \int_0^T (T-s)^{q-1} (v_n(s) - v_*(s)) ds \\
&\quad -\frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \sum_{i=1}^{n} \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-r)^{q-1} \frac{(v_n(r) - v_*(r))}{s} drds \Bigg\| \to 0,
\end{aligned}$$

as $n \to \infty$. Thus, it follows by Lemma 1.2 that $\Theta \circ S_{F,x}$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v_*(s) ds + \frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \int_0^T (T-s)^{q-1} v_*(s) ds$$

$$-\frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \sum_{i=1}^{n} \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-r)^{q-1} \frac{v_*(r)}{s} drds,$$

for some $v_* \in S_{F,x_*}$.

Finally, we show there exists an open set $U \subseteq \mathscr{E}_0$ with $x \notin \theta\mathscr{B}(x)$ for any $\theta \in (0,1)$ and all $x \in \partial U$. Let $\theta \in (0,1)$ and $x \in \theta\mathscr{B}(x)$. Then there exists $v \in L^1([0,T],\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in [0,T]$, we have

$$x(t) = \frac{\theta}{\Gamma(q)}\int_0^t (t-s)^{q-1}v(s)ds + \frac{\theta t^{q-1}}{\Lambda_1\Gamma(q)}\int_0^T (T-s)^{q-1}v(s)ds$$

$$-\frac{\theta t^{q-1}}{\Lambda_1\Gamma(q)}\sum_{i=1}^n \frac{\alpha_i}{\Gamma(p_i)}\int_0^{\eta_i}\int_0^s \left(\log\frac{\eta_i}{s}\right)^{p_i-1}(s-r)^{q-1}\frac{v(r)}{s}drds.$$

As in the second step, we can obtain

$$\|x\| \le \psi(\|x\|)\|p\|\Phi_1,$$

which implies that

$$\frac{\|x\|}{\psi(\|x\|)\|p\|\Phi_1} \le 1.$$

In view of (5.8.3), there exists M such that $\|x\| \ne M$. Let us set

$$U = \{x \in \mathscr{E}_0 : \|x\| < M\}.$$

Note that the operator $\mathscr{B} : \overline{U} \to \mathscr{P}(\mathscr{E}_0)$ is upper semi-continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \theta\mathscr{B}(x)$ for some $\theta \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 1.15), we deduce that $\mathscr{B}$ has a fixed point $x \in \overline{U}$, which is a solution of the problem (5.20). This completes the proof. □

Example 5.8 Consider the boundary value problem for Riemann-Liouville fractional differential inclusions with nonlocal Hadamard fractional integral boundary conditions studied in Example 5.7 with the values of $F(t,x)$ as follows:

(a) Let $F : [0,5/2] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ be a multivalued map given by

$$x \to F(t,x) = \left[\frac{|x|}{1+\sin^2 2x + |x|} + t^2 + \frac{1}{2}, 1 + e^{-x^2} + \left(t + \frac{93}{2}\right)^{\frac{1}{2}}\right]. \tag{5.23}$$

For $f \in F$, we have

$$|f| \le \max\left(\frac{|x|}{1+\sin^2 2x + |x|} + t^2 + \frac{1}{2}, 1 + e^{-x^2} + \left(t + \frac{93}{2}\right)^{\frac{1}{2}}\right) \le 9,\ x \in \mathbb{R}.$$

Thus, $\|F(t,x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t,x)\} \le 9 = p(t)\psi(\|x\|),\ x \in \mathbb{R}$, with $p(t) = 1, \psi(\|x\|) = 9$. Further, using the condition (5.8.3), we find that $M > 96.400854$. Therefore, all the conditions of Theorem 5.8 are satisfied. So, the problem (5.21) with $F(t,x)$ given by (5.23) has at least one solution on $[0,5/2]$.

(b) Let $F : [0, 5/2] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ be a multivalued map given by

$$x \to F(t, x) = \left[e^{-x^4} + \frac{t}{3}, \frac{|x|}{1 + 2|x|} + t + \frac{3}{2} \right]. \tag{5.24}$$

For $f \in F$, we have

$$|f| \leq \max \left(e^{-x^4} + \frac{t}{3}, \frac{|x|}{1 + 2|x|} + t + \frac{3}{2} \right) \leq 2 + t, \ x \in \mathbb{R}.$$

Here, $\|F(t, x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t, x)\} \leq (2 + t) = p(t)\psi(\|x\|), \ x \in \mathbb{R}$, with $p(t) = 2 + t$, $\psi(\|x\|) = 1$. It is easy to verify that $M > 48.200427$. Then, by Theorem 5.8, the problem (5.21) with $F(t, x)$ given by (5.24) has at least one solution on $[0, 5/2]$.

5.3.3 The Lower Semicontinuous Case

In the next result, it is assumed that F is not necessarily convex valued. Our strategy to deal with this problem is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo (Lemma 1.3) for lower semi-continuous maps with decomposable values.

Theorem 5.9 *Assume that (5.8.2), (5.8.3) and the following condition hold:*

(5.9.1) $F : [0, T] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*

- *(a)* $(t, x) \longmapsto F(t, x)$ *is* $\mathscr{L} \otimes \mathscr{B}$ *measurable,*
- *(b)* $x \longmapsto F(t, x)$ *is lower semi-continuous for each* $t \in [0, T]$.

Then the problem (5.20) has at least one solution on $[0, T]$.

Proof It follows from (5.8.2) and (5.9.1) that F is of l.s.c. type. Then from Lemma 1.3, there exists a continuous function $f : \mathscr{E}_0 \to L^1([0, T], \mathbb{R})$ such that $f(x) \in F(x)$ for all $x \in \mathscr{E}_0$.

Consider the problem

$$\begin{cases} {}_{RL}D^q x(t) = f(x(t)), \ \ 0 < t < T, \ \ 1 < q \leq 2, \\ x(0) = 0, \qquad x(T) = \displaystyle\sum_{i=1}^{n} \alpha_{i H} I^{p_i} x(\eta_i). \end{cases} \tag{5.25}$$

Observe that if $x \in \mathscr{C}^2([0, T], \mathbb{R})$ is a solution of (5.25), then x is a solution to the problem (5.20). In order to transform the problem (5.25) into a fixed point problem, we define the operator $\overline{\mathscr{B}}_F$ as

$$\overline{\mathscr{B}}_F x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(x(s))ds + \frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \int_0^T (T-s)^{q-1} f(x(s))ds$$

$$-\frac{t^{q-1}}{\Lambda_1 \Gamma(q)} \sum_{i=1}^{n} \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s} \right)^{p_i - 1} (s-r)^{q-1} \frac{f(x(r))}{s} drds.$$

It can easily be shown that $\overline{\mathscr{B}}_F$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 5.8. So, we omit it. This completes the proof. □

5.4 Riemann-Liouville Fractional Differential Equations and Inclusions with Nonlocal Hadamard Fractional Integral Boundary Conditions

In this section, we introduce the general form of nonlocal conditions by replacing $x(T)$ by $g(x)$ in (5.2) and consider the following boundary value problem

$$_{RL}D^q x(t) = f(t, x(t)), \quad t \in (0, T), \tag{5.26}$$

$$x(0) = 0, \qquad g(x) = \sum_{i=1}^{n} \alpha_{i\,H} I^{p_i} x(\eta_i), \tag{5.27}$$

where $g : C([0, T], \mathbb{R}) \to \mathbb{R}$.

Also, we study the multivalued analogue of the above problem

$$_{RL}D^q x(t) \in F(t, x(t)), \quad t \in (0, T), \tag{5.28}$$

$$x(0) = 0, \qquad g(x) = \sum_{i=1}^{n} \alpha_{i\,H} I^{p_i} x(\eta_i), \tag{5.29}$$

where $F : [0, T] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map, $\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

Lemma 5.2 *Let* $\Lambda_2 := \sum_{i=1}^{n} \frac{\alpha_i \eta_i^{q-1}}{(q-1)^{p_i}} \neq 0,\ 1 < q \leq 2,\ p_i > 0,\ \alpha_i \in \mathbb{R},\ \eta_i \in (0, T)$, $i = 1, 2, 3, \ldots, n$ *and* $h \in \mathscr{E}_0$. *Then, the nonlocal Hadamard fractional boundary value problem for linear Riemann-Liouville fractional differential equation*

$$_{RL}D^q x(t) = h(t), \qquad 0 \leq t \leq T, \tag{5.30}$$

subject to the boundary conditions (5.27) is equivalent to the following fractional integral equation

$$x(t) = {}_{RL}I^q h(t) - \frac{t^{q-1}}{\Lambda_2}\left(\sum_{i=1}^{n} \alpha_i ({}_H I^{p_i}{}_{RL}I^q h)(\eta_i) - g(x)\right). \tag{5.31}$$

Proof We omit the proof as it is similar to that of Lemma 5.1. □

5.4.1 Existence Results: The Single-Valued Case

In view of Lemma 5.2, we define an operator $\mathcal{Q} : \mathcal{E}_0 \to \mathcal{E}_0$ by

$$\begin{aligned}(\mathcal{Q}x)(t) = &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\ &-\frac{t^{q-1}}{\Lambda_2 \Gamma(q)} \sum_{i=1}^{n} \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i}\int_0^{s} \left(\log\frac{\eta_i}{s}\right)^{p_i-1} (s-\tau)^{q-1} \frac{f(\tau,x(\tau))}{s} d\tau ds \\ &+\frac{t^{q-1}}{\Lambda_2} g(x), \qquad t\in[0,T].\end{aligned} \tag{5.32}$$

For convenience, we set:

$$p_0 = \frac{T^q}{\Gamma(q+1)} + \frac{T^{q-1}}{|\Lambda_2|\Gamma(q+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^q}{q^{p_i}}, \tag{5.33}$$

and

$$k_0 = \frac{T^{q-1}}{|\Lambda_2|}. \tag{5.34}$$

In the next, we prove an existence and uniqueness result for the problem (5.26)–(5.27) by means of Banach's fixed point theorem.

Theorem 5.10 *Let $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ and $g : C([0,T],\mathbb{R}) \to \mathbb{R}$ be continuous functions. Assume that:*

(5.10.1) $|f(t,x) - f(t,y)| \le L|x-y|, \forall t \in [0,T],\ L > 0,\ x,y\in\mathbb{R}$;
(5.10.2) $|g(u) - g(v)| \le \ell\|u-v\|,\ \ell < k_0^{-1}$ *for all* $u,v \in C([0,T],\mathbb{R})$;
(5.10.3) $\gamma := Lp_0 + \ell k_0 < 1$.

Then the problem (5.26)–(5.27) has a unique solution on $[0,T]$.

Proof For $x,y \in \mathcal{E}_0$ and for each $t\in[0,T]$, from the definition of the operator $\mathcal{Q}$ defined by (5.32), and assumptions (5.10.1), (5.10.2), we obtain

$$
\begin{aligned}
&|(\mathscr{Q}x)(t)-(\mathscr{Q}y)(t)| \\
&\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))-f(s,y(s))|ds \\
&\quad +\frac{T^{q-1}}{|\Lambda_2|\Gamma(q)}\sum_{i=1}^{n}\frac{|\alpha_i|}{\Gamma(p_i)}\int_0^{\eta_i}\int_0^s\left(\log\frac{\eta_i}{s}\right)^{p_i-1}(s-\tau)^{q-1}\frac{|f(\tau,x(\tau))-f(\tau,y(\tau))|}{s}d\tau ds \\
&\quad +\frac{T^{q-1}}{|\Lambda_2|}|g(x)-g(y)| \\
&\le L\|x-y\|\Bigg[\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}ds \\
&\quad +\frac{T^{q-1}}{|\Lambda_2|\Gamma(q)}\sum_{i=1}^{n}\frac{|\alpha_i|}{\Gamma(p_i)}\int_0^{\eta_i}\int_0^s\left(\log\frac{\eta_i}{s}\right)^{p_i-1}(s-\tau)^{q-1}\frac{d\tau}{s}ds\Bigg]+\frac{T^{q-1}}{|\Lambda_2|}\ell\|x-y\| \\
&\le L\|x-y\|\left\{\frac{T^q}{\Gamma(q+1)}+\frac{T^{q-1}}{|\Lambda_2|\Gamma(q+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^q}{q^{p_i}}\right\}+\frac{T^{q-1}}{|\Lambda_2|}\ell\|x-y\| \\
&= (Lp_0+\ell k_0)\|x-y\|.
\end{aligned}
$$

Hence

$$\|\mathscr{Q}x-\mathscr{Q}y\| \le \gamma\|x-y\|.$$

As $\gamma < 1$ by (5.10.3), the operator $\mathscr{Q}$ is a contraction from the Banach space $\mathscr{E}_0$ into itself. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Example 5.9 Consider the following nonlocal boundary value problem:

$$
\begin{cases}
{}_{RL}D^{3/2}x(t)=\dfrac{e^{-t^2}}{3(\sqrt{3}+t)^2}\cdot\dfrac{|x|}{1+|x|}-1, \quad t\in(0,\pi), \\
x(0)=0, \quad \dfrac{1}{8}x(\pi)+\dfrac{\sqrt{2}}{2}=\dfrac{1}{2}{}_HI^{\sqrt{3}}x\left(\dfrac{\pi}{2}\right)-\dfrac{4}{5}{}_HI^{2/3}x\left(\dfrac{\pi}{3}\right).
\end{cases}
\tag{5.35}
$$

Here $q=3/2,\ T=\pi,\ n=2,\ \alpha_1=1/2,\ \alpha_2=-4/5,\ \rho_1=\sqrt{3},\ \rho_2=2/3,\ \eta_1=\pi/2,\ \eta_2=\pi/3,\ g(x)=(1/8)x(\pi)+(\sqrt{2}/2)$ and $f(t,x)=(e^{-t^2}|x|)/(3(\sqrt{3}+t)^2)(1+|x|))-1$. By using computer program, we find that $\Lambda_2\approx 0.78220904\neq 0,\ p_0\approx 6.13531215$ and $k_0\approx 2.2659593$. As $|f(t,x)-f(t,y)\le(1/9)|x-y|$ and $|g(t,x)-g(t,y)\le(1/4)|x-y|$, therefore, (5.10.1) and (5.10.2) are satisfied with $L=1/9$ and $\ell=1/4<0.44131419=k_0^{-1}$, respectively. Also $\gamma=Lp_0+\ell k_0\approx 0.96494626<1$. By the conclusion of Theorem 5.10, the nonlocal boundary value problem (5.35) has a unique solution on $[0,\pi]$.

Our next existence result is based on O'Regan's fixed point theorem (Theorem 1.6).

Theorem 5.11 *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that (5.10.2) holds. In addition, we assume that:*

(5.11.1) $g(0) = 0$;

(5.11.2) there exists a nonnegative function $m \in C([0,T], \mathbb{R}^+)$ and a nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that

$$|f(t,u)| \le m(t)\psi(\|u\|) \text{ for any } (t,u) \in [0,T] \times \mathbb{R};$$

(5.11.3) $\sup_{r\in(0,\infty)} \frac{r}{p_0\|m\|\psi(r)} > \frac{1}{1-k_0\ell}$,

where p_0 and k_0 are defined by (5.33) and (5.34) respectively. Then, the problem (5.26)–(5.27) has at least one solution on $[0,T]$.

Proof Consider the operator $\mathcal{Q} : \mathcal{E}_0 \to \mathcal{E}_0$ defined by (5.32). We decompose $\mathcal{Q}$ into a sum of two operators

$$(\mathcal{Q}x)(t) = (\mathcal{Q}_1x)(t) + (\mathcal{Q}_2x)(t), \quad t \in [0,T], \tag{5.36}$$

where

$$\begin{aligned}(\mathcal{Q}_1x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds \\ &- \frac{t^{q-1}}{\Lambda_2\Gamma(q)} \sum_{i=1}^n \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i}\int_0^s \left(\log\frac{\eta_i}{s}\right)^{p_i-1}(s-\tau)^{q-1}\frac{f(\tau,x(\tau))}{s}d\tau ds, \quad t \in [0,T],\end{aligned} \tag{5.37}$$

and

$$(\mathcal{Q}_2x)(t) = \frac{t^{q-1}}{\Lambda_2} g(x), \quad t \in [0,T]. \tag{5.38}$$

Let

$$K_r = \{x \in \mathcal{E}_0 : \|x\| < r\}.$$

From (5.11.3), there exists a number $r_0 > 0$ such that

$$\frac{r_0}{p_0\|m\|\psi(r_0)} > \frac{1}{1-k_0\ell}. \tag{5.39}$$

We shall prove that operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ satisfy all the conditions of Theorem 1.6.

Step 1. *The set $\mathscr{Q}(\bar{K}_{r_0})$ is bounded.* For any $x \in \bar{K}_{r_0}$, we have

$$\begin{aligned}
\|\mathscr{Q}_1 x\| &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds \\
&\quad + \frac{T^{q-1}}{|\Lambda_2|\Gamma(q)} \sum_{i=1}^n \frac{|\alpha_i|}{\Gamma(p_i)} \int_0^{\eta_i}\int_0^s \Big(\log\frac{\eta_i}{s}\Big)^{p_i-1}(s-\tau)^{q-1}\frac{|f(\tau,x(\tau))|}{s}d\tau ds \\
&\leq \|m\|\psi(r_0)\left\{\frac{T^q}{\Gamma(q+1)} + \frac{T^{q-1}}{|\Lambda_2|\Gamma(q+1)}\sum_{i=1}^n \frac{|\alpha_i|\eta_i^q}{q^{p_i}}\right\} \\
&= \|m\|\psi(r_0)p_0.
\end{aligned}$$

This shows that $\mathscr{Q}_1(\bar{K}_{r_0})$ is uniformly bounded.
The conditions (5.10.2) and (5.11.2) imply that

$$\|\mathscr{Q}_2(x)\| \leq \frac{T^{q-1}}{|\Lambda_2|}\ell\, r_0,$$

for any $x \in \bar{K}_{r_0}$. Thus, the set $\mathscr{Q}(\bar{K}_{r_0})$ is bounded.

Step 2. *The operator $\mathscr{Q}_1$ is continuous and completely continuous.*
By Step 1, $\mathscr{Q}_1(\bar{K}_{r_0})$ is uniformly bounded. In addition, for any $t_1, t_2 \in [0,T]$, we have:

$$\begin{aligned}
&|(\mathscr{Q}_1 x)(t_2) - (\mathscr{Q}_1 x)(t_1)| \\
&\leq \int_0^{t_1} \frac{[(t_2-s)^{q-1} - (t_1-s)^{q-1}]}{\Gamma(q)}|f(s,x(s))|ds + \int_{t_1}^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds \\
&\quad + \frac{|t_2^{q-1}-t_1^{q-1}|}{|\Lambda_2|\Gamma(q)}\sum_{i=1}^n \frac{|\alpha_i|}{\Gamma(p_i)}\int_0^{\eta_i}\int_0^s\Big(\log\frac{\eta_i}{s}\Big)^{p_i-1}(s-\tau)^{q-1}\frac{|f(\tau,x(\tau))|}{s}d\tau ds \\
&\leq \frac{\|m\|\psi(r_0)}{\Gamma(q+1)}[2(t_2-t_1)^q + |t_2^q - t_1^q|] + \frac{\|m\|\psi(r_0)|t_2^{q-1}-t_1^{q-1}|}{|\Lambda_2|\Gamma(q+1)}\sum_{i=1}^n\frac{|\alpha_i|\eta_i^q}{q^{p_i}},
\end{aligned}$$

which is independent of x, and tends to zero as $t_2 - t_1 \to 0$. Thus, $\mathscr{Q}_1$ is equicontinuous. Hence, by the Arzelá-Ascoli Theorem, $\mathscr{Q}_1(\bar{K}_{r_0})$ is a relatively compact set. Now, let $x_n, x \in \bar{K}_{r_0}$ with $\|x_n - x\| \to 0$. Then the limit $|x_n(t) - x(t)| \to 0$ is uniformly valid on $[0,T]$. From the uniform continuity of $f(t,x)$ on the compact set $[0,T]\times[-r_0,r_0]$, it follows that $\|f(t,x_n(t)) - f(t,x(t))\| \to 0$ is uniformly valid on $[0,T]$. Hence $\|\mathscr{Q}_1 x_n - \mathscr{Q}_1 x\| \to 0$ as $n \to \infty$ which proves the continuity of $\mathscr{Q}_1$. Consequently the operator $\mathscr{Q}_1$ is continuous and completely continuous

Step 3. *The operator* $\mathscr{Q}_2 : \bar{K}_{r_0} \to \mathscr{E}_0$ *is contractive*. Observe that

$$|(\mathscr{Q}_2 x)(t) - (\mathscr{Q}_2 y)(t)| = \frac{t^{q-1}}{|\Lambda_2|}|g(x) - g(y)| \le \frac{T^{q-1}}{|\Lambda_2|}\ell\|x - y\| = \hat{\theta}\|x - y\|,$$

with $\hat{\theta} = k_0\ell < 1$ by (5.10.2). Hence $\mathscr{Q}_2$ is contractive.

Step 4. *Finally, it will be shown that the case (C2) in Theorem 1.6 does not occur*. For that, we suppose that (C2) holds. Then, we have that there exist $\kappa \in (0, 1)$ and $x \in \partial K_{r_0}$ such that $x = \kappa\mathscr{Q}x$. So, we have $\|x\| = r_0$ and for $t \in [0, T]$,

$$x(t) = \kappa\Bigg\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s))ds$$
$$- \frac{t^{q-1}}{\Lambda_2\Gamma(q)}\sum_{i=1}^n \frac{\alpha_i}{\Gamma(p_i)}\int_0^{\eta_i}\int_0^s \left(\log\frac{\eta_i}{s}\right)^{p_i-1}(s-\tau)^{q-1}\frac{f(\tau, x(\tau))}{s}d\tau ds + \frac{t^{q-1}}{\Lambda_2}g(x)\Bigg\}.$$

Using the hypotheses (5.11.1)–(5.11.3), we get

$$|x(t)| \le \psi(\|x\|)\Bigg\{\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} m(s)ds$$
$$+ \frac{T^{q-1}}{|\Lambda_2|\Gamma(q)}\sum_{i=1}^n \frac{|\alpha_i|}{\Gamma(p_i)}\int_0^{\eta_i}\int_0^s \left(\log\frac{\eta_i}{s}\right)^{p_i-1}(s-\tau)^{q-1}\frac{m(\tau)}{s}d\tau ds\Bigg\}$$
$$+ \frac{T^{q-1}}{|\Lambda_2|}\ell\|x\|.$$

Taking the supremum over $t \in [0, T]$, we obtain

$$\|x\| \le \psi(\|x\|)\Bigg\{\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} m(s)ds$$
$$+ \frac{T^{q-1}}{|\Lambda_2|\Gamma(q)}\sum_{i=1}^n \frac{|\alpha_i|}{\Gamma(p_i)}\int_0^{\eta_i}\int_0^s \left(\log\frac{\eta_i}{s}\right)^{p_i-1}(s-\tau)^{q-1}\frac{m(\tau)}{s}d\tau ds\Bigg\}$$
$$+ \frac{T^{q-1}}{|\Lambda_2|}\ell\|x\|,$$

or

$$r_0 \le \|m\|\psi(r_0)\left\{\frac{T^q}{\Gamma(q+1)} + \frac{T^{q-1}}{|\Lambda_2|\Gamma(q+1)}\sum_{i=1}^n \frac{|\alpha_i|\eta_i^q}{q^{p_i}}\right\} + \frac{T^{q-1}}{|\Lambda_2|}\ell r_0,$$

which implies that

$$r_0 \le p_0\|m\|\psi(r_0) + k_0\ell\, r_0.$$

Thus,

$$\frac{r_0}{p_0\|m\|\psi(r_0)} \leq \frac{1}{1-k_0\ell},$$

which contradicts (5.39). Thus we have shown that the operators $\mathscr{Q}_1$ and $\mathscr{Q}_2$ satisfy all the conditions of Theorem 1.6. Hence, the operator $\mathscr{Q}$ has at least one fixed point $x \in \bar{K}_{r_0}$, which is the solution of the problem (5.26)–(5.27). The proof is completed. □

Example 5.10 Consider the following nonlocal nonlinear fractional boundary value problem:

$$\begin{cases} {}_{RL}D^{5/3}x(t) = \dfrac{t}{2}\left(\dfrac{|x|+1}{|x|+2}+|x|\right), \quad t \in (0,1/3), \\ x(0)=0, \quad \dfrac{1}{4}\sin\left(x\left(\dfrac{1}{6}\right)\right) = \dfrac{4}{7}{}_HI^{\pi}x\left(\dfrac{1}{12}\right) \\ \qquad + \dfrac{1}{4}{}_HI^{1/2}x\left(\dfrac{1}{8}\right) + {}_HI^{3/2}x\left(\dfrac{1}{5}\right). \end{cases} \tag{5.40}$$

Here $q = 5/3,\ T = 1/3,\ n = 3,\ \alpha_1 = 4/7,\ \alpha_2 = 1/4,\ \alpha_3 = 1,\ \rho_1 = \pi,\ \rho_2 = 1/2,\ \rho_3 = 3/2,\ \eta_1 = 1/12,\ \eta_2 = 1/8,\ \eta_3 = 1/5,\ g(x) = (1/4)\sin(x)$ and $f(t,x) = (t/2)((|x|+1)/(|x|+2)+|x|)$. By using computer program, we find that $\Lambda_2 \approx 1.09451783 \neq 0,\ p_0 \approx 0.11808820,\ k_0 \approx 0.43923438$. As $|g(x)-g(y)| \leq (1/4)|x-y|$ with $\ell = (1/4) < 0.43923438 = k_0^{-1}$ and $g(0) = 0$, therefore, (5.10.2) and (5.11.1) are satisfied respectively. Since $|f(t,x)| = |(t/2)((|x|+1)/(|x|+2)+|x|)| \leq (t/2)(x^2+3|x|+1)$, we choose $m(t) = t/2$ and $\psi(|x|) = x^2+3|x|+1$, and find that

$$\sup_{r\in(0,\infty)} \frac{r}{p_0\|m\|\psi(r)} \approx 10.16189578 > 1.123353914 = \frac{1}{1-k_0\ell}.$$

Therefore, by Theorem 5.11, the problem (5.40) has at least one solution on $[0,1/3]$.

5.4.2 *Existence Results: The Multivalued Case*

In this section, we will prove an existence result for the problem (5.28)–(5.29) by using the nonlinear alternative for contractive maps (Theorem 1.17).

Definition 5.2 A function $x \in \mathscr{C}^2([0,T],\mathbb{R})$ is a solution of the problem (5.28)–(5.29) if $x(0) = 0,\ g(x) = \sum_{i=1}^{n}\alpha_{iH}I^{p_i}x(\eta_i)$, and there exists a function $f \in L^1([0,T],\mathbb{R})$ such that $f(t) \in F(t,x(t))$ a.e. on $[0,T]$ and

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds \\ -\frac{t^{q-1}}{\Lambda_2 \Gamma(q)} \sum_{i=1}^{n} \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-\tau)^{q-1} \frac{f(\tau)}{s} d\tau ds \\ +\frac{t^{q-1}}{\Lambda_2} g(x). \tag{5.41}$$

Theorem 5.12 *Assume that (5.10.2) holds. In addition, we suppose that:*

(5.11.1) $F : [0, T] \times \mathbb{R} \to \mathscr{P}_{cp,c}(\mathbb{R})$ *is* L^1*−Carathéodory multivalued map;*

(5.11.2) there exists a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ *and a function* $m \in C([0, T], \mathbb{R}^+)$ *such that*

$$\|F(t, x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t, x)\} \le m(t)\psi(|x|) \ \textit{for each} \ (t, x) \in [0, T] \times \mathbb{R};$$

(5.11.3) there exists a number $M > 0$ *such that*

$$\frac{(1 - k_0\ell)M}{p_0\|m\|\psi(M)} > 1, \tag{5.42}$$

where p_0 *and* k_0 *are defined by (5.33) and (5.34) respectively.*

Then the problem (5.28)–(5.29) has at least one solution on $[0, T]$.

Proof To transform the problem (5.28)–(5.29) into a fixed point problem, we consider the operator $\mathscr{N} : \mathscr{E}_0 \longrightarrow \mathscr{P}(\mathscr{E}_0)$ defined by

$$\mathscr{N}(x) = \left\{ \begin{array}{l} h \in \mathscr{E}_0 : \\ h(t) = \left\{ \begin{array}{l} \displaystyle\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds \\ \displaystyle -\frac{t^{q-1}}{\Lambda_2 \Gamma(q)} \sum_{i=1}^{n} \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-\tau)^{q-1} \frac{f(\tau)}{s} drds \\ \displaystyle +\frac{t^{q-1}}{\Lambda_2} g(x), \end{array} \right. \end{array} \right\}$$

for $f \in S_{F,x}$.

Next, we introduce two operators: $\mathscr{N}_1 : \mathscr{E}_0 \longrightarrow \mathscr{E}_0$ by

$$\mathscr{N}_1 x(t) = \frac{t^{q-1}}{\Lambda_2} g(x), \tag{5.43}$$

and the multivalued operator $\mathscr{N}_2 : \mathscr{E}_0 \longrightarrow \mathscr{P}(\mathscr{E}_0)$ by

$$\mathscr{N}_2(x) = \left\{ \begin{array}{l} h \in \mathscr{E}_0 : \\ h(t) = \left\{ \begin{array}{l} \displaystyle\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds \\ \displaystyle -\frac{t^{q-1}}{\Lambda_2 \Gamma(q)} \sum_{i=1}^{n} \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-\tau)^{q-1} \frac{f(\tau)}{s} drds. \end{array} \right. \end{array} \right\} \tag{5.44}$$

Observe that $\mathscr{N} = \mathscr{N}_1 + \mathscr{N}_2$. We shall show that the operators $\mathscr{N}_1$ and $\mathscr{N}_2$ satisfy all the conditions of Theorem 1.17 on $[0, T]$. For that, we consider the operators $\mathscr{N}_1, \mathscr{N}_2 : B_r \to \mathscr{P}_{cp,c}(\mathscr{E}_0)$, where $B_r = \{x \in \mathscr{E}_0 : \|x\| \leq r\}$ is a bounded set in $\mathscr{E}_0$. First, we prove that $\mathscr{N}_2$ is compact-valued on B_r. Note that the operator $\mathscr{N}_2$ is equivalent to the composition $\mathscr{L} \circ S_F$, where $\mathscr{L}$ is the continuous linear operator on $L^1([0, T], \mathbb{R})$ into $\mathscr{E}_0$, defined by

$$\mathscr{L}(v)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v(s)ds$$
$$-\frac{t^{q-1}}{\Lambda_2 \Gamma(q)} \sum_{i=1}^{n} \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i - 1} (s-\tau)^{q-1} \frac{v(\tau)}{s} drds.$$

Suppose that $x \in B_r$ is arbitrary and let $\{v_n\}$ be a sequence in $S_{F,x}$. Then, by definition of $S_{F,x}$, we have $v_n(t) \in F(t, x(t))$ for almost all $t \in [0, T]$. Since $F(t, x(t))$ is compact for all $t \in J$, there is a convergent subsequence of $\{v_n(t)\}$, (we denote it by $\{v_n(t)\}$ again) that converges in measure to some $v(t) \in S_{F,x}$ for almost all $t \in J$. On the other hand, $\mathscr{L}$ is continuous, so $\mathscr{L}(v_n)(t) \to \mathscr{L}(v)(t)$ pointwise on $[0, T]$.

In order to show that the convergence is uniform, we have to show that $\{\mathscr{L}(v_n)\}$ is an equi-continuous sequence. Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Then, we have

$$|\mathscr{L}(v_n)(t_2) - \mathscr{L}(v_n)(t_1)|$$
$$\leq \left| \int_0^{t_1} \frac{[(t_2-s)^{q-1} - (t_1-s)^{q-1}]}{\Gamma(q)} v_n(s)ds \right| + \left| \int_{t_1}^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)} v_n(s)ds \right|$$
$$+ \frac{|t_2^{q-1} - t_1^{q-1}|}{|\Lambda_2|\Gamma(q)} \sum_{i=1}^{n} \frac{|\alpha_i|}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-\tau)^{q-1} \frac{|v_n(\tau)|}{s} d\tau ds$$
$$\leq \frac{\|m\|\psi(r)}{\Gamma(q+1)} [2(t_2-t_1)^q + |t_2^q - t_1^q|] + \frac{\|m\|\psi(r)|t_2^{q-1} - t_1^{q-1}|}{|\Lambda_2|\Gamma(q+1)} \sum_{i=1}^{n} \frac{|\alpha_i|\eta_i^q}{q^{p_i}}.$$

We see that the right hand of the above inequality tends to zero as $t_2 \to t_1$. Thus, the sequence $\{\mathscr{L}(v_n)\}$ is equi-continuous and by using the Arzelá-Ascoli Theorem, we get that there is a uniformly convergent subsequence. So, there is a subsequence of $\{v_n\}$, (we denote it again by $\{v_n\}$) such that $\mathscr{L}(v_n) \to \mathscr{L}(v)$. Note that $\mathscr{L}(v) \in \mathscr{L}(S_{F,x})$. Hence, $\mathscr{N}_2(x) = \mathscr{L}(S_{F,x})$ is compact for all $x \in B_r$. So $\mathscr{N}_2(x)$ is compact.

Now, we show that $\mathscr{N}_2(x)$ is convex for all $x \in \mathscr{E}_0$. Let $z_1, z_2 \in \mathscr{N}_2(x)$. We select $f_1, f_2 \in S_{F,x}$ such that

$$z_i(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_i(s)ds$$
$$-\frac{t^{q-1}}{\Lambda_2 \Gamma(q)} \sum_{i=1}^{n} \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-\tau)^{q-1} \frac{f_i(\tau)}{s} d\tau ds,$$

$i = 1, 2$, for almost all $t \in [0, T]$. Let $0 \le \nu \le 1$. Then, we have

$$[\nu z_1 + (1-\nu)z_2](t)$$
$$= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}[\nu f_1(s) + (1-\nu)f_2(s)]ds$$
$$-\frac{t^{q-1}}{\Lambda_2\Gamma(q)}\sum_{i=1}^n \frac{\alpha_i}{\Gamma(p_i)}\int_0^{\eta_i}\int_0^s \left(\log\frac{\eta_i}{s}\right)^{p_i-1}(s-\tau)^{q-1}\frac{[\nu f_1(\tau) + (1-\nu)f_2(\tau)]}{s}d\tau ds.$$

Since F has convex values, so $S_{F,x}$ is convex and $\nu f_1(s) + (1-\nu)f_2(s) \in S_{F,x}$. Thus

$$\nu z_1 + (1-\nu)z_2 \in \mathscr{N}_2(x).$$

Consequently, $\mathscr{N}_2$ is convex-valued. Obviously, $\mathscr{N}_1$ is compact and convex-valued.

The rest of the proof consists of several steps and claims.

Step 1: We show that $\mathscr{N}_1$ *is a contraction on* $\mathscr{E}_0$. This is a consequence of (5.10.2), and the proof is similar to the one for the operator $\mathscr{Q}_2$ in Step 2 of Theorem 5.11.

Step 2: We shall show that the operator $\mathscr{N}_2$ *is compact and upper semicontinuous*. This will be given in several claims.

Claim I: $\mathscr{N}_2$ *maps bounded sets into bounded sets in* $\mathscr{E}_0$. To see this, let $B_r = \{x \in \mathscr{E}_0 : \|x\| \le r\}$ be a bounded set in $\mathscr{E}_0$. Then, for each $h \in \mathscr{N}_2(x),\ x \in B_r$, there exists $f \in S_{F,x}$ such that

$$h(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds$$
$$-\frac{t^{q-1}}{\Lambda_2\Gamma(q)}\sum_{i=1}^n \frac{\alpha_i}{\Gamma(p_i)}\int_0^{\eta_i}\int_0^s \left(\log\frac{\eta_i}{s}\right)^{p_i-1}(s-\tau)^{q-1}\frac{f(\tau)}{s}d\tau ds.$$

Then, for $t \in [0, T]$, we have

$$|h(t)| \le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s)|ds$$
$$+\frac{T^{q-1}}{|\Lambda_2|\Gamma(q)}\sum_{i=1}^n \frac{|\alpha_i|}{\Gamma(p_i)}\int_0^{\eta_i}\int_0^s \left(\log\frac{\eta_i}{s}\right)^{p_i-1}(s-r)^{q-1}\frac{|f(\tau)|}{s}d\tau ds$$
$$\le \psi(\|x\|)\Bigg[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}m(s)ds$$
$$+\frac{T^{q-1}}{|\Lambda_2|\Gamma(q)}\sum_{i=1}^n \frac{|\alpha_i|}{\Gamma(p_i)}\int_0^{\eta_i}\int_0^s \left(\log\frac{\eta_i}{s}\right)^{p_i-1}(s-\tau)^{q-1}\frac{m(\tau)}{s}d\tau ds\Bigg]$$
$$\le \psi(r)\left\{\frac{T^q}{\Gamma(q+1)} + \frac{T^{q-1}}{|\Lambda_2|\Gamma(q+1)}\sum_{i=1}^n \frac{|\alpha_i|\eta_i^q}{q^{p_i}}\right\}\|m\|.$$

Thus,

$$\|h\| \leq \psi(r) p_0 \|m\|.$$

Claim II: Next, we show that $\mathcal{N}_2$ *maps bounded sets into equicontinuous sets*. Let $t_1, t_2 \in [0, T]$ and $x \in B_r$. For each $h \in \mathcal{N}_2(x)$, we obtain

$$\begin{aligned}
&|h(t_2) - h(t_1)| \\
&\leq \left| \int_0^{t_1} \frac{[(t_2 - s)^{q-1} - (t_1 - s)^{q-1}]}{\Gamma(q)} f(s)ds \right| + \left| \int_{t_1}^{t_2} \frac{(t_2 - s)^{q-1}}{\Gamma(q)} f(s)ds \right| \\
&\quad + \frac{|t_2^{q-1} - t_1^{q-1}|}{|\Lambda_2|\Gamma(q)} \sum_{i=1}^{n} \frac{|\alpha_i|}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s} \right)^{p_i - 1} (s - \tau)^{q-1} \frac{|f(\tau)|}{s} d\tau ds \\
&\leq \frac{\|m\|\psi(r)}{\Gamma(q+1)} [2(t_2 - t_1)^q + |t_2^q - t_1^q|] + \frac{\|m\|\psi(\rho)|t_2^{q-1} - t_1^{q-1}|}{|\Lambda_2|\Gamma(q+1)} \sum_{i=1}^{n} \frac{|\alpha_i|\eta_i^q}{q^{p_i}}.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero, independently of $x \in B_r$ as $t_2 - t_1 \to 0$. Therefore it follows by the Arzelá-Ascoli Theorem that $\mathcal{N}_2 : \mathcal{E}_0 \to \mathcal{P}(\mathcal{E}_0)$ is completely continuous.

By Lemma 1.1, $\mathcal{N}_2$ will be upper semi-continuous (u.s.c.) if we prove that it has a closed graph since $\mathcal{N}_2$ is already shown to be completely continuous. We establish it in the next claim.

Claim III: $\mathcal{N}_2$ *has a closed graph*. Let $x_n \to x_*, h_n \in \mathcal{N}_2(x_n)$ and $h_n \to h_*$. Then, we need to show that $h_* \in \mathcal{N}_2(x_*)$. Associated with $h_n \in \mathcal{N}_2(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in [0, T]$,

$$\begin{aligned}
h_n(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_n(s)ds \\
&\quad - \frac{t^{q-1}}{\Lambda_2 \Gamma(q)} \sum_{i=1}^{n} \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s} \right)^{p_i - 1} (s - \tau)^{q-1} \frac{f_n(\tau)}{s} d\tau ds.
\end{aligned}$$

Thus it suffices to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0, T]$,

$$\begin{aligned}
h_*(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)ds \\
&\quad - \frac{t^{q-1}}{\Lambda_2 \Gamma(q)} \sum_{i=1}^{n} \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s} \right)^{p_i - 1} (s - \tau)^{q-1} \frac{f_*(\tau)}{s} d\tau ds.
\end{aligned}$$

Let us consider the linear operator $\Theta : L^1([0, T], \mathbb{R}) \to \mathcal{E}_0$ given by

$$\begin{aligned}
f \mapsto \Theta(f)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds \\
&\quad - \frac{t^{q-1}}{\Lambda_2 \Gamma(q)} \sum_{i=1}^{n} \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s} \right)^{p_i - 1} (s - \tau)^{q-1} \frac{f(r)}{s} drds.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\|h_n(t) - h_*(t)\| \\
&= \Bigg\| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}(f_n(s) - f_*(s))ds \\
&\quad - \frac{t^{q-1}}{\Lambda_2 \Gamma(q)} \sum_{i=1}^{n} \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-\tau)^{q-1} \frac{(f_n(\tau) - f_*(\tau))}{s} d\tau ds \Bigg\| \to 0,
\end{aligned}$$

as $n \to \infty$. Thus, it follows by Lemma 1.2 that $\Theta \circ S_{F,x}$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$\begin{aligned}
h_*(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_*(s)ds \\
&\quad - \frac{t^{q-1}}{\Lambda_2 \Gamma(q)} \sum_{i=1}^{n} \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-\tau)^{q-1} \frac{f_*(\tau)}{s} d\tau ds,
\end{aligned}$$

for some $f_* \in S_{F,x_*}$. Hence $\mathcal{N}_2$ has a closed graph (and therefore has closed values). As a result $\mathcal{N}_2$ is upper semicontinuous.

Therefore the operators $\mathcal{N}_1$ and $\mathcal{N}_2$ satisfy all the conditions of Theorem 1.17 and hence its application yields either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible. If $x \in \kappa \mathcal{N}_1(x) + \kappa \mathcal{N}_2(x)$ for $\kappa \in (0, 1)$, then there exists $f \in S_{F,x}$ such that

$$\begin{aligned}
x(t) = \kappa \Bigg\{ &\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds \\
&- \frac{t^{q-1}}{\Lambda_2 \Gamma(q)} \sum_{i=1}^{n} \frac{\alpha_i}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-\tau)^{q-1} \frac{f_*(\tau)}{s} d\tau ds \\
&+ \frac{t^{q-1}}{\Lambda_2} g(x) \Bigg\}, \quad t \in [0, T].
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
|x(t)| \le \psi(\|x\|)\|m\| \Bigg[&\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} ds \\
&+ \frac{T^{q-1}}{|\Lambda_2| \Gamma(q)} \sum_{i=1}^{n} \frac{|\alpha_i|}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-\tau)^{q-1} \frac{d\tau}{s} ds \Bigg] + \frac{T^{q-1}}{|\Lambda_2|} \ell \|x\|,
\end{aligned}$$

which, on taking supremum over $t \in [0, T]$, yields

$$\begin{aligned}\|x\| \leq \psi(\|x\|)\bigg\{ \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} m(s)ds \\ + \frac{T^{q-1}}{|\Lambda_2|\Gamma(q)} \sum_{i=1}^{n} \frac{|\alpha_i|}{\Gamma(p_i)} \int_0^{\eta_i} \int_0^s \left(\log \frac{\eta_i}{s}\right)^{p_i-1} (s-r)^{q-1} \frac{m(\tau)}{s} d\tau ds \bigg\} + \frac{T^{q-1}}{|\Lambda_2|} \ell \|x\| \\ \leq \psi(\|x\|)\|m\| p_0 + k_0 \ell \|x\|.\end{aligned}$$

If condition (ii) of Theorem 1.17 holds, then there exists $\kappa \in (0, 1)$ and $x \in \partial B_M$, with $x = \kappa \mathcal{N}(x)$. Then, x is a solution of (5.36) with $\|x\| = M$. Now, the last inequality implies that

$$\frac{(1 - k_0\ell)M}{p_0\|m\|\psi(M)} \leq 1,$$

which contradicts (5.42). Hence $\mathcal{N}$ has a fixed point in $[0, T]$ by Theorem 1.17, and consequently the problem (5.28)–(5.29) has a solution. This completes the proof. □

Example 5.11 Consider the following fractional boundary value problem

$$\begin{cases} {}_{RL}D^{7/4}x(t) \in F(t, x), \quad t \in (0, 1/2), \\ x(0) = 0, \quad \dfrac{|x(1/6)|}{4(1 + |x(1/6)|)} = \dfrac{3}{5} {}_HI^{\sqrt{2}} x\left(\dfrac{1}{3}\right) + \dfrac{2}{7} {}_HI^{\sqrt{3}} x\left(\dfrac{1}{4}\right) \\ \qquad + {}_HI^{\sqrt{5}} x\left(\dfrac{1}{5}\right), \end{cases} \tag{5.45}$$

where $F : [0, 1/2] \times \mathbb{R}^2 \to \mathscr{P}(\mathbb{R})$ is a multivalued map given by

$$x \to F(t, x) = \left[\frac{t|x|(1 + \cos^2 2x)}{5(1 + |x|)}, \frac{(1 + t)(1 + |x|)e^{-2x^2}}{6(1 + \sin^2 4x)} \right].$$

Here $q = 7/2$, $T = 1/2$, $n = 3$, $\alpha_1 = 3/5$, $\alpha_2 = 2/7$, $\alpha_3 = 1$, $\rho_1 = \sqrt{2}$, $\rho_2 = \sqrt{3}$, $\rho_3 = \sqrt{5}$, $\eta_1 = 1/3$, $\eta_2 = 1/4$, $\eta_3 = 1/5$, $g(x) = (1/4)(|x|/(1 + |x|))$. Using computer program, we find that $\Lambda_2 \approx 1.13066832 \neq 0$, $p_0 \approx 0.20657749$, $k_0 \approx 0.52588681$.

As $|g(x) - g(y)| \leq (1/4)|x - y|$, therefore, (5.10.2) is satisfied with $\ell = (1/4) < (1/0.52588681) = k_0^{-1}$. For $f \in F$ and $x, y \in \mathbb{R}$, we have

$$|f| \leq \max\left(\frac{t|x|(1 + \cos^2 2x)}{5(1 + |x|)}, \frac{(1 + t)(1 + |x|)e^{-2x^2}}{6(1 + \sin^2 4x)} \right) \leq \frac{t + 1}{6}(1 + |x|), \quad x \in \mathbb{R}.$$

Thus,

$$\|F(t, x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t, x)\} \leq m(t)\psi(\|x\|), \quad x \in \mathbb{R},$$

with $m(t) = (t + 1)/6$, $\psi(\|x\|) = 1 + \|x\|$. By computing directly, we find that there exists a constant $M > 0.06322119$ such that (5.12.3) holds. Clearly, all the conditions of Theorem 5.12 are satisfied. Hence the problem (5.45) has at least one solution on $[0, 1/2]$.

5.5 Riemann-Liouville Fractional Differential Equations with Multiple Hadamard Fractional Integral Conditions

In this section, we study the following boundary value problem of Riemann-Liouville fractional differential equations supplemented with multiple Hadamard fractional integral conditions:

$$
{}_{RL}D^{\alpha}x(t) = f(t, x(t)), \quad 1 < \alpha \leq 2, \ t \in (0, T), \tag{5.46}
$$

$$
x(0) = 0, \qquad \sum_{i=1}^{m} \mu_{i\,H}I^{\beta_i}x(\eta_i) = \sum_{j=1}^{n} \delta_{j\,H}I^{\gamma_j}x(\xi_j) + \lambda, \tag{5.47}
$$

where ${}_{RL}D^{\alpha}$ denotes the Riemann-Liouville fractional derivative of order α, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, η_i, $\xi_j \in (0, T)$, λ, μ_i, $\delta_j \in \mathbb{R}$, for all $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, and ${}_{H}I^{\psi}$ is the Hadamard fractional integral of order $\psi > 0$ ($\psi = \beta_i, \gamma_j, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$).

First of all, we consider the following lemma, which deals with the linear variant of the problem (5.46)–(5.47) and plays a pivotal role in developing the existence theory for the problem at hand.

Lemma 5.3 *Let $1 < \alpha \leq 2$, β_i, $\gamma_j > 0$, η_i, $\xi_j \in (0, T)$, $\lambda, \mu_i, \delta_j \in \mathbb{R}$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ and $h \in \mathscr{E}_0$. Then the solution of the following Riemann-Liouville fractional differential equation*

$$
{}_{RL}D^{\alpha}x(t) = h(t), \quad t \in (0, T), \tag{5.48}
$$

subject to the multiple Hadamard fractional integral conditions

$$
x(0) = 0, \qquad \sum_{i=1}^{m} \mu_{i\,H}I^{\beta_i}x(\eta_i) = \sum_{j=1}^{n} \delta_{j\,H}I^{\gamma_j}x(\xi_j) + \lambda, \tag{5.49}
$$

is equivalent to the following integral equation

$$
\begin{aligned}
x(t) = {} & \frac{t^{\alpha-1}}{\Lambda_3}\left(\sum_{j=1}^{n} \delta_{j\,H}I^{\gamma_j}{}_{RL}I^{\alpha}h(\xi_j) - \sum_{i=1}^{m} \mu_{i\,H}I^{\beta_i}{}_{RL}I^{\alpha}h(\eta_i) + \lambda\right) \\
& + {}_{RL}I^{\alpha}h(t),
\end{aligned} \tag{5.50}
$$

where

$$\Lambda_3 = \sum_{i=1}^{m} \mu_i (\alpha-1)^{-\beta_i} \eta_i^{\alpha-1} - \sum_{j=1}^{n} \delta_j (\alpha-1)^{-\gamma_j} \xi_j^{\alpha-1} \neq 0. \tag{5.51}$$

Proof Applying the Riemann-Liouville fractional integral of order α to both sides of (5.48), we get

$$x(t) = k_1 t^{\alpha-1} + k_2 t^{\alpha-2} + {}_{RL}I^{\alpha} h(t). \tag{5.52}$$

where $k_1,\ k_2 \in \mathbb{R}$.

Using first condition of (5.49) in (5.52), we find that $k_2 = 0$. In consequence, (5.52) reduces to

$$x(t) = k_1 t^{\alpha-1} + {}_{RL}I^{\alpha} h(t). \tag{5.53}$$

For any $p > 0$, by Lemma 1.6, it follows that

$${}_{H}I^{p} x(t) = k_1 (\alpha-1)^{-p} t^{\alpha-1} + {}_{H}I^{p}{}_{RL}I^{\alpha} h(t). \tag{5.54}$$

The second condition of (5.49) together with (5.54) yields

$$k_1 = \frac{1}{\Lambda_3} \left(\sum_{j=1}^{n} \delta_j {}_{H}I^{\gamma_j}{}_{RL}I^{\alpha} h(\xi_j) - \sum_{i=1}^{m} \mu_i {}_{H}I^{\beta_i}{}_{RL}I^{\alpha} h(\eta_i) + \lambda \right), \tag{5.55}$$

where Λ_3 is defined by (5.51). Substituting the value of k_1 into (5.53), we obtain (5.50). The converse follows by direct computation. The proof is completed. □

Throughout this section, for convenience, we use the following notations:

$${}_{RL}I^{\alpha} f(s, x(s))(z) = \frac{1}{\Gamma(\alpha)} \int_0^z (z-s)^{\alpha-1} f(s, x(s)) ds \quad \text{for } z \in [0, T],$$

and

$${}_{H}I^{u}{}_{RL}I^{\alpha} f(s, x(s))(v) = \frac{1}{\Gamma(u)\Gamma(\alpha)} \int_0^v \int_0^t \left(\log \frac{v}{t}\right)^{u-1} (t-s)^{\alpha-1} \frac{f(s, x(s))}{t} ds dt,$$

for $v \in (0, T]$, where $z \in \{t, T, \nu_1, \nu_2\}$, $u \in \{\beta_i, \gamma_j\}$ and $v = \{\eta_i, \xi_j\}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

By Lemma 5.3, we define an operator $\mathscr{F} : \mathscr{E}_0 \to \mathscr{E}_0$ associated with the problem (5.46)–(5.47) by

$$(\mathscr{F}x)(t) = \frac{t^{\alpha-1}}{\Lambda_3}\left(\sum_{j=1}^{n}\delta_{jH}I^{\gamma_j}{}_{RL}I^{\alpha}f(s,x(s))(\xi_j) - \sum_{i=1}^{m}\mu_{iH}I^{\beta_i}{}_{RL}I^{\alpha}f(s,x(s))(\eta_i) + \lambda\right) + {}_{RL}I^{\alpha}f(s,x(s))(t), \quad (5.56)$$

with $\Lambda_3 \neq 0$. It should be noticed that problem (5.46)–(5.47) has solutions if and only if the operator $\mathscr{F}$ has fixed points.

For the sake of convenience, we put

$$\Phi_2 = \frac{1}{\Gamma(\alpha+1)}\left(T^{\alpha} + \frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{i=1}^{m}|\mu_i|\alpha^{-\beta_i}\eta_i^{\alpha} + \frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{j=1}^{n}|\delta_j|\alpha^{-\gamma_j}\xi_j^{\alpha}\right), \quad (5.57)$$

$$\Omega_2 = \frac{T^{\alpha-1}|\lambda|}{|\Lambda_3|}. \quad (5.58)$$

In our first result, we prove the existence and uniqueness of solutions for the problem (5.46)–(5.47).

Theorem 5.13 *Let $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption (5.1.1). If*

$$L\Phi_2 < 1, \quad (5.59)$$

where Φ_2 is given by (5.57), then the problem (5.46)–(5.47) has a unique solution on $[0,T]$.

Proof We transform the problem (5.46)–(5.47) into a fixed point problem, $x = \mathscr{F}x$, where the operator $\mathscr{F}$ is defined by (5.56). By using the Banach's contraction mapping principle, we shall show that $\mathscr{F}$ has a fixed point, which is the unique solution of problem (5.46)–(5.47).

Let us define $\sup_{t\in[0,T]}|f(t,0)| = M < \infty$ and choose $r \geq \dfrac{M\Phi_2+\Omega_2}{1-L\Phi_2}$. Then, we show that $\mathscr{F}B_r \subset B_r$, where $B_r = \{x \in \mathscr{E}_0 : \|x\| \leq r\}$. For any $x \in B_r$, and taking into account Lemma 1.6, we get

$$\begin{aligned}\|\mathscr{F}x\| &\leq \sup_{t\in[0,T]}\Bigg\{{}_{RL}I^{\alpha}|f(s,x(s))|(t) + \frac{t^{\alpha-1}}{|\Lambda_3|}\Bigg(\sum_{j=1}^{n}|\delta_j|_H I^{\gamma_j}{}_{RL}I^{\alpha}|f(s,x(s))|(\xi_j)\\ &\quad + \sum_{i=1}^{m}|\mu_i|_H I^{\beta_i}{}_{RL}I^{\alpha}|f(s,x(s))|(\eta_i) + |\lambda|\Bigg)\Bigg\}\\ &\leq {}_{RL}I^{\alpha}\big(|f(s,x(s)) - f(s,0)| + |f(s,0)|\big)(T)\\ &\quad + \frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{i=1}^{m}|\mu_i|_H I^{\beta_i}{}_{RL}I^{\alpha}\big(|f(s,x(s)) - f(s,0)| + |f(s,0)|\big)(\eta_i)\end{aligned}$$

$$+\frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{j=1}^{n}|\delta_j|_H I^{\gamma_j}{}_{RL}I^{\alpha}\big(|f(s,x(s))-f(s,0)|+|f(s,0)|\big)(\xi_j)+\frac{T^{\alpha-1}|\lambda|}{|\Lambda_3|}$$

$$\leq (Lr+M)\left\{\frac{1}{\Gamma(\alpha+1)}\left(T^{\alpha}+\frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{i=1}^{m}|\mu_i|\alpha^{-\beta_i}\eta_i^{\alpha}+\frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{j=1}^{n}|\delta_j|\alpha^{-\gamma_j}\xi_j^{\alpha}\right)\right\}$$

$$+\frac{T^{\alpha-1}|\lambda|}{|\Lambda_3|}$$

$$=(Lr+M)\Phi_2+\Omega_2\leq r,$$

which implies that $\mathscr{F}B_r \subset B_r$.

For $x, y \in \mathscr{E}_0$ and for each $t \in [0, T]$, we have

$$|\mathscr{F}x(t)-\mathscr{F}y(t)|$$

$$\leq {}_{RL}I^{\alpha}(|f(s,x(s))-f(s,y(s))|)(t)$$

$$+\frac{t^{\alpha-1}}{|\Lambda_3|}\sum_{i=1}^{m}|\mu_i|_H I^{\beta_i}{}_{RL}I^{\alpha}(|f(s,x(s))-f(s,y(s))|)(\eta_i)$$

$$+\frac{t^{\alpha-1}}{|\Lambda_3|}\sum_{j=1}^{n}|\delta_j|_H I^{\gamma_j}{}_{RL}I^{\alpha}(|f(s,x(s))-f(s,y(s))|)(\xi_j)$$

$$\leq L\|x-y\|\left\{\frac{1}{\Gamma(\alpha+1)}\left(T^{\alpha}+\frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{i=1}^{m}|\mu_i|\alpha^{-\beta_i}\eta_i^{\alpha}+\frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{j=1}^{n}|\delta_j|\alpha^{-\gamma_j}\xi_j^{\alpha}\right)\right\}$$

$$=L\Phi_2\|x-y\|,$$

which consequently implies that $\|\mathscr{F}x-\mathscr{F}y\| \leq L\Phi_2\|x-y\|$. As $L\Phi_2 < 1$, $\mathscr{F}$ is a contraction. Hence, by the Banach's contraction mapping principle, we deduce that the operator $\mathscr{F}$ has a fixed point which corresponds to the unique solution of the problem (5.46)–(5.47). This completes the proof. □

Example 5.12 Consider the following nonlocal boundary value problem of Riemann-Liouville fractional differential equation with Hadamard fractional integral boundary conditions

$$\begin{cases} {}_{RL}D^{4/3}x(t)=\dfrac{\cos^2(2\pi t)}{(t^2+3)^2}\dfrac{|x(t)|}{|x(t)|+1}+\dfrac{5}{4}, \quad 0<t<2, \\ x(0)=0, \\ 2{}_HI^{1/2}x(1)+\dfrac{3}{2}{}_HI^{5/4}x\left(\dfrac{3}{4}\right)-\pi{}_HI^{\frac{5}{3}}x\left(\sqrt{2}\right) \\ =\sqrt{3}{}_HI^{3/2}x\left(\dfrac{\pi}{2}\right)-e^2{}_HI^{1/3}x\left(\dfrac{3}{2}\right)+\sqrt{\pi}{}_HI^{7/4}x\left(\dfrac{1}{3}\right)+5. \end{cases} \tag{5.60}$$

Here $\alpha = 4/3$, $m = 3$, $n = 3$, $\lambda = 5$, $T = 2$, $\mu_1 = 2$, $\mu_2 = 3/2$, $\mu_3 = -\pi$, $\beta_1 = 1/2$, $\beta_2 = 5/4$, $\beta_3 = 5/3$, $\eta_1 = 1$, $\eta_2 = 3/4$, $\eta_3 = \sqrt{2}$, $\delta_1 = \sqrt{3}$, $\delta_2 = -e^2$, $\delta_3 = \sqrt{\pi}$, $\gamma_1 = 3/2$, $\gamma_2 = 1/3$, $\gamma_3 = 7/4$, $\xi_1 = \pi/2$, $\xi_2 = 3/2$, $\xi_3 = 1/3$ and $f(t,x) = (\cos^2(2\pi t)|x|)/((t^2+3)^2(|x|+1)) + 5/4$. Since $|f(t,x) - f(t,y)| \le (1/9)|x-y|$, (5.13.1) is satisfied with $L = 1/9$. Further, we find that $\Lambda_3 \approx -19.82738586$, $\Phi_2 \approx 3.14973662$ and $L\Phi_2 \approx 0.3499707356 < 1$. Hence, by Theorem 5.13, the problem (5.60) has a unique solution on $[0,2]$.

Next, we prove the second existence and uniqueness result by means of nonlinear contractions.

Theorem 5.14 *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption:*

(5.14.1) $|f(t,x) - f(t,y)| \le g(t)\dfrac{|x-y|}{G^* + |x-y|}$, $t \in [0,T]$, $x, y \ge 0$, *where $g : [0,T] \to \mathbb{R}^+$ is continuous and the constant G^* is defined by*

$$G^* = {}_{RL}I^{\alpha} g(T) + \frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{i=1}^{m} |\mu_i| {}_H I^{\beta_i} {}_{RL}I^{\alpha} g(\eta_i) + \frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{j=1}^{n} |\delta_j| {}_H I^{\gamma_j} {}_{RL}I^{\alpha} g(\xi_j). \tag{5.61}$$

Then the problem (5.46)–(5.47) has a unique solution on $[0,T]$.

Proof We consider the operator $\mathscr{F} : \mathscr{E}_0 \to \mathscr{E}_0$ given by (5.56) and a continuous nondecreasing function $\Psi : \mathbb{R}_0^+ \to \mathbb{R}^+$ defined by $\Psi(\theta) = \dfrac{G^*\theta}{G^* + \theta}$, $\forall \theta \ge 0$. Note that the function Ψ satisfies $\Psi(0) = 0$ and $\Psi(\theta) < \theta$ for all $\theta > 0$.

For any $x, y \in \mathscr{E}_0$ and for each $t \in [0,T]$, we have

$$\begin{aligned}
|\mathscr{F}x(t) - \mathscr{F}y(t)| &\le {}_{RL}I^{\alpha}(|f(s,x(s)) - f(s,y(s))|)(t) \\
&\quad + \frac{t^{\alpha-1}}{|\Lambda_3|}\sum_{i=1}^{m} |\mu_i| {}_H I^{\beta_i} {}_{RL}I^{\alpha}(|f(s,x(s)) - f(s,y(s))|)(\eta_i) \\
&\quad + \frac{t^{\alpha-1}}{|\Lambda_3|}\sum_{j=1}^{n} |\delta_j| {}_H I^{\gamma_j} {}_{RL}I^{\alpha}(|f(s,x(s)) - f(s,y(s))|)(\xi_j) \\
&\le {}_{RL}I^{\alpha}\left(g(s)\frac{|x(s)-y(s)|}{G^* + |x(s)-y(s)|}\right)(T) \\
&\quad + \frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{i=1}^{m} |\mu_i| {}_H I^{\beta_i} {}_{RL}I^{\alpha}\left(g(s)\frac{|x(s)-y(s)|}{G^* + |x(s)-y(s)|}\right)(\eta_i) \\
&\quad + \frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{j=1}^{n} |\delta_j| {}_H I^{\gamma_j} {}_{RL}I^{\alpha}\left(g(s)\frac{|x(s)-y(s)|}{G^* + |x(s)-y(s)|}\right)(\xi_j)
\end{aligned}$$

$$\leq \frac{\Psi(\|x-y\|)}{G^*}\Bigg({}_{RL}I^{\alpha}g(T) + \frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{i=1}^{m}|\mu_i|_H I^{\beta_i}{}_{RL}I^{\alpha}g(\eta_i)$$
$$+ \frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{j=1}^{n}|\delta_j|_H I^{\gamma_j}{}_{RL}I^{\alpha}g(\xi_j)\Bigg)$$
$$= \Psi(\|x-y\|).$$

This implies that $\|\mathscr{F}x-\mathscr{F}y\| \leq \Psi(\|x-y\|)$. Therefore, $\mathscr{F}$ is a nonlinear contraction. Hence, by Theorem 1.11, the operator $\mathscr{F}$ has a fixed point which is the unique solution of the problem (5.46)–(5.47). This completes the proof. □

Example 5.13 Consider the following nonlocal boundary value problem for Riemann-Liouville fractional differential equation with Hadamard fractional integral boundary conditions

$$\begin{cases} {}_{RL}D^{3/2}x(t) = \dfrac{t}{(t+4)^2}\dfrac{|x(t)|}{|x(t)|+1} + \dfrac{t^2}{2} + \dfrac{3}{4}, \quad 0 < t < 3, \\ x(0) = 0, \\ \dfrac{1}{2}{}_H I^{1/3}x\left(\dfrac{5}{2}\right) + 3{}_H I^{7/4}x\left(\dfrac{9}{4}\right) - \dfrac{3}{4}{}_H I^{1/2}x\left(\dfrac{1}{3}\right) - \pi^2{}_H I^{4/3}x\left(\sqrt{3}\right) \\ = \sqrt{5}{}_H I^{11/6}x\left(\sqrt{7}\right) - \dfrac{5}{2}{}_H I^{7/5}x\left(\dfrac{1}{4}\right) + 2{}_H I^{2/3}x(e) - 4. \end{cases} \tag{5.62}$$

Here $\alpha = 3/2$, $m = 4$, $n = 3$, $\lambda = -4$, $T = 2$, $\mu_1 = 1/2$, $\mu_2 = 3$, $\mu_3 = -3/4$, $\mu_4 = -\pi^2$, $\beta_1 = 1/3$, $\beta_2 = 7/4$, $\beta_3 = 1/2$, $\beta_4 = 4/3$, $\eta_1 = 5/2$, $\eta_2 = 9/4$, $\eta_3 = 1/3$, $\eta_4 = \sqrt{3}$, $\delta_1 = \sqrt{5}$, $\delta_2 = -5/2$, $\delta_3 = 2$, $\gamma_1 = 11/6$, $\gamma_2 = 7/5$, $\gamma_3 = 2/3$, $\xi_1 = \sqrt{7}$, $\xi_2 = 1/4$, $\xi_3 = e$ and $f(t,x) = (t|x|)/((t+4)^2(|x|+1)) + t^2/2 + 3/4$. We choose $g(t) = t/16$ and find $\Lambda \approx -32.10761052$ and $G^* \approx 0.3314426952$. Clearly,

$$|f(t,x) - f(t,y)| = \frac{t}{(t+4)^2}\left(\frac{|x|-|y|}{1+|x|+|y|+|x||y|}\right)$$
$$\leq \frac{t}{16}\left(\frac{|x-y|}{0.3314426952+|x-y|}\right).$$

Hence, by Theorem 5.14, the problem (5.62) has a unique solution on $[0, 3]$.

Next, we present an existence result by means of Krasnoselskii's fixed point theorem.

Theorem 5.15 *Assume that* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function satisfying the assumptions (5.1.1) and (5.4.1). If*

$$\frac{LT^{\alpha}}{\Gamma(\alpha+1)} < 1, \tag{5.63}$$

then the problem (5.46)–(5.47) has at least one solution on $[0, T]$.

Proof We define $\sup_{t\in[0,T]} |\varphi(t)| = \|\varphi\|$ and choose a suitable constant $\bar{r}$ such that $\bar{r} \geq \|\varphi\|\Phi_2 + \Omega_2$, where Φ_2 and Ω_2 are defined by (5.57) and (5.58), respectively. Furthermore, we define the operators $\mathscr{P}$ and $\mathscr{Q}$ on $B_{\bar{r}} = \{x \in \mathscr{E}_0 : \|x\| \leq \bar{r}\}$ as

$$
\begin{aligned}
(\mathscr{P}x)(t) &= \frac{t^{\alpha-1}}{\Lambda_3}\left(\sum_{j=1}^{n} \delta_{jH}I^{\gamma_j}{}_{RL}I^{\alpha} f(s,x(s))(\xi_j)\right. \\
&\qquad \left. - \sum_{i=1}^{m} \mu_{iH}I^{\beta_i}{}_{RL}I^{\alpha} f(s,x(s))(\eta_i) + \lambda\right), \quad t \in [0,T], \\
(\mathscr{Q}x)(t) &= {}_{RL}I^{\alpha} f(s,x(s))(t), \quad t \in [0,T].
\end{aligned}
$$

For $x, y \in B_{\bar{r}}$, we have

$$
\begin{aligned}
&\|\mathscr{P}x + \mathscr{Q}y\| \\
&\leq \|\varphi(t)\|\left\{\frac{1}{\Gamma(\alpha+1)}\left(T^{\alpha} + \frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{i=1}^{m}|\mu_i|\alpha^{-\beta_i}\eta_i^{\alpha} + \frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{j=1}^{n}|\delta_j|\alpha^{-\gamma_j}\xi_j^{\alpha}\right)\right\} \\
&\quad + \frac{T^{\alpha-1}|\lambda|}{|\Lambda_3|} \\
&= \|\varphi\|\Phi_2 + \Omega_2 \\
&\leq \bar{r}.
\end{aligned}
$$

This shows that $\mathscr{P}x + \mathscr{Q}y \in B_{\bar{r}}$. By using the assumption (5.13.1) together with (5.63), it is easy to show that $\mathscr{Q}$ is a contraction. Since the function f is continuous, we have that the operator $\mathscr{P}$ is continuous. Further, we have

$$
\|\mathscr{P}x\| \leq \|\varphi\|\left\{\frac{T^{\alpha-1}}{|\Lambda_3|\Gamma(\alpha+1)}\left(\sum_{i=1}^{m}|\mu_i|\alpha^{-\beta_i}\eta_i^{\alpha} + \sum_{j=1}^{n}|\delta_j|\alpha^{-\gamma_j}\xi_j^{\alpha}\right)\right\} + \frac{T^{\alpha-1}|\lambda|}{|\Lambda_3|}.
$$

Therefore, $\mathscr{P}$ is uniformly bounded on $B_{\bar{r}}$. Next, we prove the compactness of the operator $\mathscr{P}$. Let us set $\sup_{(t,x)\in[0,T]\times B_{\bar{r}}} |f(t,x)| = \bar{f} < \infty$. Consequently, we get

$$
\begin{aligned}
&|(\mathscr{P}x)(t_1) - (\mathscr{P}x)(t_2)| \\
&= \left|\frac{t_1^{\alpha-1}}{\Lambda_3}\left(\sum_{j=1}^{n} \delta_{jH}I^{\gamma_j}{}_{RL}I^{\alpha} f(s,x(s))(\xi_j) - \sum_{i=1}^{m} \mu_{iH}I^{\beta_i}{}_{RL}I^{\alpha} f(s,x(s))(\eta_i) + \lambda\right)\right. \\
&\quad \left. - \frac{t_2^{\alpha-1}}{\Lambda_3}\left(\sum_{j=1}^{n} \delta_{jH}I^{\gamma_j}{}_{RL}I^{\alpha} f(s,x(s))(\xi_j) - \sum_{i=1}^{m} \mu_{iH}I^{\beta_i}{}_{RL}I^{\alpha} f(s,x(s))(\eta_i) + \lambda\right)\right| \\
&\leq \frac{\left|t_1^{\alpha-1} - t_2^{\alpha-1}\right|}{|\Lambda_3|}\left(\frac{\bar{f}}{\Gamma(\alpha+1)}\sum_{i=1}^{m}|\mu_i|\alpha^{-\beta_i}\eta_i^{\alpha} + \frac{\bar{f}}{\Gamma(\alpha+1)}\sum_{j=1}^{n}|\delta_j|\alpha^{-\gamma_j}\xi_j^{\alpha} + |\lambda|\right),
\end{aligned}
$$

which is independent of x, and tends to zero as $t_2 \to t_1$ $(0 < t_2 < t_1 < T)$. Thus, $\mathscr{P}$ is equicontinuous. So $\mathscr{P}$ is relatively compact on $B_{\bar{r}}$. Hence, by the Arzelá-Ascoli Theorem, $\mathscr{P}$ is compact on $B_{\bar{r}}$. Thus all the assumptions of Theorem 1.2 are satisfied. So the problem (5.46)–(5.47) has at least one solution on $[0, T]$. The proof is completed. □

Remark 5.1 In the above theorem, we can interchange the roles of the operators $\mathscr{P}$ and $\mathscr{Q}$ to obtain a second result replacing (5.63) by the following condition:

$$\frac{LT^{\alpha-1}}{|\Lambda_3|\Gamma(\alpha+1)}\left(\sum_{i=1}^{m}|\mu_i|\alpha^{-\beta_i}\eta_i^{\alpha}+\sum_{j=1}^{n}|\delta_j|\alpha^{-\gamma_j}\xi_j^{\alpha}\right)<1.$$

Our next existence result relies on Leray-Schauder's nonlinear alternative.

Theorem 5.16 *Assume that* $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function satisfying the assumption (5.5.1). In addition, we suppose that:*

(5.16.1) there exists a constant $N > 0$ *such that*

$$\frac{N}{\|p\|\psi(N)\Phi_2+\Omega_2}>1,$$

where Φ_2 *and* Ω_2 *are defined by (5.57) and (5.58), respectively.*

Then the problem (5.46)–(5.47) has at least one solution on $[0, T]$.

Proof Firstly, we shall show that the operator $\mathscr{F}$, defined by (5.56), *maps bounded sets (balls) into bounded sets in* $\mathscr{E}_0$. For a positive number R, let $B_R = \{x \in \mathscr{E}_0 : \|x\| \le R\}$ be a bounded ball in $\mathscr{E}_0$. Then, for $t \in [0, T]$, we have

$$\begin{aligned}
&|\mathscr{F}x(t)|\\
&\le {}_{RL}I^{\alpha}|f(s,x(s))|(T)+\frac{T^{\alpha-1}}{|\Lambda_3|}\left(\sum_{j=1}^{n}|\delta_j|{}_HI^{\gamma_j}{}_{RL}I^{\alpha}|f(s,x(s))|(\xi_j)\right.\\
&\quad\left.+\sum_{i=1}^{m}|\mu_i|{}_HI^{\beta_i}{}_{RL}I^{\alpha}|f(s,x(s))|(\eta_i)+|\lambda|\right)\\
&\le \|p\|\psi(\|x\|)\left\{\frac{1}{\Gamma(\alpha+1)}\left(T^{\alpha}+\frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{i=1}^{m}|\mu_i|\alpha^{-\beta_i}\eta_i^{\alpha}+\frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{j=1}^{n}|\delta_j|\alpha^{-\gamma_j}\xi_j^{\alpha}\right)\right\}\\
&\quad+\frac{T^{\alpha-1}|\lambda|}{|\Lambda_3|}\\
&\le \|p\|\psi(R)\left\{\frac{1}{\Gamma(\alpha+1)}\left(T^{\alpha}+\frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{i=1}^{m}|\mu_i|\alpha^{-\beta_i}\eta_i^{\alpha}+\frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{j=1}^{n}|\delta_j|\alpha^{-\gamma_j}\xi_j^{\alpha}\right)\right\}\\
&\quad+\frac{T^{\alpha-1}|\lambda|}{|\Lambda_3|}\\
&:=\hat{K}.
\end{aligned}$$

Therefore, we conclude that $\|\mathscr{F}x\| \le \hat{K}$.

Secondly, we show that $\mathscr{F}$ *maps bounded sets into equicontinuous sets of* $\mathscr{E}_0$. Let $\sup_{(t,x)\in[0,T]\times B_R}|f(t,x)| = f^* < \infty$, $\nu_1, \nu_2 \in [0, T]$ with $\nu_1 < \nu_2$ and $x \in B_R$. Then, we have

$$\begin{aligned}
&|(\mathscr{F}x)(\nu_2) - (\mathscr{F}x)(\nu_1)| \\
= &\Bigg| {}_{RL}I^\alpha f(s,x(s))(\nu_2) - {}_{RL}I^\alpha f(s,x(s))(\nu_1) \\
&+ \frac{\nu_2^{\alpha-1}}{\Lambda_3}\left(\sum_{j=1}^n \delta_{jH}I^{\gamma_j}{}_{RL}I^\alpha f(s,x(s))(\xi_j) - \sum_{i=1}^m \mu_{iH}I^{\beta_i}{}_{RL}I^\alpha f(s,x(s))(\eta_i) + \lambda\right) \\
&- \frac{\nu_1^{\alpha-1}}{\Lambda_3}\left(\sum_{j=1}^n \delta_{jH}I^{\gamma_j}{}_{RL}I^\alpha f(s,x(s))(\xi_j) - \sum_{i=1}^m \mu_{iH}I^{\beta_i}{}_{RL}I^\alpha f(s,x(s))(\eta_i) + \lambda\right)\Bigg| \\
\le &\frac{f^*}{\Gamma(\alpha+1)}[2(\nu_2-\nu_1)^\alpha + |\nu_2^\alpha - \nu_1^\alpha|] \\
&+ \frac{|\nu_2^{\alpha-1} - \nu_1^{\alpha-1}|}{|\Lambda_3|}\left(\frac{f^*}{\Gamma(\alpha+1)}\sum_{i=1}^m |\mu_i|\alpha^{-\beta_i}\eta_i^\alpha + \frac{f^*}{\Gamma(\alpha+1)}\sum_{j=1}^n |\delta_j|\alpha^{-\gamma_j}\xi_j^\alpha + |\lambda|\right).
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_R$ as $\nu_2 \to \nu_1$. Therefore it follows by the Arzelá-Ascoli Theorem that $\mathscr{F} : \mathscr{E}_0 \to \mathscr{E}_0$ is completely continuous.

Finally, we show that there exists an open set $U \subset \mathscr{E}_0$ with $x \neq \theta\mathscr{F}x$ for $\theta \in (0, 1)$ and $x \in \partial U$. Let x be a solution. Then, for $t \in [0, T]$, and following the similar computations as in the first step, we obtain

$$\begin{aligned}
\|x\| \le &\|p\|\psi(\|x\|)\left\{\frac{1}{\Gamma(\alpha+1)}\left(T^\alpha + \frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{i=1}^m |\mu_i|\alpha^{-\beta_i}\eta_i^\alpha + \frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{j=1}^n |\delta_j|\alpha^{-\gamma_j}\xi_j^\alpha\right)\right\} \\
&+ \frac{T^{\alpha-1}|\lambda|}{|\Lambda_3|} \\
= &\|p\|\psi(\|x\|)\Phi_2 + \Omega_2.
\end{aligned}$$

Consequently, we have

$$\frac{\|x\|}{\|p\|\psi(\|x\|)\Phi_2 + \Omega_2} \le 1.$$

In view of (5.16.2), there exists N such that $\|x\| \neq N$. Let us set

$$U = \{x \in \mathscr{E}_0 : \|x\| < N\}. \tag{5.64}$$

Note that the operator $\mathscr{F} : \overline{U} \to \mathscr{E}_0$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \theta\mathscr{F}x$ for some $\theta \in (0, 1)$. Consequently, by nonlinear alternative of Leray-Schauder type (Theorem 1.4), we deduce that $\mathscr{F}$ has a fixed point in $\overline{U}$, which is a solution of the problem (5.46)–(5.47). This completes the proof. □

Example 5.14 Consider the following nonlocal boundary value problem

$$\begin{cases} {}_{RL}D^{11/6}x(t) = \dfrac{6\sin(x/6)}{4\pi + (2^x+1)^2} + \dfrac{3+\cos(t)}{8\pi+1}, \quad 0 < t < \pi, \\ x(0) = 0, \\ \dfrac{3}{2}{}_HI^{2/3}x(e) - 7{}_HI^{5/4}x(3) - \dfrac{2}{3}{}_HI^{9/2}x\left(\dfrac{2}{\pi}\right) \\ \qquad = e^2{}_HI^{9/4}x\left(\sqrt{8}\right) + 2{}_HI^{11/3}x(1) \\ \qquad\quad + \pi{}_HI^{1/3}x\left(\dfrac{\sqrt{7}}{2}\right) - \dfrac{7}{4}{}_HI^{9/7}x\left(\dfrac{4}{e}\right) + \dfrac{5}{3}. \end{cases} \tag{5.65}$$

Here $\alpha = 11/6, m = 3, n = 4, \lambda = 5/3, T = \pi, \mu_1 = 3/2, \mu_2 = -7, \mu_3 = -2/3$, $\beta_1 = 2/3$, $\beta_2 = 5/4$, $\beta_3 = 9/2$, $\eta_1 = e$, $\eta_2 = 3$, $\eta_3 = 2/\pi$, $\delta_1 = e^2$, $\delta_2 = 2$, $\delta_3 = \pi$, $\delta_4 = -7/4$, $\gamma_1 = 9/4$, $\gamma_2 = 11/3$, $\gamma_3 = 1/3$, $\gamma_4 = 9/7$, $\xi_1 = \sqrt{8}$, $\xi_2 = 1$, $\xi_3 = \sqrt{7}/2$, $\xi_4 = 4/e$ and $f(t,x) = (6\sin(x/6))/(4\pi + (2^x+1)^2) + (3 + \cos(t))/(8\pi+1)$. Clearly

$$|f(t,x)| = \left|\frac{6\sin(x/6)}{4\pi + (2^x+1)^2} + \frac{3+\cos(t)}{8\pi+1}\right| \le (3 + \cos(\pi t))\left(\frac{|x|+1}{8\pi}\right).$$

Choosing $p(t) = 3 + \cos(t)$ and $\psi(|x|) = (|x|+1)/(8\pi)$, we find that $\Lambda_3 \approx -50.6564991$, $\Phi_2 \approx 6.20634772$, $\Omega_2 \approx 0.08540929522$, and $N > 87.75640147$. Hence, by Theorem 5.16, the problem (5.65) has at least one solution on $[0, \pi]$.

Now, we apply the Leray-Schauder degree theory to obtain the final existence result.

Theorem 5.17 *Assume that* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function. Suppose that:*

(5.17.1) there exist constants $0 \le \kappa < \Phi_2^{-1}$, *and* $K > 0$ *such that*

$$|f(t,x) \le \kappa|x| + K \ \textit{for all} \ (t,x) \in [0,T] \times \mathbb{R},$$

where Φ_2 *is defined by (5.57).*

Then the problem (5.46)–(5.47) has at least one solution on $[0,T]$.

Proof Consider the fixed point problem

$$x = \mathscr{F}x, \tag{5.66}$$

where the operator $\mathscr{F}$ is given by (5.56). We will show that there exists a fixed point $x \in \mathscr{E}_0$ satisfying (5.66). It is sufficient to show that $\mathscr{F} : \overline{B}_\rho \to \mathscr{E}_0$ satisfies

$$x \ne \mu\mathscr{F}x, \quad \forall x \in \partial B_\rho, \quad \forall \mu \in [0,1], \tag{5.67}$$

where $B_\rho = \{x \in \mathscr{E}_0 : \|x\| < \rho, \rho > 0\}$. Let us define

$$H(\mu, x) = \mu \mathscr{F} x, \quad x \in \mathscr{E}_0, \quad \mu \in [0, 1].$$

It is easy to see that the operator $\mathscr{F}$ is continuous, uniformly bounded and equicontinuous. Then, by the Arzelá-Ascoli Theorem, a continuous map h_μ defined by $h_\mu(x) = x - H(\mu, x) = x - \mu\mathscr{F}x$ is completely continuous. If (5.67) is true, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, we have

$$\begin{aligned}\deg(h_\mu, B_\rho, 0) &= \deg(I - \mu\mathscr{F}, B_\rho, 0) = \deg(h_1, B_\rho, 0)\\ &= \deg(h_0, B_\rho, 0) = \deg(I, B_\rho, 0) = 1 \neq 0, \quad 0 \in B_\rho,\end{aligned}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, $h_1(x) = x - \mathscr{F}x = 0$ for at least one $x \in B_\rho$. In order to prove (5.67), we assume that $x = \mu\mathscr{F}x$ for some $\mu \in [0, 1]$. Then

$$\begin{aligned}
&|\mathscr{F}x(t)|\\
&\le \sup_{t\in[0,T]} \Bigg\{ {}_{RL}I^{\alpha}|f(s, x(s))|(t) + \frac{t^{\alpha-1}}{|\Lambda_3|}\Bigg(\sum_{j=1}^{n} |\delta_j| {}_H I^{\gamma_j} {}_{RL}I^{\alpha}|f(s, x(s))|(\xi_j)\\
&\quad + \sum_{i=1}^{m} |\mu_i| {}_H I^{\beta_i} {}_{RL}I^{\alpha}|f(s, x(s))|(\eta_i) + |\lambda| \Bigg)\Bigg\}\\
&\le (\kappa\|x\| + K)\Bigg\{ \frac{1}{\Gamma(\alpha+1)} \Bigg(T^{\alpha} + \frac{T^{\alpha-1}}{|\Lambda_3|} \sum_{i=1}^{m} |\mu_i| \alpha^{-\beta_i} \eta_i^{\alpha} + \frac{T^{\alpha-1}}{|\Lambda_3|} \sum_{j=1}^{n} |\delta_j| \alpha^{-\gamma_j} \xi_j^{\alpha} \Bigg)\Bigg\}\\
&\quad + \frac{T^{\alpha-1}|\lambda|}{|\Lambda_3|}\\
&= (\kappa\|x\| + K)\Phi_2 + \Omega_2,
\end{aligned}$$

which implies that

$$\|x\| \le \frac{K\Phi_2 + \Omega_2}{1 - \kappa\Phi_2},$$

where Ω_2 is defined by (5.58). If $\rho = \dfrac{K\Phi_2 + \Omega_2}{1 - \kappa\Phi_2} + 1$, the inequality (5.67) holds. This completes the proof. □

Example 5.15 Consider the following nonlocal boundary value problem

$$\begin{cases} {}_{RL}D^{9/5}x(t) = \dfrac{(t+2e^{-t})x}{3x^2+10(\pi^t+5)} + \dfrac{4}{13(e^{3t}+3)}, \quad 0<t<4, \\ x(0)=0, \\ 3{}_HI^{5/2}x\left(\dfrac{1}{2}\right) + 4{}_HI^{1/3}x(\pi) + e{}_HI^{5/4}x\left(\dfrac{10}{3}\right) + \dfrac{7}{3}{}_HI^{11/5}x\left(\dfrac{9}{4}\right) \\ \qquad = \dfrac{\pi}{2}{}_HI^{9/4}x\left(\dfrac{11}{6}\right) - \tan^2(4){}_HI^{10/3}x\left(\dfrac{12}{7}\right) \\ \qquad\qquad - \dfrac{3}{e}{}_HI^{2/5}x\left(\dfrac{13}{4}\right) - \sqrt{11}{}_HI^{3/7}x\left(\dfrac{9}{8}\right) + \dfrac{11}{9}. \end{cases} \tag{5.68}$$

Here $\alpha = 9/5$, $m = 4$, $n = 4$, $\lambda = 11/9$, $T = 4$, $\mu_1 = 3$, $\mu_2 = 4$, $\mu_3 = e$, $\mu_4 = 7/3$, $\beta_1 = 5/2$, $\beta_2 = 1/3$, $\beta_3 = 5/4$, $\beta_4 = 11/5$, $\eta_1 = 1/2$, $\eta_2 = \pi$, $\eta_3 = 10/3$, $\eta_4 = 9/4$, $\delta_1 = \pi/2$, $\delta_2 = -\tan^2(4)$, $\delta_3 = -3/e$, $\delta_4 = -\sqrt{11}$, $\gamma_1 = 9/4$, $\gamma_2 = 10/3$, $\gamma_3 = 2/5$, $\gamma_4 = 3/7$, $\xi_1 = 11/6$, $\xi_2 = 12/7$, $\xi_3 = 13/4$, $\xi_4 = 9/8$ and $f(t,x) = ((t+2e^{-t})x)/(3x^2+10(\pi^t+5)) + (4)/(13(e^{3t}+3))$. Clearly $|f(t,x)| \le \frac{1}{10}|x| + \frac{1}{13}$, so (5.17.1) is satisfied with $\kappa = 1/10$ and $K = 1/13$. Further, we compute $\Lambda_3 \approx 37.7176876$, $\Phi_2 \approx 9.743207923$ and $\kappa\Phi_2 \approx 0.9743207923 < 1$. Hence, by Theorem 5.17, the problem (5.68) has at least one solution on $[0,4]$.

5.6 Riemann-Liouville Fractional Differential Inclusions with Multiple Hadamard Fractional Integral Conditions

In this section, we discuss the existence of solutions for Riemann-Liouville fractional differential inclusions supplemented with multiple Hadamard fractional integral conditions:

$$\begin{cases} {}_{RL}D^{\alpha}x(t) \in F(t,x(t)), \quad 0<t<T, \quad 1<\alpha\le 2, \\ x(0)=0, \qquad \displaystyle\sum_{i=1}^{m} \mu_i {}_HI^{\beta_i}x(\eta_i) = \sum_{j=1}^{n} \delta_j {}_HI^{\gamma_j}x(\xi_j) + \lambda, \end{cases} \tag{5.69}$$

where $F : [0,T]\times\mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map, and $\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

5.6.1 The Carathéodory Case

In this subsection, we consider the case when F has convex values and is of Carathéodory type. An existence result for the given problem is obtained by applying the nonlinear alternative of Leray-Schauder type.

Theorem 5.18 *Assume that the assumptions (5.8.1) and (5.8.2) hold. In addition, we suppose that:*

(5.18.1) there exists a constant $M > 0$ such that

$$\frac{M}{\psi(M)\|p\|\Phi_2 + T^{\alpha-1}|\lambda|/|\Lambda_3|} > 1,$$

where Φ_2 is defined by (5.57).

Then the problem (5.69) has at least one solution on $[0, T]$.

Proof Define an operator $\mathscr{F} : \mathscr{E}_0 \to \mathscr{P}(\mathscr{E}_0)$ by

$$\mathscr{F}(x) = \left\{ \begin{array}{l} h \in \mathscr{E}_0 : \\ h(t) = \left\{ \begin{array}{l} {}_{RL}I^{\alpha} v(t) \\ + \dfrac{t^{\alpha-1}}{\Lambda_3}\left(\displaystyle\sum_{j=1}^{n} \delta_{jH} I^{\gamma_j}{}_{RL}I^{\alpha} v(\xi_j) \right. \\ \left. - \displaystyle\sum_{i=1}^{m} \mu_{iH} I^{\beta_i}{}_{RL}I^{\alpha} v(\eta_i) + \lambda \right) \end{array} \right. \end{array} \right\} \tag{5.70}$$

for $v \in S_{F,x}$. It is obvious that the fixed points of $\mathscr{F}$ are solutions of the problem (5.69).

We will show that $\mathscr{F}$ satisfies the assumptions of Leray-Schauder nonlinear alternative (Theorem 1.15). The proof consists of several steps.

Step 1. *$\mathscr{F}(x)$ is convex for each $x \in \mathscr{E}_0$.*

This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore, we omit the proof.

Step 2. *$\mathscr{F}$ maps bounded sets (balls) into bounded sets in $\mathscr{E}_0$.*

For a positive number ρ, let $B_\rho = \{x \in \mathscr{E}_0 : \|x\| \le \rho\}$ be a bounded ball in $\mathscr{E}_0$. Then, for each $h \in \mathscr{F}(x), x \in B_\rho$, there exists $v \in S_{F,x}$ such that

$$h(t) = {}_{RL}I^{\alpha} v(t) + \frac{t^{\alpha-1}}{\Lambda_3}\left(\sum_{j=1}^{n} \delta_{jH} I^{\gamma_j}{}_{RL}I^{\alpha} v(\xi_j) - \sum_{i=1}^{m} \mu_{iH} I^{\beta_i}{}_{RL}I^{\alpha} v(\eta_i) + \lambda \right).$$

Then, we have

$$\begin{aligned} |h(t)| &\le {}_{RL}I^{\alpha}|v(t)| + \frac{t^{\alpha-1}}{\Lambda_3}\left(\sum_{j=1}^{n} \delta_{jH} I^{\gamma_j}{}_{RL}I^{\alpha}|v(\xi_j)| - \sum_{i=1}^{m} \mu_{iH} I^{\beta_i}{}_{RL}I^{\alpha}|v(\eta_i)| + |\lambda| \right) \\ &\le \frac{\|p\|\psi(\|x\|)}{\Gamma(\alpha+1)}\left(T^{\alpha} + \frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{i=1}^{m}|\mu_i|\alpha^{-\beta_i}\eta_i^{\alpha} + \frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{j=1}^{n}|\delta_j|\alpha^{-\gamma_j}\xi_j^{\alpha} \right) \\ &\quad + T^{\alpha-1}|\lambda|/|\Lambda_3| \\ &= \Phi_2\|p\|\psi(\|x\|) + T^{\alpha-1}|\lambda|/|\Lambda_3|. \end{aligned}$$

Thus

$$\|h\| \leq \Phi_2\|p\|\psi(\rho) + T^{\alpha-1}|\lambda|/|\Lambda_3|.$$

Step 3. $\mathscr{F}$ *maps bounded sets into equicontinuous sets of* $\mathscr{E}_0$.

Let $\tau_1, \tau_2 \in [0, T]$ with $\tau_1 < \tau_2$ and $x \in B_\rho$. For each $h \in \mathscr{F}(x)$, we obtain

$$\begin{aligned}
&|h(\tau_2) - h(\tau_1)| \\
&\leq \frac{1}{\Gamma(q)}\left|\int_0^{\tau_1}[(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}]v(s)ds + \int_{\tau_1}^{\tau_2}(\tau_2 - s)^{\alpha-1}v(s)ds\right| \\
&\quad + \frac{(\tau_2^{\alpha-1} - \tau_1^{\alpha-1})}{|\lambda|}\left(\sum_{j=1}^{n}\delta_{jH}I^{\gamma_j}{}_{RL}I^{\alpha}|v(\xi_j)| - \sum_{i=1}^{m}\mu_{iH}I^{\beta_i}{}_{RL}I^{\alpha}|v(\eta_i)|\right) \\
&\leq \frac{\|p\|\psi(\rho)}{\Gamma(q+1)}[2(\tau_2 - \tau_1)^q + |\tau_2^q - \tau_1^q|] \\
&\quad + \frac{(\tau_2^{q-1} - \tau_1^{\alpha-1})\|p\|\psi(\rho)}{|\lambda|\Gamma(\alpha+1)}\left(\sum_{i=1}^{m}|\mu_i|\alpha^{-\beta_i}\eta_i^{\alpha} + \sum_{j=1}^{n}|\delta_j|\alpha^{-\gamma_j}\xi_j^{\alpha}\right).
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero, independently of $x \in B_\rho$ as $\tau_2 - \tau_1 \to 0$. As $\mathscr{F}$ satisfies the above three assumptions, it follows by the Arzelá-Ascoli Theorem that $\mathscr{F} : \mathscr{E}_0 \to \mathscr{P}(\mathscr{E}_0)$ is completely continuous.

Since $\mathscr{F}$ is completely continuous, in order to prove that it is u.s.c., it is enough to show that it has a closed graph.

Step 4. $\mathscr{F}$ *has a closed graph.*

Let $x_n \to x_*, h_n \in \mathscr{F}(x_n)$ and $h_n \to h_*$. Then, we need to show that $h_* \in \mathscr{F}(x_*)$. Associated with $h_n \in \mathscr{F}(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in [0, T]$,

$$h_n(t) = {}_{RL}I^{\alpha}v_n(t) + \frac{t^{\alpha-1}}{\Lambda_3}\left(\sum_{j=1}^{n}\delta_{jH}I^{\gamma_j}{}_{RL}I^{\alpha}v_n(\xi_j) - \sum_{i=1}^{m}\mu_{iH}I^{\beta_i}{}_{RL}I^{\alpha}v_n(\eta_i) + \lambda\right).$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [0, T]$,

$$h_*(t) = {}_{RL}I^{\alpha}v_*(t) + \frac{t^{\alpha-1}}{\Lambda_3}\left(\sum_{j=1}^{n}\delta_{jH}I^{\gamma_j}{}_{RL}I^{\alpha}v_*(\xi_j) - \sum_{i=1}^{m}\mu_{iH}I^{\beta_i}{}_{RL}I^{\alpha}v_*(\eta_i) + \lambda\right).$$

Let us consider the linear operator $\Theta : L^1([0, T], \mathbb{R}) \to \mathscr{E}_0$ given by

$$f \mapsto \Theta(v)(t) = {}_{RL}I^{\alpha}v(t) + \frac{t^{\alpha-1}}{\Lambda_3}\left(\sum_{j=1}^{n}\delta_{jH}I^{\gamma_j}{}_{RL}I^{\alpha}v(\xi_j) - \sum_{i=1}^{m}\mu_{iH}I^{\beta_i}{}_{RL}I^{\alpha}v(\eta_i) + \lambda\right).$$

Observe that

$$\|h_n(t) - h_*(t)\| = \Bigg\| {}_{RL}I^{\alpha}(v_n(t) - v_*(t)) + \frac{t^{\alpha-1}}{\Lambda_3}\Bigg(\sum_{j=1}^{n} \delta_{jH}I^{\gamma_j}{}_{RL}I^{\alpha}(v_n)\xi_j) - v_*(\xi_j)) - \sum_{i=1}^{m} \mu_{iH}I^{\beta_i}{}_{RL}I^{\alpha}(v_n(\eta_i) - v_*(\eta_i))\Bigg)\Bigg\| \to 0,$$

as $n \to \infty$.

Thus, it follows by Lemma 1.2 that $\Theta \circ S_{F,x}$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = {}_{RL}I^{\alpha}v_*(t) + \frac{t^{\alpha-1}}{\Lambda_3}\Bigg(\sum_{j=1}^{n} \delta_{jH}I^{\gamma_j}{}_{RL}I^{\alpha}v_*(\xi_j) - \sum_{i=1}^{m} \mu_{iH}I^{\beta_i}{}_{RL}I^{\alpha}v_*(\eta_i) + \lambda\Bigg),$$

for some $v_* \in S_{F,x_*}$.

Step 5. We show there exists an open set $U \subseteq \mathscr{E}_0$ with $x \notin \theta\mathscr{F}(x)$ for any $\theta \in (0,1)$ and all $x \in \partial U$.

Let $\theta \in (0,1)$ and $x \in \theta\mathscr{F}(x)$. Then there exists $v \in L^1([0,T],\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in [0,T]$, we have

$$x(t) = \theta\, {}_{RL}I^{\alpha}v(t) + \theta\frac{t^{\alpha-1}}{\Lambda_3}\Bigg(\sum_{j=1}^{n} \delta_{jH}I^{\gamma_j}{}_{RL}I^{\alpha}v(\xi_j) - \sum_{i=1}^{m} \mu_{iH}I^{\beta_i}{}_{RL}I^{\alpha}v(\eta_i) + \lambda\Bigg).$$

As before, we can obtain

$$\begin{aligned}\|x\| &\le \psi(\|x\|)\|p\|\left\{\frac{1}{\Gamma(\alpha+1)}\left(T^{\alpha} + \frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{i=1}^{m}|\mu_i|\alpha^{-\beta_i}\eta_i^{\alpha} + \frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{j=1}^{n}|\delta_j|\alpha^{-\gamma_j}\xi_j^{\alpha}\right)\right\} \\ &\quad + T^{\alpha-1}|\lambda|/|\Lambda_3| \\ &= \psi(\|x\|)\|p\|\Phi_2 + T^{\alpha-1}|\lambda|/|\Lambda_3|,\end{aligned}$$

which implies that

$$\frac{\|x\|}{\psi(\|x\|)\|p\|\Phi_2 + T^{\alpha-1}|\lambda|/|\Lambda_3|} \le 1.$$

In view of (5.18.3), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in \mathscr{E}_0 : \|x\| < M\}.$$

Note that the operator $\mathscr{F} : \overline{U} \to \mathscr{P}(\mathscr{E}_0)$ is a compact multivalued map, u.s.c. with convex closed values. From the choice of U, there is no $x \in \partial U$ such that $x \in \theta\mathscr{F}(x)$

for some $\theta \in (0, 1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 1.15), we deduce that $\mathscr{F}$ has a fixed point $x \in \overline{U}$, which is a solution of the problem (5.69). This completes the proof. □

5.6.2 The Lower Semicontinuous Case

Here we assume that the multivalued map F is not necessarily convex valued.

Theorem 5.19 *Assume that (5.8.2), (5.18.1) and (5.9.1) hold.*
Then the boundary value problem (5.69) has at least one solution on $[0, T]$.

Proof It follows from (5.8.2) and (5.9.1) that F is of l.s.c. type. Then from Lemma 1.3, there exists a continuous function $f : \mathscr{E}_0 \to L^1([0, T], \mathbb{R})$ such that $f(x) \in \mathscr{F}(x)$ for all $x \in \mathscr{E}_0$.

Consider the problem

$$\begin{cases} {}_{RL}D^q x(t) = f(x(t)), \quad 0 < t < T, \quad 1 < \alpha \leq 2, \\ x(0) = 0, \qquad \displaystyle\sum_{i=1}^{m} \mu_{iH} I^{\beta_i} x(\eta_i) = \sum_{j=1}^{n} \delta_{jH} I^{\gamma_j} x(\xi_j) + \lambda. \end{cases} \tag{5.71}$$

Observe that if $x \in \mathscr{C}^2([0, T], \mathbb{R})$ is a solution of (5.71), then x is a solution to the problem (5.69). In order to transform the problem (5.71) into a fixed point problem, we define the operator $\overline{\mathscr{F}}$ as

$$\overline{\mathscr{F}}x(t) = {}_{RL}I^{\alpha} f(x(t)) + \frac{t^{\alpha-1}}{\Lambda_3}\left(\sum_{j=1}^{n} \delta_{jH} I^{\gamma_j}{}_{RL}I^{\alpha} f(x(\xi_j)) - \sum_{i=1}^{m} \mu_{iH} I^{\beta_i}{}_{RL}I^{\alpha} f(x((\eta_i)) + \lambda\right).$$

It can easily be shown that $\overline{\mathscr{F}}$ is continuous and completely continuous. We omit the rest of the proof as it is similar to that of Theorem 5.18. This completes the proof. □

5.6.3 The Lipschitz Case

In this subsection, we prove the existence of solutions for the problem (5.69) with a not necessary nonconvex valued right hand side, by applying a fixed point theorem for multivalued maps due to Covitz and Nadler (Theorem 1.18).

Theorem 5.20 *Assume that the assumptions (5.7.1) and (5.7.2) are satisfied. Then the problem (5.69) has at least one solution on* $[0, T]$ *if* $\|m\|\Phi_2 < 1$.

Proof Consider the operator $\mathscr{F}$ defined by (5.70). Observe that the set $S_{F,x}$ is nonempty for each $x \in \mathscr{E}_0$ by the assumption (5.7.1), so F has a measurable selection (see [57, Theorem III.6]). Now, we show that the operator $\mathscr{F}$ satisfies the assumptions of Theorem 1.18. We show that $\mathscr{F}(x) \in \mathscr{P}_{cl}(\mathscr{E}_0)$ for each $x \in \mathscr{E}_0$. Let $\{u_n\}_{n\geq 0} \in \mathscr{F}(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $\mathscr{E}_0$. Then $u \in \mathscr{E}_0$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [0,T]$,

$$u_n(t) = {}_{RL}I^{\alpha} v_n(t) + \frac{t^{\alpha-1}}{\Lambda_3}\left(\sum_{j=1}^{n} \delta_{jH} I^{\gamma_j}{}_{RL}I^{\alpha} v_n(\xi_j) - \sum_{i=1}^{m} \mu_{iH} I^{\beta_i}{}_{RL}I^{\alpha} v_n(\eta_i) + \lambda\right).$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([0,T],\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0,T]$, we have

$$u_n(t) \to v(t) = {}_{RL}I^{\alpha} v(t) + \frac{t^{\alpha-1}}{\Lambda_3}\left(\sum_{j=1}^{n} \delta_{jH} I^{\gamma_j}{}_{RL}I^{\alpha} v(\xi_j) - \sum_{i=1}^{m} \mu_{iH} I^{\beta_i}{}_{RL}I^{\alpha} v(\eta_i) + \lambda\right).$$

Hence, $u \in \mathscr{F}(x)$.

Next, we show that there exists $\bar{\delta} < 1$ $(\bar{\delta} := \|m\|\Phi_2)$, such that

$$H_d(\mathscr{F}(x), \mathscr{F}(\bar{x})) \leq \bar{\delta}\|x - \bar{x}\| \quad \text{for each} \quad x, \bar{x} \in \mathscr{E}_0.$$

Let $x, \bar{x} \in \mathscr{E}_0$ and $h_1 \in \mathscr{F}(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in [0,T]$,

$$h_1(t) = {}_{RL}I^{\alpha} v_1(t) + \frac{t^{\alpha-1}}{\Lambda_3}\left(\sum_{j=1}^{n} \delta_{jH} I^{\gamma_j}{}_{RL}I^{\alpha} v_1(\xi_j) - \sum_{i=1}^{m} \mu_{iH} I^{\beta_i}{}_{RL}I^{\alpha} v_1(\eta_i) + \lambda\right).$$

By (5.7.2), we have

$$H_d(F(t,x), F(t,\bar{x})) \leq m(t)|x(t) - \bar{x}(t)|.$$

So, there exists $w \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \leq m(t)|x(t) - \bar{x}(t)|, \quad t \in [0,T].$$

Define $U : [0,T] \to \mathscr{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t)|x(t) - \bar{x}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{x}(t))$ is measurable [57, Proposition III.4], there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [0,T]$, we have $|v_1(t) - v_2(t)| \leq m(t)|x(t) - \bar{x}(t)|$.

For each $t \in [0, T]$, let us define

$$h_2(t) = {}_{RL}I^{\alpha} v_2(t) + \frac{t^{\alpha-1}}{\Lambda_3}\left(\sum_{j=1}^{n} \delta_{jH} I^{\gamma_j}{}_{RL}I^{\alpha} v_2(\xi_j) - \sum_{i=1}^{m} \mu_{iH} I^{\beta_i}{}_{RL}I^{\alpha} v_2(\eta_i) + \lambda\right).$$

Thus,

$$\begin{aligned}
&|h_1(t) - h_2(t)| \\
\le\ & {}_{RL}I^{\alpha}|v_1(t) - v_2(t)| \\
&+ \frac{t^{\alpha-1}}{|\Lambda_3|}\left(\sum_{j=1}^{n} \delta_{jH} I^{\gamma_j}{}_{RL}I^{\alpha}|v_1(\xi_j) - v_2(\xi_j)| + \sum_{i=1}^{m} \mu_{iH} I^{\beta_i}{}_{RL}I^{\alpha}|v_1(\eta_i) - v_2(\eta_i)|\right) \\
\le\ & \|m\|\left\{\frac{1}{\Gamma(\alpha+1)}\left(T^{\alpha} + \frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{i=1}^{m}|\mu_i|\alpha^{-\beta_i}\eta_i^{\alpha} + \frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{j=1}^{n}|\delta_j|\alpha^{-\gamma_j}\xi_j^{\alpha}\right)\right\}\|x - \bar{x}\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|h_1 - h_2\| \\
\le\ & \|m\|\left\{\frac{1}{\Gamma(\alpha+1)}\left(T^{\alpha} + \frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{i=1}^{m}|\mu_i|\alpha^{-\beta_i}\eta_i^{\alpha} + \frac{T^{\alpha-1}}{|\Lambda_3|}\sum_{j=1}^{n}|\delta_j|\alpha^{-\gamma_j}\xi_j^{\alpha}\right)\right\}\|x - \bar{x}\|.
\end{aligned}$$

Analogously, interchanging the roles of x and $\bar{x}$, we obtain

$$H_d(\mathscr{F}(x), \mathscr{F}(\bar{x})) \le \bar{\delta}\|x - \bar{x}\|,$$

where $\bar{\delta} = \|m\|\Phi_2\| < 1$.

So $\mathscr{F}$ is a contraction. Therefore, it follows by Theorem 1.18 that $\mathscr{F}$ has a fixed point x which is a solution of (5.69). This completes the proof. □

Example 5.16 Let us consider the following nonlocal boundary value problem

$$\begin{cases} {}_{RL}D^{5/3}x(t) \in F(t, x(t)), \quad t \in \left(0, \frac{3}{2}\right), \\ x(0) = 0, \\ 4_H I^{2/5} x\left(\frac{1}{3}\right) + 3_H I^{1/2} x\left(\frac{2}{3}\right) = \frac{2}{3}{}_H I^{1/2} x\left(\frac{1}{4}\right) + \frac{3}{5}{}_H I^{1/3} x\left(\frac{4}{3}\right) + \frac{1}{7}. \end{cases} \tag{5.72}$$

Here we have $\alpha = 5/3$, $T = 3/2$, $m = 2$, $n = 2$, $\mu_1 = 4$, $\mu_2 = 3$, $\beta_1 = 2/5$, $\beta_2 = 1/2$, $\eta_1 = 1/3$, $\eta_2 = 2/3$, $\delta_1 = 2/3$, $\delta_2 = 3/5$, $\gamma_1 = 1/2$, $\gamma_2 = 1/3$, $\xi_1 = 1/4$, $\xi_2 = 4/3$, $\lambda = 1/7$. By using computer program, we find $\Lambda_3 \approx 6.221625494 \neq 0$.

(a) Let $F : [0, 3/2] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ be a multivalued map defined by

$$x \to F(t, x) = \left[\frac{1 + \cos^2 x}{2 + \sin^2 x}, \frac{e^x}{2e^x + 3} + \frac{2t^3}{9} + 1\right]. \tag{5.73}$$

For $f \in F$, we have

$$|f| \leq \max\left(\frac{1+\cos^2 x}{2+\sin^2 x}, \frac{e^x}{2e^x+3} + \frac{2t^3}{9} + 1\right) \leq \frac{9}{4},\ x \in \mathbb{R}.$$

Thus,

$$\|F(t,x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t,x)\} \leq \frac{9}{4} = p(t)\psi(\|x\|),\ x \in \mathbb{R},$$

with $p(t) = 9$, $\psi(\|x\|) = 1/4$. Further, using the condition (5.18.3), we find that $M > 3.779988106$. Therefore, all the conditions of Theorem 5.18 are satisfied. So, the problem (5.72) with $F(t,x)$ given by (5.73) has at least one solution on $[0, 3/2]$.

(b) Let $F : [0, 3/2] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ be a multivalued map defined by

$$x \to F(t,x) = \left[0, \frac{3\sin^2 x}{(\sqrt{8}+2t)^2} + \frac{3}{128}\right]. \tag{5.74}$$

Then, we have

$$\sup\{|x| : x \in F(t,x)\} \leq \frac{3}{(\sqrt{8}+2t)^2} + \frac{3}{128},$$

and

$$H_d(F(t,x), F(t,\bar{x})) \leq \frac{3}{(\sqrt{8}+2t)^2}|x-\bar{x}|.$$

Let $m(t) = 3/(\sqrt{8}+2t)^2$. Then $H_d(F(t,x), F(t,\bar{x})) \leq m(t)|x-\bar{x}|$ with $d(0, F(t,0) \leq m(t)$ and $\|m\| = 3/8$. Further $\|m\|\Phi_2 \approx 0.624983363 < 1$. Thus all the conditions of Theorem 5.20 are satisfied. Therefore, by the conclusion of Theorem 5.20, the problem (5.72) with $F(t,x)$ given by (5.74) has at least one solution on $[0, 3/2]$.

5.7 Hadamard-Type Fractional Differential Equations with Multiple Nonlocal Fractional Integral Boundary Conditions

In this section, we investigate the following boundary value problem of Hadamard fractional differential equations equipped with multiple nonlocal fractional integral boundary conditions

$$D^q x(t) = f(t, x(t)), \quad 1 < q \le 2, \ t \in (1, e), \tag{5.75}$$

$$x(1) = 0, \quad \sum_{i=1}^{m} \lambda_i J^{\alpha_i} x(\eta_i) = \sum_{j=1}^{n} \mu_j \left(J^{\beta_j} x(e) - J^{\beta_j} x(\xi_j) \right), \tag{5.76}$$

where D^q denotes the Hadamard fractional derivative of order $q, f : [1, e] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\eta_i,\ \xi_j \in (1, e)$, $\lambda_i,\ \mu_j \in \mathbb{R}$, for all $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$, $\eta_1 < \eta_2 < \cdots < \eta_m$, $\xi_1 < \xi_2 < \cdots < \xi_n$, and J^{ϕ} is the Hadamard fractional integral of order $\phi > 0$ ($\phi = \alpha_i, \beta_j, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$).

We emphasize that integral boundary conditions in (5.76) are encountered in various applications such as population dynamics, blood flow models, chemical engineering, cellular systems, heat transmission, plasma physics, thermoelasticity, etc.

Moreover, the condition (5.76) is a general form of the integral boundary conditions and covers many special cases. For example, if $\alpha_i = \beta_j = 1$, for all $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$, then the condition (5.76) reduces to

$$x(1) = 0,$$
$$\lambda_1 \int_1^{\eta_1} x(s) \frac{ds}{s} + \cdots + \lambda_m \int_1^{\eta_m} x(s) \frac{ds}{s} = \mu_1 \int_{\xi_1}^{e} x(s) \frac{ds}{s} + \cdots + \mu_n \int_{\xi_n}^{e} x(s) \frac{ds}{s}.$$

For the sake of computational convenience, we set

$$\Lambda_4 = \sum_{i=1}^{m} \lambda_i \frac{\Gamma(q)}{\Gamma(q + \alpha_i)} (\log \eta_i)^{q+\alpha_i-1} - \sum_{j=1}^{n} \mu_j \frac{\Gamma(q)}{\Gamma(q + \beta_j)} \left(1 - (\log \xi_j)^{q+\beta_j-1} \right). \tag{5.77}$$

Lemma 5.4 *Let $\Lambda_4 \neq 0$, $1 < q \le 2$, $\alpha_i,\ \beta_j > 0$, $\eta_i,\ \xi_j \in (1, e)$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ and $h \in C([1, e], \mathbb{R})$. Then x is a solution of the following fractional differential equation*

$$D^q x(t) = h(t), \quad t \in (1, e), \tag{5.78}$$

subject to the boundary conditions

$$x(1) = 0, \qquad \sum_{i=1}^{m} \lambda_i J^{\alpha_i} x(\eta_i) = \sum_{j=1}^{n} \mu_j \left(J^{\beta_j} x(e) - J^{\beta_j} x(\xi_j) \right), \tag{5.79}$$

if and only if it is a solution of the following integral equation

$$\begin{aligned} x(t) = {} & \frac{(\log t)^{q-1}}{\Lambda_4} \sum_{j=1}^{n} \mu_j \left(J^{q+\beta_j} h(e) - J^{q+\beta_j} h(\xi_j) \right) \\ & - \frac{(\log t)^{q-1}}{\Lambda_4} \sum_{i=1}^{m} \lambda_i J^{q+\alpha_i} h(\eta_i) + J^q h(t). \end{aligned} \tag{5.80}$$

Proof Applying the Hadamard fractional integral of order q to both sides of (5.78), we have

$$x(t) = z_1 (\log t)^{q-1} + z_2 (\log t)^{q-2} + J^q h(t), \tag{5.81}$$

where $z_1, z_2 \in \mathbb{R}$.

Using the condition $x(1) = 0$ in (5.81) implies that $z_2 = 0$. Therefore (5.81) becomes

$$x(t) = z_1 (\log t)^{q-1} + J^q h(t). \tag{5.82}$$

For any $p > 0$, it follows that

$$J^p x(t) = z_1 \frac{\Gamma(q)}{\Gamma(q+p)} (\log t)^{q+p-1} + J^{q+p} h(t). \tag{5.83}$$

Using the second condition of (5.79) with (5.83) in (5.82) leads to

$$z_1 = \frac{1}{\Lambda_4} \sum_{j=1}^{n} \mu_j \left(J^{q+\beta_j} h(e) - J^{q+\beta_j} h(\xi_j) \right) - \frac{1}{\Lambda_4} \sum_{i=1}^{m} \lambda_i J^{q+\alpha_i} h(\eta_i). \tag{5.84}$$

Substituting the value of z_1 into (5.82), we obtain (5.80) as required. The converse follows by direct computation. The proof is completed. □

By Lemma 5.4, we define an operator $\mathscr{F} : C([1,e],\mathbb{R}) \to C([1,e],\mathbb{R})$ associated with the problem (5.75)–(5.76) by

$$\begin{aligned}(\mathscr{F}x)(t) &= J^q f(s,x(s))(t) - \frac{(\log t)^{q-1}}{\Lambda_4} \sum_{i=1}^{m} \lambda_i J^{\alpha_i+q} f(s,x(s))(\eta_i) \\ &+ \frac{(\log t)^{q-1}}{\Lambda_4} \sum_{j=1}^{n} \mu_j \left(J^{\beta_j+q} f(s,x(s))(e) - J^{\beta_j+q} f(s,x(s))(\xi_j) \right), \end{aligned} \tag{5.85}$$

with $\Lambda_4 \neq 0$. It should be noticed that problem (5.75)–(5.76) has solutions if and only if the operator $\mathscr{F}$ has fixed points.
In the sequel, we set $\mathscr{E}_1 = C([1,e],\mathbb{R})$ and

$$\Phi_3 = \frac{1}{\Gamma(q+1)} + \frac{1}{|\Lambda_4|} \sum_{i=1}^{m} |\lambda_i| \frac{(\log \eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} + \frac{1}{|\Lambda_4|} \sum_{j=1}^{n} |\mu_j| \frac{1+(\log \xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)}. \tag{5.86}$$

The first existence and uniqueness result is based on the Banach contraction mapping principle.

Theorem 5.21 *Let $f : [1,e] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption (5.1.1). Then the problem (5.75)–(5.76) has a unique solution on $[1,e]$ if*

$$L\Phi_3 < 1, \tag{5.87}$$

where Φ_3 is given by (5.86).

Proof We transform the problem (5.75)–(5.76) into a fixed point problem, $x = \mathscr{F}x$, where the operator $\mathscr{F}$ is defined by (5.85).

We set $\sup_{t\in[1,e]} |f(t,0)| = M < \infty$ and choose $r \geq \dfrac{M\Phi_3}{1-L\Phi_3}$ to show that $\mathscr{F}B_r \subset B_r$, where $B_r = \{x \in \mathscr{E}_1 : \|x\| \leq r\}$. For any $x \in B_r$, we have

$$\begin{aligned}
\|\mathscr{F}x\| \leq& \sup_{t\in[1,e]} \Bigg\{ J^q|f(s,x(s))|(t) + \frac{(\log t)^{q-1}}{|\Lambda_4|}\sum_{i=1}^m |\lambda_i| J^{\alpha_i+q}|f(s,x(s))|(\eta_i) \\
&+ \frac{(\log t)^{q-1}}{|\Lambda_4|}\sum_{j=1}^n |\mu_j| \left(J^{\beta_j+q}|f(s,x(s))|(e) + J^{\beta_j+q}|f(s,x(s))|(\xi_j)\right)\Bigg\} \\
\leq& J^q\big(|f(s,x(s)) - f(s,0)| + |f(s,0)|\big)(e) \\
&+ \frac{1}{|\Lambda_4|}\sum_{i=1}^m |\lambda_i| J^{\alpha_i+q}\big(|f(s,x(s)) - f(s,0)| + |f(s,0)|\big)(\eta_i) \\
&+ \frac{1}{|\Lambda_4|}\sum_{j=1}^n |\mu_j| \Big(J^{\beta_j+q}\big(|f(s,x(s)) - f(s,0)| + |f(s,0)|\big)(e) \\
&+ J^{\beta_j+q}\big(|f(s,x(s)) - f(s,0)| + |f(s,0)|\big)(\xi_j)\Big) \\
\leq& (Lr+M)\Bigg\{\frac{1}{\Gamma(q+1)} + \frac{1}{|\Lambda_4|}\sum_{i=1}^m |\lambda_i| \frac{(\log \eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\
&+ \frac{1}{|\Lambda_4|}\sum_{j=1}^n |\mu_j| \frac{1 + (\log \xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\Bigg\} \\
=& (Lr+M)\Phi_3 \leq r.
\end{aligned}$$

It follows that $\mathscr{F}B_r \subset B_r$.

For $x, y \in \mathscr{E}_1$ and for each $t \in [1,e]$, we have

$$\begin{aligned}
&|\mathscr{F}x(t) - \mathscr{F}y(t)| \\
\leq& J^q(|f(s,x(s)) - f(s,y(s))|)(t) \\
&+ \frac{(\log t)^{q-1}}{|\Lambda_4|}\sum_{i=1}^m |\lambda_i| J^{\alpha_i+q}(|f(s,x(s)) - f(s,y(s))|)(\eta_i) \\
&+ \frac{(\log t)^{q-1}}{|\Lambda_4|}\sum_{j=1}^n |\mu_j| \Bigg(J^{\beta_j+q}(|f(s,x(s)) - f(s,y(s))|)(e) \\
&+ J^{\beta_j+q}(|f(s,x(s)) - f(s,y(s))|)(\xi_j)\Bigg)
\end{aligned}$$

$$\leq L\|x-y\|\left\{\frac{1}{\Gamma(q+1)}+\frac{1}{|\Lambda_4|}\sum_{i=1}^{m}|\lambda_i|\frac{(\log\eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)}\frac{1}{|\Lambda_4|}\sum_{j=1}^{n}|\mu_j|\frac{1+(\log\xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\right\}$$
$$= L\Phi_3\|x-y\|.$$

The above inequality implies that $\|\mathscr{F}x-\mathscr{F}y\|\leq L\Phi_3\|x-y\|$. As $L\Phi_3<1$, $\mathscr{F}$ is a contraction. Hence, by the Banach contraction mapping principle, we deduce that $\mathscr{F}$ has a fixed point which is the unique solution of the problem (5.75)–(5.76). This completes the proof. □

Example 5.17 Consider the following boundary value problem of Hadamard fractional differential equation with fractional integral boundary conditions

$$\begin{cases} D^{3/2}x(t)=\dfrac{\log t^5}{e^t(t+2)^2}\dfrac{|x(t)|}{(3+|x(t)|)}+\dfrac{1}{10}, \quad t\in J=[1,e],\\ x(1)=0,\\ 2J^{1/4}x\left(\dfrac{5}{4}\right)+\dfrac{1}{5}J^{3/2}x\left(\dfrac{9}{5}\right)+3J^2x\left(\dfrac{15}{7}\right)=J^{2/3}x(e)-J^{2/3}x\left(\dfrac{10}{7}\right)\\ \qquad +5\left(J^{9/7}x(e)-J^{9/7}x(2)\right)-2\left(J^{11/4}x(e)-J^{11/4}x\left(\dfrac{9}{4}\right)\right). \end{cases} \tag{5.88}$$

Here $q=3/2$, $\lambda_1=2$, $\lambda_2=1/5$, $\Lambda_4=3$, $\alpha_1=1/4$, $\alpha_2=3/2$, $\alpha_3=2$, $\eta_1=5/4$, $\eta_2=9/5$, $\eta_3=15/7$, $\mu_1=1$, $\mu_2=5$, $\mu_3=-2$, $\beta_1=2/3$, $\beta_2=9/7$, $\beta_3=11/4$, $\xi_1=10/7$, $\xi_2=2$, $\xi_3=9/4$ and $f(t,x)=(\log t^5|x|)/(e^t(t+2)^2(3+|x|))1/10$. Since $|f(t,x)-f(t,y)|\leq(5/27e)|x-y|$, (5.21.1) is satisfied with $L=5/27e$. With the given values, it is found that $\Lambda_4\approx-0.6895040549$, $\Phi_3\approx 3.975680952$ and $L\Phi_3\approx 0.2708465347<1$. Hence, by Theorem 5.21, the boundary value problem (5.88) has a unique solution on $[1,e]$.

Next, we establish the second existence and uniqueness result by means of nonlinear contractions.

Theorem 5.22 *Let $f:[1,e]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the assumption:*

(5.22.1) $|f(t,x)-f(t,y)|\leq h(t)\dfrac{|x-y|}{H^*+|x-y|}$, $t\in[1,e]$, $x,y\geq 0$, *where* $h:[1,e]\to\mathbb{R}^+$ *is continuous and a constant* H^* *is defined by*

$$H^*=J^qh(e)+\frac{1}{|\Lambda_4|}\sum_{i=1}^{m}|\lambda_i|J^{\alpha_i+q}h(\eta_i)$$
$$+\frac{1}{|\Lambda_4|}\sum_{j=1}^{n}|\mu_j|\left(J^{\beta_j+q}h(e)+J^{\beta_j+q}h(\xi_j)\right). \tag{5.89}$$

Then the problem (5.75)–(5.76) has a unique solution on $[1,e]$.

Proof We consider the operator $\mathscr{F}:\mathscr{E}_1\to\mathscr{E}_1$ given by (5.85) and a continuous nondecreasing function $\Psi:\mathbb{R}_0^+\to\mathbb{R}^+$ defined by $\Psi(\theta)=\dfrac{H^*\theta}{H^*+\theta}$, $\forall\theta\geq 0$, such that $\Psi(0)=0$ and $\Psi(\theta)<\theta$ for all $\theta>0$.

For any $x,y\in\mathscr{E}_1$ and for each $t\in[1,e]$, we have

$$\begin{aligned}
|\mathscr{F}x(t)-\mathscr{F}y(t)| &\leq J^q(|f(s,x(s))-f(s,y(s))|)(t)\\
&\quad+\frac{(\log t)^{q-1}}{|\Lambda_4|}\sum_{i=1}^{m}|\lambda_i|J^{\alpha_i+q}(|f(s,x(s))-f(s,y(s))|)(\eta_i)\\
&\quad+\frac{(\log t)^{q-1}}{|\Lambda_4|}\sum_{j=1}^{n}|\mu_j|\Big(J^{\beta_j+q}(|f(s,x(s))-f(s,y(s))|)(e)\\
&\quad+J^{\beta_j+q}(|f(s,x(s))-f(s,y(s))|)(\xi_j)\Big)\\
&\leq J^q\left(h(s)\frac{|x(s)-y(s)|}{H^*+|x(s)-y(s)|}\right)(e)\\
&\quad+\frac{1}{|\Lambda_4|}\sum_{i=1}^{m}|\lambda_i|J^{\alpha_i+q}\left(h(s)\frac{|x(s)-y(s)|}{H^*+|x(s)-y(s)|}\right)(\eta_i)\\
&\quad+\frac{1}{|\Lambda_4|}\sum_{j=1}^{n}|\mu_j|\left\{J^{\beta_j+q}\left(h(s)\frac{|x(s)-y(s)|}{H^*+|x(s)-y(s)|}\right)(e)\right.\\
&\quad\left.+J^{\beta_j+q}\left(h(s)\frac{|x(s)-y(s)|}{H^*+|x(s)-y(s)|}\right)(\xi_j)\right\}\\
&\leq\frac{\Psi(\|x-y\|)}{H^*}\left(J^qh(e)+\frac{1}{|\Lambda_4|}\sum_{i=1}^{m}|\lambda_i|J^{\alpha_i+q}h(\eta_i)\right.\\
&\quad\left.+\frac{1}{|\Lambda_4|}\sum_{j=1}^{n}|\mu_j|\left(J^{\beta_j+q}h(e)+J^{\beta_j+q}h(\xi_j)\right)\right)\\
&=\Psi(\|x-y\|),
\end{aligned}$$

which implies that $\|\mathscr{F}x-\mathscr{F}y\|\leq\Psi(\|x-y\|)$. Therefore $\mathscr{F}$ is a nonlinear contraction. Hence, by Theorem 1.11, the operator $\mathscr{F}$ has a fixed point which is the unique solution of the problem (5.75)–(5.76). □

Example 5.18 Consider the following boundary value problem for Hadamard fractional differential equation with integral boundary conditions

$$\begin{cases} D^{7/4}x(t) = \dfrac{e^t}{(t+1)^2}\dfrac{|x(t)|}{(2+|x(t)|)} + \dfrac{1}{7}, \quad t \in J = [1,e], \\ x(1) = 0, \\ \dfrac{1}{4}J^{6/7}x\left(\dfrac{7}{3}\right) - \dfrac{2}{3}J^3x\left(\dfrac{7}{5}\right) - 2J^{5/2}x(2) = 4\left(J^5x(e) - J^5x\left(\dfrac{11}{5}\right)\right) \\ \qquad + \dfrac{11}{4}\left(J^{3/4}x(e) - J^{3/4}x\left(\dfrac{16}{13}\right)\right). \end{cases} \tag{5.90}$$

Here $q = 7/4$, $\lambda_1 = 1/4$, $\lambda_2 = -2/3$, $\Lambda_4 = -2$, $\alpha_1 = 6/7$, $\alpha_2 = 3$, $\alpha_3 = 5/2$, $\eta_1 = 7/3$, $\eta_2 = 7/5$, $\eta_3 = 2$, $\mu_1 = 4$, $\mu_2 = 11/4$, $\beta_1 = 5$, $\beta_2 = 3/4$, $\xi_1 = 11/5$, $\xi_2 = 16/13$, and $f(t,x) = (e^t|x|)/((t+1)^2(2+|x|)) + 1/7$. We choose $h(t) = e^t/4$ and find that $\Lambda_4 \approx -1.672972140$ and $H^* \approx 1.295076743$. Clearly,

$$|f(t,x) - f(t,y)| = \frac{e^t}{(1+t)^2}\left(\frac{2|x| - 2|y|}{4 + 2|x| + 2|y| + |x||y|}\right) \le \frac{e^t}{4}\left(\frac{|x-y|}{1.295076743 + |x-y|}\right).$$

Hence, by Theorem 5.22, the problem (5.90) has a unique solution on $[1, e]$.

The following existence result is based on Krasnoselskii's fixed point theorem (Theorem 1.2).

Theorem 5.23 *Assume that $f : [1,e] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the assumptions (5.1.1) and (5.15.1). If*

$$\frac{L}{\Gamma(q+1)} < 1, \tag{5.91}$$

then the problem (5.75)–(5.76) has at least one solution on $[1, e]$.

Proof We define $\sup_{t\in[1,e]}|\phi(t)| = \|\phi\|$ and choose a suitable constant $\bar{r}$ as $\bar{r} \ge \|\phi\|\Phi_3$, where Φ_3 is defined by (5.86). Next, we define the operators $\mathscr{P}$ and $\mathscr{Q}$ on $B_{\bar{r}} = \{x \in \mathscr{E}_1 : \|x\| \le \bar{r}\}$ as

$$\begin{aligned} (\mathscr{P}x)(t) &= \frac{(\log t)^{q-1}}{\Lambda_4}\sum_{j=1}^{n}\mu_j\left(J^{\beta_j+q}f(s,x(s))(e) - J^{\beta_j+q}f(s,x(s))(\xi_j)\right) \\ &\quad - \frac{(\log t)^{q-1}}{\Lambda_4}\sum_{i=1}^{m}\lambda_i J^{\alpha_i+q}f(s,x(s))(\eta_i), \quad t \in [1,e], \\ (\mathscr{Q}x)(t) &= J^q f(s,x(s))(t), \quad t \in [1,e]. \end{aligned}$$

For $x, y \in B_{\bar{r}}$, we have

$$\begin{aligned} \|\mathscr{P}x + \mathscr{Q}y\| \le \|\phi\|\Bigg(&\frac{1}{\Gamma(q+1)} + \frac{1}{|\Lambda_4|}\sum_{i=1}^{m}|\lambda_i|\frac{(\log\eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\ &+ \frac{1}{|\Lambda_4|}\sum_{j=1}^{n}|\mu_j|\frac{1 + (\log\xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\Bigg) \end{aligned}$$

$$
\begin{aligned}
&= \|\phi\| \Phi_3 \\
&\leq \bar{r}.
\end{aligned}
$$

This shows that $\mathscr{P}x + \mathscr{Q}y \in B_{\bar{r}}$. Using the assumption (5.21.1) together with (5.91), it can easily be shown that $\mathscr{Q}$ is a contraction mapping. Since the function f is continuous, we have that the operator $\mathscr{P}$ is continuous and

$$
\|\mathscr{P}x\| \leq \|\phi\| \left(\frac{1}{|\Lambda_4|} \sum_{i=1}^{m} |\lambda_i| \frac{(\log \eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} + \frac{1}{|\Lambda_4|} \sum_{j=1}^{n} |\mu_j| \frac{1+(\log \xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)} \right).
$$

Therefore, $\mathscr{P}$ is uniformly bounded on $B_{\bar{r}}$.

Now, we prove the compactness of the operator $\mathscr{P}$. Let us set $\sup_{(t,x)\in[1,e]\times B_{\bar{r}}} |f(t,x)| = \bar{f} < \infty$. Consequently, we get

$$
\begin{aligned}
&|(\mathscr{P}x)(t_1) - (\mathscr{P}x)(t_2)| \\
&= \Bigg| \frac{(\log t_1)^{q-1}}{\Lambda_4} \sum_{j=1}^{n} \mu_j \left(J^{\beta_j+q} f(s,x(s))(e) - J^{\beta_j+q} f(s,x(s))(\xi_j) \right) \\
&\quad - \frac{(\log t_1)^{q-1}}{\Lambda_4} \sum_{i=1}^{m} \lambda_i J^{\alpha_i+q} f(s,x(s))(\eta_i) \\
&\quad - \frac{(\log t_2)^{q-1}}{\Lambda_4} \sum_{j=1}^{n} \mu_j \left(J^{\beta_j+q} f(s,x(s))(e) - J^{\beta_j+q} f(s,x(s))(\xi_j) \right) \\
&\quad + \frac{(\log t_2)^{q-1}}{\Lambda_4} \sum_{i=1}^{m} \lambda_i J^{\alpha_i+q} f(s,x(s))(\eta_i) \Bigg| \\
&\leq \bar{f} \frac{|(\log t_2)^{q-1} - (\log t_1)^{q-1}|}{|\Lambda_4|} \left[\sum_{i=1}^{m} |\lambda_i| \frac{(\log \eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} + \sum_{j=1}^{n} |\mu_j| \frac{1-(\log \xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)} \right],
\end{aligned}
$$

which is independent of x and tends to zero as $t_2 \to t_1$. Thus, $\mathscr{P}$ is equicontinuous. So $\mathscr{P}$ is relatively compact on $B_{\bar{r}}$. Hence, by the Arzelá-Ascoli Theorem, $\mathscr{P}$ is compact on $B_{\bar{r}}$. Thus all the assumptions of Theorem 1.2 are satisfied. So the problem (5.75)–(5.76) has at least one solution on $[1,e]$. The proof is completed. □

Remark 5.2 In the above theorem, we can interchange the roles of the operators $\mathscr{P}$ and $\mathscr{Q}$ to obtain a second result by replacing (5.91) with the following condition:

$$
\frac{L}{|\Lambda_4|} \sum_{i=1}^{m} |\lambda_i| \frac{(\log \eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} + \frac{L}{|\Lambda_4|} \sum_{j=1}^{n} |\mu_j| \frac{1+(\log \xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)} < 1.
$$

Our last existence result is based on Leray-Schauder's nonlinear alternative.

Theorem 5.24 *Assume that $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the assumption (5.5.1). In addition, we suppose that:*

(5.24.1) there exists a constant $N > 0$ such that

$$\frac{N}{\|p\|\psi(N)\Phi_3} > 1,$$

where Φ_3 is defined by (5.86).

Then the problem (5.75)–(5.76) has at least one solution on $[1, e]$.

Proof In the first step, we show that the operator $\mathscr{F}$, defined by (5.85), *maps bounded sets (balls) into bounded sets in $\mathscr{E}_1$*. For a positive number R, let $B_R = \{x \in \mathscr{E}_1 : \|x\| \le R\}$ be a bounded ball in $\mathscr{E}_1$. Then, for $t \in [1, e]$, we have

$$\begin{aligned}
|\mathscr{F}x(t)| &\le J^q|f(s, x(s))|(e) + \frac{1}{|\Lambda_4|}\sum_{i=1}^{m}|\lambda_i|J^{\alpha_i+q}|f(s, x(s))|(\eta_i) \\
&\quad + \frac{1}{|\Lambda_4|}\sum_{j=1}^{n}|\mu_j|\left(J^{\beta_j+q}|f(s, x(s))|(e) + J^{\beta_j+q}|f(s, x(s))|(\xi_j)\right) \\
&\le \|p\|\psi(\|x\|)\frac{1}{\Gamma(q+1)} + \|p\|\psi(\|x\|)\frac{1}{|\Lambda_4|}\sum_{i=1}^{m}|\lambda_i|\frac{(\log\eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\
&\quad + \|p\|\psi(\|x\|)\frac{1}{|\Lambda_4|}\sum_{j=1}^{n}|\mu_j|\frac{1 + (\log\xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)} \\
&\le \|p\|\psi(R)\frac{1}{\Gamma(q+1)} + \|p\|\psi(R)\frac{1}{|\Lambda_4|}\sum_{i=1}^{m}|\lambda_i|\frac{(\log\eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\
&\quad + \|p\|\psi(R)\frac{1}{|\Lambda_4|}\sum_{j=1}^{n}|\mu_j|\frac{1 + (\log\xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)} \\
&:= K_0.
\end{aligned}$$

Therefore, we conclude that $\|\mathscr{F}x\| \le K_0$.

Secondly, we show that $\mathscr{F}$ *maps bounded sets into equicontinuous sets of $\mathscr{E}_1$*. Let $\sup_{(t,x)\in[1,e]\times B_R}|f(t, x)| = f^* < \infty$, $\nu_1, \nu_2 \in [1, e]$ with $\nu_1 < \nu_2$ and $x \in B_R$. Then, we have

$$\begin{aligned}
&|(\mathscr{F}x)(\nu_2) - (\mathscr{F}x)(\nu_1)| \\
&= \Bigg| J^q f(s, x(s))(\nu_2) - \frac{(\log\nu_2)^{q-1}}{\Lambda_4}\sum_{i=1}^{m}\lambda_i J^{\alpha_i+q}f(s, x(s))(\eta_i) \\
&\quad + \frac{(\log\nu_2)^{q-1}}{\Lambda_4}\sum_{j=1}^{n}\mu_j\left(J^{\beta_j+q}f(s, x(s))(e) - J^{\beta_j+q}f(s, x(s))(\xi_j)\right)
\end{aligned}$$

$$
\begin{aligned}
&- J^q f(s,x(s))(\nu_1) + \frac{(\log \nu_1)^{q-1}}{\Lambda_4} \sum_{i=1}^{m} \lambda_i J^{\alpha_i+q} f(s,x(s))(\eta_i) \\
&- \frac{(\log \nu_1)^{q-1}}{\Lambda_4} \sum_{j=1}^{n} \mu_j \left(J^{\beta_j+q} f(s,x(s))(e) - J^{\beta_j+q} f(s,x(s))(\xi_j) \right) \Bigg| \\
\le\, & f^* \frac{2(\log \nu_2 - \log \nu_1)^q + |(\log \nu_2)^q - (\log \nu_1)^q|}{\Gamma(q+1)} \\
&+ f^* \frac{\left|(\log \nu_2)^{q-1} - (\log \nu_1)^{q-1}\right|}{|\Lambda_4|} \left[\sum_{i=1}^{m} |\lambda_i| \frac{(\log \eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \right. \\
&\left. + \sum_{j=1}^{n} |\mu_j| \frac{1-(\log \xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)} \right].
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_R$ as $\nu_2 \to \nu_1$. Therefore it follows by the Arzelá-Ascoli Theorem that $\mathscr{F} : \mathscr{E}_1 \to \mathscr{E}_1$ is completely continuous.

Finally, we show that there exists an open set $U \subset \mathscr{E}_1$ with $x \neq \theta \mathscr{F} x$ for $\theta \in (0,1)$ and $x \in \partial U$. Let x be a solution. Then, for $t \in [1,e]$, we have

$$
\begin{aligned}
\|x\| \le\, & \|p\|\psi(\|x\|) \frac{1}{\Gamma(q+1)} + \|p\|\psi(\|x\|) \frac{1}{|\Lambda_4|} \sum_{i=1}^{m} |\lambda_i| \frac{(\log \eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\
&+ \|p\|\psi(\|x\|) \frac{1}{|\Lambda_4|} \sum_{j=1}^{n} |\mu_j| \frac{1+(\log \xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)} \\
=\, & \|p\|\psi(\|x\|)\Phi_3.
\end{aligned}
$$

Consequently, we have

$$
\frac{\|x\|}{\|p\|\psi(\|x\|)\Phi_3} \le 1.
$$

In view of (5.24.2), there exists N such that $\|x\| \neq N$. Let us set

$$
U = \{x \in \mathscr{E}_1 : \|x\| < N\}. \tag{5.92}
$$

Note that the operator $\mathscr{F} : \overline{U} \to \mathscr{E}_1$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \theta \mathscr{F} x$ for some $\theta \in (0,1)$. Hence, by nonlinear alternative of Leray-Schauder type (Theorem 1.4), we deduce that $\mathscr{F}$ has a fixed point in $\overline{U}$, which is a solution of the problem (5.75)–(5.76). This completes the proof. □

Example 5.19 Consider the following Hadamard type boundary value problem

$$\begin{cases} D^{6/5}x(t) = \dfrac{2\sin(x/4)}{5\pi + (e^x+1)^2} + \dfrac{2+\cos(\pi t)}{10\pi+3}, \quad t \in J = [1,e], \\ x(1) = 0, \\ J^4 x\left(\dfrac{3}{2}\right) - 3J^{9/4}x(2) - 10J^{1/5}x\left(\dfrac{7}{4}\right) + 6J^{7/2}x\left(\dfrac{5}{2}\right) + \dfrac{14}{3}J^5 x\left(\dfrac{11}{9}\right) \\ = 3\left(J^{3/2}x(e) - J^{3/2}x\left(\dfrac{11}{7}\right)\right) - 7\left(J^3 x(e) - J^3 x\left(\dfrac{17}{13}\right)\right) \\ +\dfrac{4}{3}\left(J^{5/3}x(e) - J^{5/3}x(2)\right). \end{cases} \tag{5.93}$$

Here $q = 6/5$, $\lambda_1 = 1$, $\lambda_2 = -3$, $\Lambda_4 = -10$, $\lambda_4 = 6$, $\lambda_5 = 14/3$, $\alpha_1 = 4$, $\alpha_2 = 9/4$, $\alpha_3 = 1/5$, $\alpha_4 = 7/2$, $\alpha_5 = 5$, $\eta_1 = 3/2$, $\eta_2 = 2$, $\eta_3 = 7/4$, $\eta_4 = 5/2$, $\eta_5 = 11/9$, $\mu_1 = 3$, $\mu_2 = -7$, $\mu_3 = 4/3$, $\beta_1 = 3/2$, $\beta_2 = 3$, $\beta_3 = 5/3$, $\xi_1 = 11/7$, $\xi_2 = 17/13$, $\xi_3 = 2$ and $f(t,x) = (2\sin(x/4))/(5\pi + (e^x+1)^2) + (2 + \cos(\pi t))/(10\pi + 3)$. Clearly,

$$|f(t,x)| = \left|\frac{2\sin(\frac{x}{4})}{5\pi + (e^x+1)^2} + \frac{2+\cos(\pi t)}{10\pi+3}\right| \le (2+\cos(\pi t))\left(\frac{|x|+1}{10\pi}\right).$$

Choosing $p(t) = 2 + \cos(\pi t)$ and $\psi(|x|) = (|x|+1)/(10\pi)$, we find that $\Lambda_4 \approx -9.148087406$, $\Phi_3 \approx 1.462649525$ and $N > 0.1623483851$. Hence, by Theorem 5.24, the problem (5.93) has at least one solution on $[1,e]$.

5.8 Notes and Remarks

The contents of Sects. 5.2–5.7 are respectively adapted from the papers [125–127, 154, 157, 162].

Chapter 6
Coupled Systems of Hadamard and Riemann-Liouville Fractional Differential Equations with Hadamard Type Integral Boundary Conditions

6.1 Introduction

In this chapter, we focus on the study of coupled systems of Hadamard and Riemann-Liouville fractional differential equations with coupled and uncoupled Hadamard type integral boundary conditions. Coupled systems of fractional order differential equations are of significant importance as such systems appear in a variety of problems of interdisciplinary fields such as synchronization phenomena [81, 84, 179], nonlocal thermoelasticity [130], bioengineering [119], etc. For details and examples, the reader is referred to the papers [11, 28, 29, 123, 151, 152, 169] and the references cited therein.

6.2 A Coupled System of Hadamard Type Fractional Differential Equations with Uncoupled Hadamard Integral Boundary Conditions

In this section, we discuss the existence of solutions for a coupled system of Hadamard type fractional differential equations equipped with uncoupled Hadamard integral boundary conditions

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), v(t)), \ \ 1 < t < e, \ \ 1 < \alpha \leq 2, \\ D^{\beta}v(t) = g(t, u(t), v(t)), \ \ 1 < t < e, \ \ 1 < \beta \leq 2, \\ u(1) = 0, \ u(e) = I^{\gamma}u(\sigma_1) = \dfrac{1}{\Gamma(\gamma)}\displaystyle\int_1^{\sigma_1}\left(\log\frac{\sigma_1}{s}\right)^{\gamma-1}\frac{u(s)}{s}ds, \ \gamma > 0, \ 1 < \sigma_1 < e, \\ v(1) = 0, \ v(e) = I^{\gamma}v(\sigma_2) = \dfrac{1}{\Gamma(\gamma)}\displaystyle\int_1^{\sigma_2}\left(\log\frac{\sigma_2}{s}\right)^{\gamma-1}\frac{v(s)}{s}ds, \ 1 < \sigma_2 < e, \end{cases} \tag{6.1}$$

B. Ahmad et al., *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, DOI 10.1007/978-3-319-52141-1_6

where D^{δ}, $\delta = \alpha, \beta$, is the Hadamard fractional derivative of fractional order δ, I^{γ} is the Hadamard fractional integral of order γ and $f, g : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Lemma 6.1 (Auxiliary Lemma) *For* $1 < q \leq 2$ *and* $z \in C([1, e], \mathbb{R})$, *the solution of the linear problem*

$$\begin{cases} D^q x(t) = z(t), \quad 1 < t < e, \\ x(1) = 0, \quad x(e) = I^{\gamma} x(\theta), \end{cases} \tag{6.2}$$

is equivalent to the integral equation

$$x(t) = I^q z(t) + \frac{(\log t)^{q-1}}{Q} \left[I^{\gamma+q} z(\theta) - I^q z(e) \right], \tag{6.3}$$

where

$$Q = \frac{1}{1 - \dfrac{\Gamma(q)}{\Gamma(\gamma + q)} (\log \theta)^{\gamma+q-1}} \tag{6.4}$$

and $\dfrac{\Gamma(q)}{\Gamma(\gamma + q)} (\log \theta)^{\gamma+q-1} \neq 1.$

Proof As before, the solution of Hadamard differential equation in (6.2) can be written as

$$x(t) = I^q z(t) + c_1 (\log t)^{q-1} + c_2 (\log t)^{q-2}, \tag{6.5}$$

where c_1 and c_2 are unknown arbitrary constants. Using the boundary conditions given by (6.2), we find that $c_2 = 0$, and

$$c_1 = \frac{1}{1 - \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_1^{\theta} \left(\log \frac{\theta}{s} \right)^{\gamma-1} \frac{(\log s)^{q-1}}{s} ds} \left[I^{\gamma+q} z(\theta) - I^q z(e) \right]. \tag{6.6}$$

Substituting the values of c_1 and c_2 in (6.5), we obtain (6.3). The converse follows by direct computation. This completes the proof. □

Let us introduce the space $X = \{u(t) | u(t) \in C([1, e], \mathbb{R})\}$ endowed with the norm $\|u\| = \max\{|u(t)|, t \in [1, e]\}$. Obviously $(X, \| \cdot \|)$ is a Banach space. Also $Y = \{v(t) | v(t) \in C([1, e], \mathbb{R})\}$ endowed with the norm $\|v\| = \max\{|v(t)|, t \in [1, e]\}$ is a Banach space. Then the product space $(X \times Y, \|(u, v)\|)$ is also a Banach space equipped with norm $\|(u, v)\| = \|u\| + \|v\|$.

In view of Lemma 6.1, for $q = \alpha, \theta = \sigma_1$ and $q = \beta, \theta = \sigma_2$ respectively, we define an operator $T : X \times Y \to X \times Y$ by

$$T(u, v)(t) = \begin{pmatrix} T_1(u, v)(t) \\ T_2(u, v)(t) \end{pmatrix}, \tag{6.7}$$

where

$$\begin{aligned} T_1(u, v)(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s, u(s), v(s))}{s} ds \\ &\quad + \frac{(\log t)^{\alpha-1}}{A} \left[\frac{1}{\Gamma(\gamma+\alpha)} \int_1^{\sigma_1} \left(\log \frac{\sigma_1}{s}\right)^{\gamma+\alpha-1} \frac{f(s, u(s), v(s))}{s} ds \right. \\ &\quad \left. - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{f(s, u(s), v(s))}{s} ds \right], \end{aligned}$$

and

$$\begin{aligned} T_2(u, v)(t) &= \frac{1}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\beta-1} \frac{g(s, u(s), v(s))}{s} ds \\ &\quad + \frac{(\log t)^{\beta-1}}{B} \left[\frac{1}{\Gamma(\gamma+\beta)} \int_1^{\sigma_2} \left(\log \frac{\sigma_2}{s}\right)^{\gamma+\beta-1} \frac{g(s, u(s), v(s))}{s} ds \right. \\ &\quad \left. - \frac{1}{\Gamma(\beta)} \int_1^e \left(\log \frac{e}{s}\right)^{\beta-1} \frac{g(s, u(s), v(s))}{s} ds \right], \end{aligned}$$

with

$$A = \frac{1}{1 - \dfrac{\Gamma(\alpha)}{\Gamma(\gamma+\alpha)} (\log \sigma_1)^{\gamma+\alpha-1}}, \quad B = \frac{1}{1 - \dfrac{\Gamma(\beta)}{\Gamma(\gamma+\beta)} (\log \sigma_2)^{\gamma+\beta-1}},$$

and $\dfrac{\Gamma(\alpha)}{\Gamma(\gamma+\alpha)} (\log \sigma_1)^{\gamma+\alpha-1} \neq 1, \quad \dfrac{\Gamma(\beta)}{\Gamma(\gamma+\beta)} (\log \sigma_2)^{\gamma+\beta-1} \neq 1.$

For computational convenience, we set

$$M_1 = \frac{1}{\Gamma(\alpha+1)} + \frac{1}{|A|} \left(\frac{(\log \sigma_1)^{\gamma+\alpha}}{\Gamma(\gamma+\alpha+1)} + \frac{1}{\Gamma(\alpha+1)} \right), \tag{6.8}$$

$$M_2 = \frac{1}{\Gamma(\beta+1)} + \frac{1}{|B|} \left(\frac{(\log \sigma_2)^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)} + \frac{1}{\Gamma(\beta+1)} \right). \tag{6.9}$$

Theorem 6.1 *Assume that:*

(6.1.1) there exist real constants $k_i,\ \lambda_i \geq 0\ (i = 1, 2)$ *and* $k_0 > 0, \lambda_0 > 0$ *such that* $\forall x_i \in \mathbb{R},\ i = 1, 2,$ *we have*

$$|f(t, x_1, x_2)| \leq k_0 + k_1|x_1| + k_2|x_2|,$$

$$|g(t, x_1, x_2)| \leq \lambda_0 + \lambda_1|x_1| + \lambda_2|x_2|.$$

In addition, it is assumed that

$$M_1k_1 + M_2\lambda_1 < 1 \ \text{ and } \ M_1k_2 + M_2\lambda_2 < 1,$$

where M_1 *and* M_2 *are given by (6.8) and (6.9) respectively. Then the system (6.1) has at least one solution on* $[1, e]$.

Proof First, we show that the operator $T : X\times Y \to X\times Y$ is completely continuous. By continuity of functions f and g, the operator T is obviously continuous.

Let $\Omega \subset X \times Y$ be bounded. Then there exist positive constants L_1 and L_2 such that

$$|f(t, u(t), v(t))| \leq L_1, \quad |g(t, u(t), v(t))| \leq L_2, \ \ \forall (u, v) \in \Omega.$$

Then, for any $(u, v) \in \Omega$, we have

$$\begin{aligned}
|T_1(u, v)(t)| \leq\ & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{|f(s, u(s), v(s))|}{s} ds \\
& + \frac{1}{|A|}\left[\frac{1}{\Gamma(\gamma+\alpha)} \int_1^{\sigma_1} \left(\log \frac{\sigma_1}{s}\right)^{\gamma+\alpha-1} \frac{|f(s, u(s), v(s))|}{s} ds \right. \\
& \left. + \frac{1}{\Gamma(\alpha)} \int_1^{e} \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{|f(s, u(s), v(s))|}{s} ds\right] \\
\leq\ & \frac{L_1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{1}{s} ds + \frac{L_1}{|A|}\left[\frac{1}{\Gamma(\gamma+\alpha)} \int_1^{\sigma_1} \left(\log \frac{\sigma_1}{s}\right)^{\gamma+\alpha-1} \frac{1}{s} ds \right. \\
& \left. + \frac{1}{\Gamma(\alpha)} \int_1^{e} \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{1}{s} ds\right],
\end{aligned}$$

which implies that

$$\|T_1(u, v)\| \leq L_1\left\{\frac{1}{\Gamma(\alpha+1)} + \frac{1}{|A|}\left(\frac{(\log \sigma_1)^{\gamma+\alpha}}{\Gamma(\gamma+\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\right)\right\} = L_1M_1.$$

Similarly, we get

$$\|T_2(u,v)\| \le L_2\left\{\frac{1}{\Gamma(\beta+1)}+\frac{1}{|B|}\left(\frac{(\log\sigma_2)^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)}+\frac{1}{\Gamma(\beta+1)}\right)\right\} = L_2M_2.$$

Thus, it follows from the above inequalities, that the operator T is uniformly bounded.

Next, we show that T is equicontinuous. Let $\tau_1,\tau_2 \in [1,e]$ with $\tau_1 < \tau_2$. Then, we have

$$\begin{aligned}
&|T_1(u(\tau_2),v(\tau_2)) - T_1(u(\tau_1),v(\tau_1))| \\
&\le \frac{1}{\Gamma(\alpha)}\left|\int_1^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{f(s,u(s),v(s))}{s}ds - \int_1^{\tau_1}\left(\log\frac{\tau_1}{s}\right)^{\alpha-1}\frac{f(s,u(s),v(s))}{s}ds\right| \\
&\quad +\left|\frac{(\log\tau_2)^{\alpha-1}-(\log\tau_1)^{\alpha-1}}{A}\left[\frac{1}{\Gamma(\beta+\alpha)}\int_1^{\sigma_1}\left(\log\frac{\sigma_1}{s}\right)^{\beta+\alpha-1}\frac{f(s,u(s),v(s))}{s}ds\right.\right. \\
&\quad \left.\left.-\frac{1}{\Gamma(\alpha)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{f(s,u(s),v(s))}{s}ds\right]\right| \\
&\le \frac{L_1}{\Gamma(\alpha+1)}[|(\log\tau_2)^{\alpha}-(\log\tau_1)^{\alpha}| + 2(\log(\tau_2/\tau_1))^{\alpha}] \\
&\quad +L_1\left|\frac{(\log\tau_2)^{\alpha-1}-(\log\tau_1)^{\alpha-1}}{A}\left[\frac{1}{\Gamma(\gamma+\alpha+1)}(\log\sigma_1)^{\gamma+\alpha}+\frac{1}{\Gamma(\alpha+1)}\right]\right|.
\end{aligned}$$

Analogously, we can obtain

$$\begin{aligned}
&|T_2(u(\tau_2),v(\tau_2)) - T_2(u(\tau_1),v(\tau_1))| \\
&\le \frac{L_2}{\Gamma(\beta+1)}[|(\log\tau_2)^{\beta}-(\log\tau_1)^{\beta}| + 2(\log(\tau_2/\tau_1))^{\beta}] \\
&\quad +L_2\left|\frac{(\log\tau_2)^{\beta-1}-(\log\tau_1)^{\beta-1}}{B}\left[\frac{1}{\Gamma(\gamma+\beta+1)}(\log\sigma_2)^{\gamma+\beta}+\frac{1}{\Gamma(\beta+1)}\right]\right|.
\end{aligned}$$

Therefore, the operator $T(u,v)$ is equicontinuous, and thus the operator $T(u,v)$ is completely continuous.

Finally, it will be shown that the set $\bar{\mathscr{E}} = \{(u,v) \in X\times Y | (u,v) = \lambda T(u,v),\ 0 \le \lambda \le 1\}$ is bounded. Let $(u,v) \in \bar{\mathscr{E}}$, then $(u,v) = \lambda T(u,v)$. For any $t \in [1,e]$, we have

$$u(t) = \lambda T_1(u,v)(t), \quad v(t) = \lambda T_2(u,v)(t).$$

Then

$$|u(t)| \leq \left\{ \frac{1}{\Gamma(\alpha+1)} + \frac{1}{|A|}\left(\frac{(\log \sigma_1)^{\gamma+\alpha}}{\Gamma(\gamma+\alpha+1)} + \frac{1}{\Gamma(\alpha+1)} \right) \right\} (k_0 + k_1\|u\| + k_2\|v\|)$$

and

$$|v(t)| \leq \left\{ \frac{1}{\Gamma(\beta+1)} + \frac{1}{|B|}\left(\frac{(\log \sigma_2)^{\gamma+\beta}}{\Gamma(\gamma+\beta+1)} + \frac{1}{\Gamma(\beta+1)} \right) \right\} (\lambda_0 + \lambda_1\|u\| + \lambda_2\|v\|).$$

Hence, we have

$$\|u\| \leq M_1(k_0 + k_1\|u\| + k_2\|v\|)$$

and

$$\|v\| \leq M_2(\lambda_0 + \lambda_1\|u\| + \lambda_2\|v\|),$$

which imply that

$$\|u\| + \|v\| \leq (M_1k_0 + M_2\lambda_0) + (M_1k_1 + M_2\lambda_1)\|u\| + (M_1k_2 + M_2\lambda_2)\|v\|.$$

Consequently,

$$\|(u,v)\| \leq \frac{M_1k_0 + M_2\lambda_0}{M_0},$$

for any $t \in [1,e]$, where M_0 is defined by

$$M_0 = \min\{1 - (M_1k_1 + M_2\lambda_1),\ 1 - (M_1k_2 + M_2\lambda_2)\},\ \ k_i,\ \lambda_i \geq 0\ (i = 1,2).$$

This proves that $\bar{\mathscr{E}}$ is bounded. Thus, by Theorem 1.3, the operator T has at least one fixed point. Hence the system (6.1) has at least one solution on $[1,e]$. The proof is complete. □

In the second result, we prove the uniqueness of solutions for the system (6.1) via Banach's contraction mapping principle.

Theorem 6.2 *Assume that:*

(6.2.1) $f, g : [1,e] \times \mathbb{R}^2 \to \mathbb{R}$ *are continuous functions, and there exist positive constants* $m_i, n_i, i = 1,2$ *such that for all* $t \in [1,e]$ *and* $u_i, v_i \in \mathbb{R}, i = 1,2,$

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \leq m_1|u_1 - v_1| + m_2|u_2 - v_2|$$

and

$$|g(t,u_1,u_2)-g(t,v_1,v_2)| \le n_1|u_1-v_1|+n_2|u_2-v_2|.$$

Then the system (6.1) has a unique solution on $[1,e]$ *if*

$$M_1(m_1+m_2)+M_2(n_1+n_2)<1,$$

where M_1 *and* M_2 *are given by (6.8) and (6.9) respectively.*

Proof Define $\sup_{t\in[1,e]} f(t,0,0)=N_1<\infty$ and $\sup_{t\in[1,e]} g(t,0,0)=N_2<\infty$, and choose

$$r \ge \frac{N_1M_1+N_2M_2}{1-M_1(m_1+m_2)-M_2(n_1+n_2)}.$$

We show that $TB_r\subset B_r$, where $B_r=\{(u,v)\in X\times Y:\|(u,v)\|\le r\}$ and the operator T is given by (6.7).

For $(u,v)\in B_r$, we have

$$\begin{aligned}
&|T_1(u,v)(t)|\\
&\le \max_{t\in[1,e]}\Bigg\{\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|f(s,u(s),v(s))|}{s}ds\\
&\quad+\frac{1}{|A|}\Bigg[\frac{1}{\Gamma(\gamma+\alpha)}\int_1^{\sigma_1}\left(\log\frac{\sigma_1}{s}\right)^{\gamma+\alpha-1}\frac{|f(s,u(s),v(s))|}{s}ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{|f(s,u(s),v(s))|}{s}ds\Bigg]\Bigg\}\\
&\le \max_{t\in[1,e]}\Bigg\{\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{\big(|f(s,u(s),v(s))-f(s,0,0)|+|f(s,0,0)|\big)}{s}ds\\
&\quad+\frac{1}{|A|}\Bigg[\frac{1}{\Gamma(\gamma+\alpha)}\int_1^{\sigma_1}\left(\log\frac{\sigma_1}{s}\right)^{\gamma+\alpha-1}\frac{\big(|f(s,u(s),v(s))-f(s,0,0)|+|f(s,0,0)|\big)}{s}ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{\big(|f(s,u(s),v(s))-f(s,0,0)|+|f(s,0,0)|\big)}{s}ds\Bigg]\Bigg\}\\
&\le\left\{\frac{1}{\Gamma(\alpha+1)}+\frac{1}{|A|}\left(\frac{(\log\sigma_1)^{\gamma+\alpha}}{\Gamma(\gamma+\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}\right)\right\}(m_1\|u\|+m_2\|v\|+N_1)\\
&\le M_1[(m_1+m_2)r+N_1].
\end{aligned}$$

Hence

$$\|T_1(u,v)\| \le M_1[(m_1+m_2)r+N_1].$$

In the same way, we can obtain

$$\|T_2(u,v)\| \le M_2[(n_1+n_2)r+N_2].$$

In consequence, we have $\|T(u,v)\| \le r$.

Now for $(u_2, v_2), (u_1, v_1) \in X \times Y$, and for any $t \in [1, e]$, we get

$$\begin{aligned}
&|T_1(u_2,v_2)(t) - T_1(u_1,v_1)(t)| \\
&\le \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{|f(s,u_2(s),v_2(s)) - f(s,u_1(s),v_1(s))|}{s}ds \\
&\quad + \frac{1}{|A|}\left[\frac{1}{\Gamma(\gamma+\alpha)}\int_1^{\sigma_1}\left(\log\frac{\sigma_1}{s}\right)^{\gamma+\alpha-1}\frac{|f(s,u_2(s),v_2(s)) - f(s,u_1(s),v_1(s))|}{s}ds\right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{|f(s,u_2(s),v_2(s)) - f(s,u_1(s),v_1(s))|}{s}ds\right] \\
&\le \left\{\frac{1}{\Gamma(\alpha+1)} + \frac{1}{|A|}\left(\frac{(\log\sigma_1)^{\gamma+\alpha}}{\Gamma(\gamma+\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\right)\right\}(m_1|u_2-u_1| + m_2|v_2-v_1|) \\
&\le M_1(m_1\|u_2-u_1\| + m_2\|v_2-v_1\|) \\
&\le M_1(m_1+m_2)(\|u_2-u_1\| + \|v_2-v_1\|),
\end{aligned}$$

and consequently, we obtain

$$\|T_1(u_2,v_2) - T_1(u_1,v_1)\| \le M_1(m_1+m_2)(\|u_2-u_1\| + \|v_2-v_1\|). \tag{6.10}$$

Similarly, one can find that

$$\|T_2(u_2,v_2) - T_2(u_1,v_1)\| \le M_2(n_1+n_2)(\|u_2-u_1\| + \|v_2-v_1\|). \tag{6.11}$$

Thus it follows from (6.10) and (6.11) that

$$\|T(u_2,v_2) - T(u_1,v_1)\| \le [M_1(m_1+m_2) + M_2(n_1+n_2)](\|u_2-u_1\| + \|v_2-v_1\|).$$

Since $M_1(m_1+m_2) + M_2(n_1+n_2) < 1$, T is a contraction. So, by Banach's fixed point theorem, the operator T has a unique fixed point, which is the unique solution of problem (6.1). This completes the proof. □

Example 6.1 Consider the following system of Hadamard differential equations and integral boundary conditions

$$\begin{cases} D^{3/2}x(t) = \dfrac{1}{4(t+2)^2}\dfrac{|u(t)|}{1+|u(t)|} + 1 + \dfrac{1}{32}\sin^2 v(t), & t \in [1, e], \\ D^{3/2}x(t) = \dfrac{1}{32\pi}\sin(2\pi u(t)) + \dfrac{|v(t)|}{16(1+|v(t)|)} + \dfrac{1}{2}, & t \in [1, e], \\ u(1) = 0, \quad u(e) = I^{3/2}u(2), \\ v(1) = 0, \quad v(e) = I^{3/2}v(5/2). \end{cases} \tag{6.12}$$

Here $\alpha = 3/2,\ \beta = 3/2,\ \gamma = 3/2,\ \sigma_1 = 2,\ \sigma_2 = 5/2,\ f(t, u, v) = \dfrac{1}{4(t+2)^2}\dfrac{|u|}{1+|u|} + 1 + \dfrac{1}{32}\sin^2 v$ and $g(t, u, v) = \dfrac{1}{32\pi}\sin(2\pi u) + \dfrac{|v|}{16(1+|v|)} + \dfrac{1}{2}$. With the given data, we find that $A \approx 1.27, B \approx 1.59$,

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \le \frac{1}{16}|u_1 - u_2| + \frac{1}{16}|v_1 - v_2|,$$

$$|g(t, u_1, u_2) - g(t, v_1, v_2)| \le \frac{1}{16}|u_1 - u_2| + \frac{1}{16}|v_1 - v_2|,$$

and

$$M_1(m_1 + m_2) + M_2(n_1 + n_2) \approx 0.43 < 1.$$

Thus all the conditions of Theorem 6.2 are satisfied and consequently, its conclusion applies to the problem (6.12).

6.3 A Coupled System of Riemann-Liouville Fractional Differential Equations with Coupled and Uncoupled Hadamard Fractional Integral Boundary Conditions

In this section, we investigate the existence of solutions for a coupled system of Riemann-Liouville fractional differential equations supplemented with coupled and uncoupled Hadamard fractional integral boundary conditions.

6.3.1 Coupled Integral Boundary Conditions Case

Consider a boundary value problem of coupled nonlinear Riemann-Liouville fractional differential equations and nonlocal coupled Hadamard fractional integral boundary conditions of the form

$$\begin{cases} {}_{RL}D^q x(t) = f(t, x(t), y(t)), & t \in [0, T], \quad 1 < q \le 2, \\ {}_{RL}D^p y(t) = g(t, x(t), y(t)), & t \in [0, T], \quad 1 < p \le 2, \\ x(0) = 0, \quad x(T) = \displaystyle\sum_{i=1}^{n} \alpha_{i\,H} I^{\rho_i} y(\eta_i), \\ y(0) = 0, \quad y(T) = \displaystyle\sum_{j=1}^{m} \beta_{j\,H} I^{\gamma_j} x(\theta_j), \end{cases} \tag{6.13}$$

where ${}_{RL}D^q$, ${}_{RL}D^p$ are the standard Riemann-Liouville fractional derivatives of orders q, p, ${}_H I^{\rho_i}$, ${}_H I^{\gamma_j}$ are the Hadamard fractional integrals of orders $\rho_i, \gamma_j > 0$, $\eta_i, \theta_j \in (0, T)$, $f, g : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ and $\alpha_i, \beta_j \in \mathbb{R}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, are real constants such that $\displaystyle\sum_{i=1}^{n} \frac{\alpha_i \eta_i^{p-1}}{(p-1)^{\rho_i}} \sum_{j=1}^{m} \frac{\beta_j \theta_j^{q-1}}{(q-1)^{\gamma_j}} \neq T^{q+p-2}$.

Lemma 6.2 *Given $\phi, \psi \in C([0, T], \mathbb{R})$, the solution of the problem*

$$\begin{cases} {}_{RL}D^q x(t) = \phi(t), & t \in [0, T], \quad 1 < q \le 2, \\ {}_{RL}D^p y(t) = \psi(t), & t \in [0, T], \quad 1 < p \le 2, \\ x(0) = 0, \quad x(T) = \displaystyle\sum_{i=1}^{n} \alpha_{i\,H} I^{\rho_i} y(\eta_i), \\ y(0) = 0, \quad y(T) = \displaystyle\sum_{j=1}^{m} \beta_{j\,H} I^{\gamma_j} x(\theta_j), \end{cases} \tag{6.14}$$

is equivalent to the integral equations

$$\begin{aligned} x(t) = {}& {}_{RL}I^q \phi(t) - \frac{t^{q-1}}{\Omega} \Bigg[\sum_{i=1}^{n} \frac{\alpha_i \eta_i^{p-1}}{(p-1)^{\rho_i}} \Bigg(\sum_{j=1}^{m} \beta_{j\,H} I^{\gamma_j}{}_{RL} I^q \phi(\theta_j) - {}_{RL}I^p \psi(T) \Bigg) \\ & + T^{p-1} \Bigg(\sum_{i=1}^{n} \alpha_{i\,H} I^{\rho_i}{}_{RL} I^p \psi(\eta_i) - {}_{RL}I^q \phi(T) \Bigg) \Bigg], \end{aligned} \tag{6.15}$$

and

$$\begin{aligned} y(t) = {}& {}_{RL}I^p \psi(t) - \frac{t^{p-1}}{\Omega} \Bigg[\sum_{j=1}^{m} \frac{\beta_j \theta_j^{q-1}}{(q-1)^{\gamma_j}} \Bigg(\sum_{i=1}^{n} \alpha_{i\,H} I^{\rho_i}{}_{RL} I^p \psi(\eta_i) - {}_{RL}I^q \phi(T) \Bigg) \\ & + T^{q-1} \Bigg(\sum_{j=1}^{m} \beta_{j\,H} I^{\gamma_j}{}_{RL} I^q \phi(\theta_j) - {}_{RL}I^p \psi(T) \Bigg) \Bigg], \end{aligned} \tag{6.16}$$

where

$$\Omega := \sum_{i=1}^{n} \frac{\alpha_i \eta_i^{p-1}}{(p-1)^{\rho_i}} \sum_{j=1}^{m} \frac{\beta_j \theta_j^{q-1}}{(q-1)^{\gamma_j}} - T^{q+p-2} \neq 0. \tag{6.17}$$

Proof Using Lemmas 1.4 and 1.5, the equations in (6.14) can be expressed into equivalent integral equations:

$$x(t) = {}_{RL}I^q\phi(t) - c_1t^{q-1} - c_2t^{q-2}, \tag{6.18}$$

$$y(t) = {}_{RL}I^p\psi(t) - d_1t^{p-1} - d_2t^{p-2}, \tag{6.19}$$

for $c_1, c_2, d_1, d_2 \in \mathbb{R}$. The conditions $x(0) = 0,\ y(0) = 0$ imply that $c_2 = 0, d_2 = 0$. Taking the Hadamard fractional integral of order $\rho_i > 0$ of (6.18) and $\gamma_j > 0$ of (6.19) and using the property of the Hadamard fractional integral given in Lemma 1.6, we get the system

$$\begin{aligned} {}_{RL}I^q\phi(T) - c_1T^{q-1} &= \sum_{i=1}^{n}\alpha_{iH}I^{\rho_i}{}_{RL}I^p\psi(\eta_i) - d_1\sum_{i=1}^{n}\frac{\alpha_i\eta_i^{p-1}}{(p-1)^{\rho_i}}, \\ {}_{RL}I^p\psi(T) - d_1T^{p-1} &= \sum_{j=1}^{m}\beta_{jH}I^{\gamma_j}{}_{RL}I^q\phi(\theta_j) - c_1\sum_{j=1}^{m}\frac{\beta_j\theta_j^{q-1}}{(q-1)^{\gamma_j}}, \end{aligned}$$

which, on solving for c_1 and d_1, yields

$$\begin{aligned} c_1 &= \frac{1}{\Omega}\Bigg[\sum_{i=1}^{n}\frac{\alpha_i\eta_i^{p-1}}{(p-1)^{\rho_i}}\left(\sum_{j=1}^{m}\beta_{jH}I^{\gamma_j}{}_{RL}I^q\phi(\theta_j) - {}_{RL}I^p\psi(T)\right) \\ &\quad + T^{p-1}\left(\sum_{i=1}^{n}\alpha_{iH}I^{\rho_i}{}_{RL}I^p\psi(\eta_i) - {}_{RL}I^q\phi(T)\right)\Bigg] \end{aligned}$$

and

$$\begin{aligned} d_1 &= \frac{1}{\Omega}\Bigg[\sum_{j=1}^{m}\frac{\beta_j\theta_j^{q-1}}{(q-1)^{\gamma_j}}\left(\sum_{i=1}^{n}\alpha_{iH}I^{\rho_i}{}_{RL}I^p\psi(\eta_i) - {}_{RL}I^q\phi(T)\right) \\ &\quad + T^{q-1}\left(\sum_{j=1}^{m}\beta_{jH}I^{\gamma_j}{}_{RL}I^q\phi(\theta_j) - {}_{RL}I^p\psi(T)\right)\Bigg]. \end{aligned}$$

Substituting the values of c_1, c_2, d_1 and d_2 in (6.18) and (6.19), we obtain (6.15) and (6.16). The converse follows by direct computation. This completes the proof. □

Throughout this subsection, we use the notations:

$${}_{RL}I^w h(s, x(s), y(s))(v) = \frac{1}{\Gamma(w)}\int_0^v (v-s)^{w-1}h(s, x(s), y(s))ds,$$

and

$$
\begin{aligned}
&{}_HI^u{}_{RL}I^w h(s,x(s),y(s))(v)\\
&\quad = \frac{1}{\Gamma(u)\Gamma(w)}\int_0^v\int_0^t\left(\log\frac{v}{t}\right)^{u-1}(t-s)^{w-1}h(s,x(s),y(s))ds\frac{dt}{t},
\end{aligned}
$$

where $u \in \{\rho_i, \gamma_j\}$, $v \in \{t, T, \eta_i, \theta_j\}$, $w = \{p, q\}$ and $h = \{f, g\}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

Denote by $X = \{u(t)|u(t) \in C([0,T],\mathbb{R})\}$ the Banach space endowed with the norm $\|u\| = \max\{|u(t)|, t \in [0,T]\}$ and similarly we can define a Banach space Y. In view of Lemma 6.2, we define an operator $\mathscr{T} : X \times Y \to X \times Y$ by

$$
\mathscr{T}(x,y)(t) = \begin{pmatrix} \mathscr{T}_1(x,y)(t) \\ \mathscr{T}_2(x,y)(t) \end{pmatrix}, \tag{6.20}
$$

where

$$
\begin{aligned}
&\mathscr{T}_1(x,y)(t)\\
&= {}_{RL}I^q f(s,x(s),y(s))(t)\\
&\quad -\frac{t^{q-1}}{\Omega}\Bigg[\sum_{i=1}^n \frac{\alpha_i\eta_i^{p-1}}{(p-1)^{\rho_i}}\left(\sum_{j=1}^m \beta_{jH}I^{\gamma_j}{}_{RL}I^q f(s,x(s),y(s))(\theta_j) - {}_{RL}I^p g(s,x(s),y(s))(T)\right)\\
&\quad +T^{p-1}\left(\sum_{i=1}^n \alpha_{iH}I^{\rho_i}{}_{RL}I^p g(s,x(s),y(s))(\eta_i) - {}_{RL}I^q f(s,x(s),y(s))(T)\right)\Bigg],
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathscr{T}_2(x,y)(t)\\
&= {}_{RL}I^p g(s,x(s),y(s))(t)\\
&\quad -\frac{t^{p-1}}{\Omega}\Bigg[\sum_{j=1}^m \frac{\beta_j\theta_j^{q-1}}{(q-1)^{\gamma_j}}\left(\sum_{i=1}^n \alpha_{iH}I^{\rho_i}{}_{RL}I^p g(s,x(s),y(s))(\eta_i) - {}_{RL}I^q f(s,x(s),y(s))(T)\right)\\
&\quad +T^{q-1}\left(\sum_{j=1}^m \beta_{jH}I^{\gamma_j}{}_{RL}I^q f(s,x(s),y(s))(\theta_j) - {}_{RL}I^p g(s,x(s),y(s))(T)\right)\Bigg].
\end{aligned}
$$

For computational convenience, we set

$$
\hat{M}_1 = \frac{T^q}{\Gamma(q+1)} + \frac{T^{q-1}}{|\Omega|\Gamma(q+1)}\sum_{i=1}^n \frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\sum_{j=1}^m \frac{|\beta_j|\theta_j^q}{q^{\gamma_j}} + \frac{T^{2q+p-2}}{|\Omega|\Gamma(q+1)}, \tag{6.21}
$$

$$\hat{M}_2 = \frac{T^{q+p-1}}{|\Omega|\Gamma(p+1)} \sum_{i=1}^{n} \frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}} + \frac{T^{q+p-2}}{|\Omega|\Gamma(p+1)} \sum_{i=1}^{n} \frac{|\alpha_i|\eta_i^{p}}{p^{\rho_i}}, \tag{6.22}$$

$$\hat{M}_3 = \frac{T^{q+p-1}}{|\Omega|\Gamma(q+1)} \sum_{j=1}^{m} \frac{|\beta_j|\theta_j^{q-1}}{(q-1)^{\gamma_j}} + \frac{T^{q+p-2}}{|\Omega|\Gamma(q+1)} \sum_{j=1}^{m} \frac{|\beta_j|\theta_j^{q}}{q^{\gamma_j}}, \tag{6.23}$$

$$\hat{M}_4 = \frac{T^{p}}{\Gamma(p+1)} + \frac{T^{p-1}}{|\Omega|\Gamma(p+1)} \sum_{i=1}^{n} \frac{|\alpha_i|\eta_i^{p}}{p^{\rho_i}} \sum_{j=1}^{m} \frac{|\beta_j|\theta_j^{q-1}}{(q-1)^{\gamma_j}} + \frac{T^{q+2p-2}}{|\Omega|\Gamma(p+1)}. \tag{6.24}$$

The first result is concerned with the existence and uniqueness of solutions for the problem (6.13) and is based on Banach's contraction mapping principle.

Theorem 6.3 *Assume that (6.2.1) holds. In addition, assume that*

$$(\hat{M}_1 + \hat{M}_3)(m_1 + m_2) + (\hat{M}_2 + \hat{M}_4)(n_1 + n_2) < 1,$$

where $\hat{M}_i, i = 1, 2, 3, 4$ *are given by (6.21)–(6.24). Then the system (6.13) has a unique solution on* $[0, T]$.

Proof Define $\sup_{t\in[0,T]} f(t, 0, 0) = N_1 < \infty$ and $\sup_{t\in[0,T]} g(t, 0, 0) = N_2 < \infty$ and choose a positive number r such that

$$r \geq \frac{(\hat{M}_1 + \hat{M}_3)N_1 + (\hat{M}_2 + \hat{M}_4)N_2}{1 - (\hat{M}_1 + \hat{M}_3)(m_1 + m_2) - (\hat{M}_2 + \hat{M}_4)(n_1 + n_2)}. \tag{6.25}$$

Now we show that $\mathscr{T} B_r \subset B_r$, where $B_r = \{(x, y) \in X \times Y : \|(x, y)\| \leq r\}$ and the operator $\mathscr{T}$ is defined by (6.20).

For $(x, y) \in B_r$, we have

$$\begin{aligned}
&|\mathscr{T}_1(x, y)(t)| \\
&= \sup_{t\in[0,T]} \Bigg\{ {}_{RL}I^q f(s, x(s), y(s))(t) - \frac{t^{q-1}}{\Omega} \Bigg[\sum_{i=1}^{n} \frac{\alpha_i \eta_i^{p-1}}{(p-1)^{\rho_i}} \\
&\quad \times \Bigg(\sum_{j=1}^{m} \beta_{jH} I^{\gamma_j}{}_{RL}I^q f(s, x(s), y(s))(\theta_j) - {}_{RL}I^p g(s, x(s), y(s))(T) \Bigg) \\
&\quad + T^{p-1} \Bigg(\sum_{i=1}^{n} \alpha_{iH} I^{\rho_i}{}_{RL}I^p g(s, x(s), y(s))(\eta_i) - {}_{RL}I^q f(s, x(s), y(s))(T) \Bigg) \Bigg] \Bigg\} \\
&\leq {}_{RL}I^q (|f(s, x(s), y(s)) - f(s, 0, 0)| + |f(s, 0, 0)|)(T) \\
&\quad + \frac{T^{q-1}}{|\Omega|} \Bigg[\sum_{i=1}^{n} \frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}} \Bigg(\sum_{j=1}^{m} |\beta_j|_H I^{\gamma_j}{}_{RL}I^q (|f(s, x(s), y(s)) - f(s, 0, 0)|
\end{aligned}$$

$$
\begin{aligned}
&+|f(s,0,0)|)(\theta_j) + {}_{RL}I^p(|g(s,x(s),y(s)) - g(s,0,0)| + |g(s,0,0)|)(T)\Bigg)\\
&+T^{p-1}\Bigg(\sum_{i=1}^{n} |\alpha_i|{}_HI^{\rho_i}{}_{RL}I^p(|g(s,x(s),y(s)) - g(s,0,0)| + |g(s,0,0)|)(\eta_i)\\
&+{}_{RL}I^q(|f(s,x(s),y(s)) - f(s,0,0)| + |f(s,0,0)|)(T)\Bigg)\Bigg]\\
\le\ & {}_{RL}I^q(m_1\|x\| + m_2\|y\| + N_1)(T) + \frac{T^{q-1}}{|\Omega|}\Bigg[\sum_{i=1}^{n} \frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\\
&\times\Bigg(\sum_{j=1}^{m} |\beta_j|{}_HI^{\gamma_j}{}_{RL}I^q(m_1\|x\| + m_2\|y\| + N_1)(\theta_j) + {}_{RL}I^p(n_1\|x\| + n_2\|y\| + N_2)(T)\Bigg)\\
&+T^{p-1}\Bigg(\sum_{i=1}^{n} |\alpha_i|{}_HI^{\rho_i}{}_{RL}I^p(n_1\|x\| + n_2\|y\| + N_2)(\eta_i)\\
&\qquad+{}_{RL}I^q(m_1\|x\| + m_2\|y\| + N_1)(T)\Bigg)\Bigg]\\
=\ & (m_1\|x\| + m_2\|y\| + N_1)\Bigg[{}_{RL}I^q(1)(T) + \frac{T^{q-1}}{|\Omega|}\sum_{i=1}^{n} \frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\sum_{j=1}^{m} |\beta_j|{}_HI^{\gamma_j}{}_{RL}I^q(1)(\theta_j)\\
&+\frac{T^{q+p-2}}{|\Omega|}{}_{RL}I^q(1)(T)\Bigg] + (n_1\|x\| + n_2\|y\| + N_2)\Bigg[\frac{T^{q-1}}{|\Omega|}\sum_{i=1}^{n} \frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}{}_{RL}I^p(1)(T)\\
&+\frac{T^{q+p-2}}{|\Omega|}\sum_{i=1}^{n} |\alpha_i|{}_HI^{\rho_i}{}_{RL}I^p(1)(\eta_i)\Bigg]\\
=\ & (m_1\|x\| + m_2\|y\| + N_1)\Bigg[\frac{T^q}{\Gamma(q+1)} + \frac{T^{q-1}}{|\Omega|\Gamma(q+1)}\sum_{i=1}^{n} \frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\sum_{j=1}^{m} \frac{|\beta_j|\theta_j^q}{q^{\gamma_j}}\\
&+\frac{T^{2q+p-2}}{|\Omega|\Gamma(q+1)}\Bigg] + (n_1\|x\| + n_2\|y\| + N_2)\Bigg[\frac{T^{q+p-1}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n} \frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\\
&+\frac{T^{q+p-2}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n} \frac{|\alpha_i|\eta_i^{p}}{p^{\rho_i}}\Bigg]\\
=\ & \hat{M}_1(m_1\|x\| + m_2\|y\| + N_1) + \hat{M}_2(n_1\|x\| + n_2\|y\| + N_2)\\
=\ & (\hat{M}_1m_1 + \hat{M}_2n_1)\|x\| + (\hat{M}_1m_2 + \hat{M}_2n_2)\|y\| + \hat{M}_1N_1 + \hat{M}_2N_2\\
\le\ & (\hat{M}_1m_1 + \hat{M}_2n_1 + \hat{M}_1m_2 + \hat{M}_2n_2)r + \hat{M}_1N_1 + \hat{M}_2N_2.
\end{aligned}
$$

Hence

$$\|\mathscr{T}_1(x,y)\| \le [\hat{M}_1(m_1+m_2)+\hat{M}_2(n_1+n_2)]r+\hat{M}_1N_1+\hat{M}_2N_2.$$

In a similar manner, we can obtain

$$\|\mathscr{T}_2(x,y)\| \le [\hat{M}_3(m_1+m_2)+\hat{M}_4(n_1+n_2)]r+\hat{M}_3N_1+\hat{M}_4N_2.$$

Consequently, $\|\mathscr{T}(x,y)\| \le r$ by (6.25).

Now for $(x_2,y_2),(x_1,y_1)\in X\times Y$, and for any $t\in[0,T]$, we get

$$\begin{aligned}
&|\mathscr{T}_1(x_2,y_2)(t)-\mathscr{T}_1(x_1,y_1)(t)|\\
\le\ & {}_{RL}I^q|f(s,x_2(s),y_2(s))-f(s,x_1(s),y_1(s))|(T)\\
&+\frac{T^{q-1}}{\Omega}\Bigg[\sum_{i=1}^{n}\frac{\alpha_i\eta_i^{p-1}}{(p-1)^{\rho_i}}\Bigg(\sum_{j=1}^{m}\beta_j{}_HI^{\gamma_j}{}_{RL}I^q|f(s,x_2(s),y_2(s))-f(s,x_1(s),y_1(s))|(\theta_j)\\
&+{}_{RL}I^p|g(s,x_2(s),y_2(s))-g(s,x_1(s),y_1(s))|(T)\Bigg)\\
&+T^{p-1}\Bigg(\sum_{i=1}^{n}\alpha_i\,{}_HI^{\rho_i}{}_{RL}I^p|g(s,x_2(s),y_2(s))-g(s,x_1(s),y_1(s))|(\eta_i)\\
&+{}_{RL}I^q|f(s,x_2(s),y_2(s))-f(s,x_1(s),y_1(s))|(T)\Bigg)\Bigg]\\
\le\ & (m_1\|x_2-x_1\|+m_2\|y_2-y_1\|)\Bigg[\frac{T^q}{\Gamma(q+1)}+\frac{T^{q-1}}{|\Omega|\Gamma(q+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^q}{q^{\gamma_j}}\\
&+\frac{T^{2q+p-2}}{|\Omega|\Gamma(q+1)}\Bigg]+(n_1\|x_2-x_1\|+n_2\|y_2-y_1\|)\Bigg[\frac{T^{q+p-1}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\\
&+\frac{T^{q+p-2}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p}}{p^{\rho_i}}\Bigg]\\
=\ & \hat{M}_1(m_1\|x_2-x_1\|+m_2\|y_2-y_1\|)+\hat{M}_2(n_1\|x_2-x_1\|+n_2\|y_2-y_1\|)\\
=\ & (\hat{M}_1m_1+\hat{M}_2n_1)\|x_2-x_1\|+(\hat{M}_1m_2+\hat{M}_2n_2)\|y_2-y_1\|.
\end{aligned}$$

Thus we have

$$\|\mathscr{T}_1(x_2,y_2)-\mathscr{T}_1(x_1,y_1)\| \le (\hat{M}_1m_1+\hat{M}_2n_1+\hat{M}_1m_2+\hat{M}_2n_2)[\|x_2-x_1\|+\|y_2-y_1\|]. \tag{6.26}$$

Similarly, we can find that

$$\|\mathscr{T}_2(x_2,y_2)-\mathscr{T}_2(x_1,y_1)\| \le (\hat{M}_3m_1+\hat{M}_4n_1+M_3m_2+\hat{M}_4n_2)[\|x_2-x_1\|+\|y_2-y_1\|]. \tag{6.27}$$

Hence it follows from (6.26) and (6.27) that

$$\|\mathscr{T}(x_2,y_2)-\mathscr{T}(x_1,y_1)\| \le [(\hat{M}_1+\hat{M}_3)(m_1+m_2)+(\hat{M}_2+\hat{M}_4)(n_1+n_2)](\|x_2-x_1\|+\|y_2-y_1\|).$$

Since $(\hat{M}_1 + \hat{M}_3)(m_1 + m_2) + (\hat{M}_2 + \hat{M}_4)(n_1 + n_2) < 1$, the operator $\mathscr{T}$ is a contraction. So, by Banach's fixed point theorem, the operator $\mathscr{T}$ has a unique fixed point, which corresponds to the unique solution of problem (6.13). This completes the proof. □

Example 6.2 Consider the following system of coupled Riemann-Liouville fractional differential equations with Hadamard type fractional integral boundary conditions

$$\begin{cases} {}_{RL}D^{3/2}x(t) = \dfrac{e^{t^2}}{(t+7)^2}\dfrac{|x(t)|}{(1+|x(t)|)} + \dfrac{\sin^2(2\pi t)}{(3e^t+1)^2}\dfrac{|y(t)|}{(1+|y(t)|)} + \dfrac{1}{3},\ t \in [0,2], \\ {}_{RL}D^{5/4}y(t) = \dfrac{1}{25}\cos x(t) + \dfrac{1}{(t+6)^2}\sin y(t) + 1, \quad t \in [0,2], \\ x(0) = 0, \quad x(2) = \dfrac{3}{2}{}_HI^{1/3}y(2/3) + \sqrt{2}{}_HI^{3/7}y(4/3), \\ y(0) = 0, \quad y(2) = \sqrt{3}{}_HI^{1/4}x(1/2) + \dfrac{1}{2}{}_HI^{4/7}x(1) + 2{}_HI^{7/10}x(3/2). \end{cases} \tag{6.28}$$

Here $q = 3/2, p = 5/4, n = 2, m = 3, T = 2, \alpha_1 = 3/2, \alpha_2 = \sqrt{2}$, $\beta_1 = \sqrt{3}$, $\beta_2 = 1/2, \beta_3 = 2, \rho_1 = 1/3, \rho_2 = 3/7, \gamma_1 = 1/4, \gamma_2 = 4/7, \gamma_3 = 7/10, \eta_1 = 2/3, \eta_2 = 4/3, \theta_1 = 1/2, \theta_2 = 1, \theta_3 = 3/2$,

$$f(t,x,y) = (e^{t^2}|x|)/(((t+7)^2)(1+|x|)) + (\sin^2(2\pi t)|y|)/(((3e^t+1)^2)(1+|y|)) + (1/3)$$

and

$$g(t,x,y) = (\cos x/25) + (\sin y)/((t+6)^2) + 1.$$

Obviously

$$|f(t,x_1,x_2) - f(t,x_2,y_2)| \le ((1/49)|x_1-y_1| + (1/16)|x_2-y_2|)$$

and

$$|g(t,x_1,y_1) - g(t,x_2,y_2)| \le ((1/25)|x_1-y_1| + (1/36)|x_2-y_2|).$$

Using the given data, we find that $\Omega \approx 28.62075873 \neq 0$, $m_1 = 1/49$, $m_2 = 1/16$, $n_1 = 1/25$, $n_2 = 1/36$, $\hat{M}_1 \simeq 2.930183476$, $\hat{M}_2 \simeq 0.64772127$, $\hat{M}_3 \simeq 0.7389741995$, $\hat{M}_4 \simeq 2.829885649$, and

$$(\hat{M}_1 + \hat{M}_3)(m_1 + m_2) + (\hat{M}_2 + \hat{M}_4)(n_1 + n_2) \simeq 0.5399075928 < 1.$$

Thus all the conditions of Theorem 6.3 are satisfied. Therefore, by the conclusion of Theorem 6.3, the problem (6.28) has a unique solution on $[0, 2]$.

In the next result, we prove the existence of solutions for the problem (6.13) by applying Leray-Schauder alternative.

Theorem 6.4 *Suppose that (6.1.1) holds. In addition it is assumed that*

$$(\hat{M}_1 + \hat{M}_3)k_1 + (\hat{M}_2 + \hat{M}_4)\lambda_1 < 1 \ \textit{and} \ (\hat{M}_1 + \hat{M}_3)k_2 + (\hat{M}_2 + \hat{M}_4)\lambda_2 < 1,$$

where $\hat{M}_i, i = 1, 2, 3, 4$ are given by (6.21)–(6.24). Then there exists at least one solution for the system (6.13) on $[0, T]$.

Proof First, we show that the operator $\mathcal{T} : X \times Y \to X \times Y$ defined by (6.20) is completely continuous. By continuity of functions f and g, the operator $\mathcal{T}$ is continuous.

Let $\Theta \subset X \times Y$ be bounded. Then there exist positive constants P_1 and P_2 such that

$$|f(t, x(t), y(t))| \leq P_1, \quad |g(t, x(t), y(t))| \leq P_2, \ \forall (x, y) \in \Theta.$$

Then, for any $(x, y) \in \Theta$, we have

$$\begin{aligned}
&\|\mathcal{T}_1(x, y)(t)\| \\
&\leq {}_{RL}I^q|f(s, x(s), y(s))|(T) + \frac{T^{q-1}}{|\Omega|}\Bigg[\sum_{i=1}^{n} \frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}} \\
&\quad \times \left(\sum_{j=1}^{m} |\beta_j|_H I^{\gamma_j}{}_{RL}I^q|f(s, x(s), y(s))|(\theta_j) + {}_{RL}I^p|g(s, x(s), y(s))|(T)\right) \\
&\quad + T^{p-1}\left(\sum_{i=1}^{n} |\alpha_i|_H I^{\rho_i}{}_{RL}I^p|g(s, x(s), y(s))|(\eta_i) + {}_{RL}I^q|f(s, x(s), y(s))|(T)\right)\Bigg] \\
&\leq \left(\frac{T^{q+p-1}}{|\Omega|\Gamma(p+1)} \sum_{i=1}^{n} \frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}} + \frac{T^{q+p-2}}{|\Omega|\Gamma(p+1)} \sum_{i=1}^{n} \frac{|\alpha_i|\eta_i^{p}}{p^{\rho_i}}\right)P_2 \\
&\quad + \left(\frac{T^q}{\Gamma(q+1)} + \frac{T^{q-1}}{|\Omega|\Gamma(q+1)} \sum_{i=1}^{n} \frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}} \sum_{j=1}^{m} \frac{|\beta_j|\theta_j^q}{q^{\gamma_j}} + \frac{T^{2q+p-2}}{|\Omega|\Gamma(q+1)}\right)P_1,
\end{aligned}$$

which implies that

$$
\begin{aligned}
&\|\mathscr{T}_1(x,y)\| \\
&\le \left(\frac{T^{q+p-1}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}+\frac{T^{q+p-2}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p}}{p^{\rho_i}}\right)P_2 \\
&\quad+\left(\frac{T^q}{\Gamma(q+1)}+\frac{T^{q-1}}{|\Omega|\Gamma(q+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q}}{q^{\gamma_j}}+\frac{T^{2q+p-2}}{|\Omega|\Gamma(q+1)}\right)P_1 \\
&= \hat{M}_2P_2+\hat{M}_1P_1.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
&\|\mathscr{T}_2(x,y)\| \\
&\le \left(\frac{T^{p}}{\Gamma(p+1)}+\frac{T^{p-1}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p}}{p^{\rho_i}}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q-1}}{(q-1)^{\gamma_j}}+\frac{T^{q+2p-2}}{|\Omega|\Gamma(p+1)}\right)L_2 \\
&\quad+\left(\frac{T^{q+p-1}}{|\Omega|\Gamma(q+1)}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q-1}}{(q-1)^{\gamma_j}}+\frac{T^{q+p-2}}{|\Omega|\Gamma(p+1)}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q}}{q^{\gamma_j}}\right)L_1 \\
&= \hat{M}_4P_2+\hat{M}_3P_1.
\end{aligned}
$$

Thus, it follows from the above inequalities that the operator $\mathscr{T}$ is uniformly bounded.

Next, we show that the operator $\mathscr{T}$ is equicontinuous. Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Then, we have

$$
\begin{aligned}
&|\mathscr{T}_1(x(t_2),y(t_2))-\mathscr{T}_1(x(t_1),y(t_1))| \\
&\le \frac{1}{\Gamma(q)}\int_0^{t_1}|(t_2-s)^{q-1}-(t_1-s)^{q-1}||f(s,x(s),y(s))|ds \\
&\quad+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}|f(s,x(s),y(s))|ds+\frac{t_2^{q-1}-t_1^{q-1}}{|\Omega|}\Bigg[\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}} \\
&\quad\times\left(\sum_{j=1}^{m}|\beta_j|{}_HI^{\gamma_j}{}_{RL}I^{q}|f(s,x(s),y(s))|(\theta_j)+{}_{RL}I^{p}|g(s,x(s),y(s))|(T)\right) \\
&\quad+T^{p-1}\left(\sum_{i=1}^{n}|\alpha_i|{}_HI^{\rho_i}{}_{RL}I^{p}|g(s,x(s),y(s))|(\eta_i)+{}_{RL}I^{q}|f(s,x(s),y(s))|(T)\right)\Bigg] \\
&\le \frac{P_1}{\Gamma(q+1)}[2(t_2-t_1)^q+|t_2^q-t_1^q|]
\end{aligned}
$$

$$+\frac{t_2^{q-1}-t_1^{q-1}}{|\Omega|}\Bigg[\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\left(\frac{P_1}{\Gamma(q+1)}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^q}{q^{\gamma_j}}+\frac{T^p}{\Gamma(p+1)}P_2\right)$$
$$+T^{p-1}\left(\frac{P_2}{\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^p}{p^{\rho_i}}+\frac{T^q}{\Gamma(q+1)}P_1\right)\Bigg].$$

Analogously, we can obtain

$$\begin{aligned}&|\mathcal{T}_2(x(t_2),y(t_2))-\mathcal{T}_2(x(t_1),y(t_1))|\\ \leq\ &\frac{P_2}{\Gamma(p+1)}[2(t_2-t_1)^p+|t_2^p-t_1^p|]\\ &+\frac{t_2^{q-1}-t_1^{q-1}}{|\Omega|}\Bigg[\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q-1}}{(q-1)^{\gamma_j}}\left(\frac{P_2}{\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^p}{p^{\rho_i}}+\frac{T^q}{\Gamma(q+1)}P_1\right)\\ &+T^{q-1}\left(\frac{P_1}{\Gamma(q+1)}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^q}{q^{\gamma_j}}+\frac{T^p}{\Gamma(p+1)}P_2\right)\Bigg].\end{aligned}$$

Therefore, the operator $\mathcal{T}(x,y)$ is equicontinuous, and hence it is completely continuous.

Finally, it will be verified that the set $\bar{\mathcal{E}}=\{(x,y)\in X\times Y|(x,y)=\lambda\mathcal{T}(x,y),\ 0\leq\lambda\leq 1\}$ is bounded. Let $(x,y)\in\bar{\mathcal{E}}$, then $(x,y)=\lambda\mathcal{T}(x,y)$. For any $t\in[0,T]$, we have

$$x(t)=\lambda\mathcal{T}_1(x,y)(t),\quad y(t)=\lambda\mathcal{T}_2(x,y)(t).$$

Then

$$\begin{aligned}|x(t)|\leq(k_0+k_1\|x\|+k_2\|y\|)&\left(\frac{T^q}{\Gamma(q+1)}+\frac{T^{q-1}}{|\Omega|\Gamma(q+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^q}{q^{\gamma_j}}\right.\\ &\left.+\frac{T^{2q+p-2}}{|\Omega|\Gamma(q+1)}\right)+(\lambda_0+\lambda_1\|x\|+\lambda_2\|y\|)\left(\frac{T^{q+p-1}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^{p-1}}{(p-1)^{\rho_i}}\right.\\ &\left.+\frac{T^{q+p-2}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^p}{p^{\rho_i}}\right)\end{aligned}$$

and

$$|y(t)|\leq(\lambda_0+\lambda_1\|x\|+\lambda_2\|y\|)\left(\frac{T^p}{\Gamma(p+1)}+\frac{T^{p-1}}{|\Omega|\Gamma(p+1)}\sum_{i=1}^{n}\frac{|\alpha_i|\eta_i^p}{p^{\rho_i}}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q-1}}{(q-1)^{\gamma_j}}\right.$$

$$+\frac{T^{q+2p-2}}{|\Omega|\Gamma(p+1)}\Bigg) + (k_0 + k_1\|x\| + k_2\|y\|)\Bigg(\frac{T^{q+p-1}}{|\Omega|\Gamma(q+1)}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q-1}}{(q-1)^{\gamma_j}}$$
$$+\frac{T^{q+p-2}}{|\Omega|\Gamma(q+1)}\sum_{j=1}^{m}\frac{|\beta_j|\theta_j^{q}}{q^{\gamma_j}}\Bigg).$$

In consequence, we get

$$\|x\| \le (k_0 + k_1\|x\| + k_2\|y\|)\hat{M}_1 + (\lambda_0 + \lambda_1\|x\| + \lambda_2\|y\|)\hat{M}_2$$

and

$$\|y\| \le (k_0 + k_1\|x\| + k_2\|y\|)\hat{M}_3 + (\lambda_0 + \lambda_1\|x\| + \lambda_2\|y\|)\hat{M}_4,$$

which imply that

$$\|x\| + \|y\| \le (\hat{M}_1 + \hat{M}_3)k_0 + (\hat{M}_2 + \hat{M}_4)\lambda_0 + [(\hat{M}_1 + \hat{M}_3)k_1 + (\hat{M}_2 + \hat{M}_4)\lambda_1]\|x\|$$
$$+[(\hat{M}_1 + \hat{M}_3)k_2 + (\hat{M}_2 + \hat{M}_4)\lambda_2]\|y\|.$$

Consequently,

$$\|(x,y)\| \le \frac{(\hat{M}_1 + \hat{M}_3)k_0 + (\hat{M}_2 + \hat{M}_4)\lambda_0}{\hat{M}_0},$$

where

$$\hat{M}_0 = \min\{1 - (\hat{M}_1 + \hat{M}_3)k_1 - (\hat{M}_2 + \hat{M}_4)\lambda_1,\ 1 - (\hat{M}_1 + \hat{M}_3)k_2 - (\hat{M}_2 + \hat{M}_4)\lambda_2\},$$

$k_i,\ \lambda_i \ge 0$ $(i = 1, 2)$, which shows that $\bar{\mathscr{E}}$ is bounded. Thus, by Theorem 1.3, the operator $\mathscr{T}$ has at least one fixed point. Hence the system (6.13) has at least one solution on $[0, T]$. The proof is complete. □

Example 6.3 Consider the following system of coupled Riemann-Liouville fractional differential equations supplemented with Hadamard type fractional integral boundary conditions

$$\begin{cases} {}_{RL}D^{\sqrt{2}}x(t) = 1 + \dfrac{\sqrt{2}}{81}x(t)\cos y(t) + \dfrac{\sqrt{3}}{36\pi}y(t),\ t \in [0,\pi], \\ {}_{RL}D^{\sqrt{3}}y(t) = \dfrac{3}{2} + \dfrac{\sqrt{3}}{64\pi}\sin x(t) + \dfrac{1}{63\pi}y(t),\ t \in [0,\pi], \\ x(0) = 0, \quad x(\pi) = \dfrac{\sqrt{3}}{2}{}_HI^{1/2}y(\pi/4) + \dfrac{2}{17}{}_HI^{2/3}y(\pi/3) + \dfrac{4}{9}{}_HI^{3/4}y(\pi/2), \\ y(0) = 0, \quad y(\pi) = \dfrac{1}{2}{}_HI^{3/5}x(\pi/6) + \dfrac{\sqrt{5}}{14}{}_HI^{5/6}x(\pi/3). \end{cases} \tag{6.29}$$

Here $q = \sqrt{2}, p = \sqrt{3}, n = 3, m = 2, T = \pi, \alpha_1 = \sqrt{3}/2, \alpha_2 = 2/17$, $\alpha_3 = 4/9$, $\beta_1 = 1/2, \beta_2 = \sqrt{5}/14, \rho_1 = 1/2, \rho_2 = 2/3, \rho_3 = 3/4$, $\gamma_1 = 3/5$, $\gamma_2 = 5/6, \eta_1 = \pi/4, \eta_2 = \pi/3, \eta_3 = \pi/2, \theta_1 = \pi/6, \theta_2 = \pi/3$, $f(t,x,y) = 1 + (\sqrt{2}x\cos y)/(81) + (\sqrt{3}y)/(36\pi)$ and $g(t,x,y) = (3/2) + (\sqrt{3}\sin x)/(64\pi) + (y)/(63)$. By using computer program, we get $\Omega \approx -1.955428761 \neq 0$. Since $|f(t,x,y)| \leq k_0 + k_1|x| + k_2|y|, |g(t,x,y)| \leq \lambda_0 + \lambda_1|x| + \lambda_2|y|$, where $k_0 = 1, k_1 = \sqrt{2}/81, k_2 = \sqrt{3}/36\pi, \lambda_0 = 3/2, \lambda_1 = \sqrt{3}/64\pi$, $\lambda_2 = 1/63\pi$, it is found that $\hat{M}_1 \simeq 12.01088124, \hat{M}_2 \simeq 8.095664081, \hat{M}_3 \simeq 5.051706267, \hat{M}_4 \simeq 14.14407333$. Furthermore, $(\hat{M}_1 + \hat{M}_3)k_1 + (\hat{M}_2 + \hat{M}_4)\lambda_1 \approx 0.6297371340 < 1$, and $(\hat{M}_1 + \hat{M}_3)k_2 + (\hat{M}_2 + \hat{M}_4)\lambda_2 \approx 0.3736753802 < 1$. Thus all the conditions of Theorem 6.4 hold true and consequently the conclusion of Theorem 6.4, applies to the problem (6.29) on $[0, \pi]$.

6.3.2 Uncoupled Integral Boundary Conditions Case

In this subsection, we consider the following coupled system of Riemann-Liouville fractional differential equations equipped with uncoupled Hadamard type fractional integral boundary conditions

$$\begin{cases} {}_{RL}D^q x(t) = f(t, x(t), y(t)), & t \in [0,T], \quad 1 < q \leq 2, \\ {}_{RL}D^p y(t) = g(t, x(t), y(t)), & t \in [0,T], \quad 1 < p \leq 2, \\ x(0) = 0, \quad x(T) = \displaystyle\sum_{i=1}^{n} \alpha_{i\,H} I^{\rho_i} x(\eta_i), \\ y(0) = 0, \quad y(T) = \displaystyle\sum_{j=1}^{m} \beta_{j\,H} I^{\gamma_j} y(\theta_j). \end{cases} \tag{6.30}$$

Lemma 6.3 (Auxiliary Lemma) *For $h \in C([0,T], \mathbb{R})$, the solution of the problem*

$$\begin{cases} {}_{RL}D^q x(t) = h(t), \quad 1 < q \leq 2, \quad t \in [0,T], \\ x(0) = 0, \quad x(T) = \displaystyle\sum_{i=1}^{n} \alpha_{i\,H} I^{\rho_i} x(\eta_i), \end{cases} \tag{6.31}$$

is equivalent to the following integral equation

$$x(t) = {}_{RL}I^q h(t) - \frac{t^{q-1}}{\Lambda}\left({}_{RL}I^q h(T) - \sum_{i=1}^{n} \alpha_i ({}_H I^{\rho_i}\,{}_{RL}I^q h)(\eta_i) \right), \tag{6.32}$$

where

$$\Lambda := T^{q-1} - \sum_{i=1}^{n} \frac{\alpha_i \eta_i^{q-1}}{(q-1)^{\rho_i}} \neq 0. \tag{6.33}$$

Similarly, one can find

$$y(t) = {}_{RL}I^p h_1(t) - \frac{t^{p-1}}{\Phi}\left({}_{RL}I^p h_1(T) - \sum_{j=1}^{m} \beta_{jH} I^{\gamma_j}{}_{RL}I^p h_1(\theta_j)\right), \tag{6.34}$$

where

$$\Phi := T^{p-1} - \sum_{j=1}^{m} \frac{\beta_j \theta_j^{p-1}}{(p-1)^{\gamma_j}} \neq 0. \tag{6.35}$$

Next, we define an operator $\mathfrak{T} : X \times Y \to X \times Y$ by

$$\mathfrak{T}(x, y)(t) = \begin{pmatrix} \mathfrak{T}_1(x, y)(t) \\ \mathfrak{T}_2(x, y)(t) \end{pmatrix},$$

where

$$\begin{aligned}\mathfrak{T}_1(x, y)(t) &= {}_{RL}I^q f(s, x(s), y(s))(t) \\ &\quad - \frac{t^{q-1}}{\Lambda}\left({}_{RL}I^q f(s, x(s), y(s))(T) - \sum_{i=1}^{n} \alpha_{iH} I^{\rho_i}{}_{RL}I^q f(s, x(s), y(s))(\eta_i)\right),\end{aligned}$$

and

$$\begin{aligned}\mathfrak{T}_2(x, y)(t) &= {}_{RL}I^p g(s, x(s), y(s))(t) \\ &\quad - \frac{t^{p-1}}{\Phi}\left({}_{RL}I^p g(s, x(s), y(s))(T) - \sum_{j=1}^{m} \beta_{jH} I^{\gamma_j}{}_{RL}I^p g(s, x(s), y(s))(\theta_j)\right),\end{aligned}$$

with Λ and Φ respectively given by (6.33) and (6.35).

In the sequel, we set

$$\delta_1 = \frac{T^q}{\Gamma(q+1)} + \frac{T^{2q-1}}{|\Lambda|\Gamma(q+1)} + \frac{T^{q-1}}{|\Lambda|\Gamma(q+1)} \sum_{i=1}^{n} \frac{|\alpha_i|\eta_i^q}{q^{\rho_i}}, \tag{6.36}$$

$$\delta_2 = \frac{T^p}{\Gamma(p+1)} + \frac{T^{2p-1}}{|\Phi|\Gamma(p+1)} + \frac{T^{p-1}}{|\Phi|\Gamma(p+1)} \sum_{j=1}^{m} \frac{|\beta_j|\theta_j^p}{p^{\gamma_j}}. \tag{6.37}$$

Now, we present the existence and uniqueness result for the problem (6.30). We do not provide the proof of this result as it is similar to that of Theorem 6.3.

Theorem 6.5 *Assume that* $f, g : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ *are continuous functions and there exist positive constants* $\bar{m}_i, \bar{n}_i, i = 1, 2$ *such that for all* $t \in [0, T]$ *and* $x_i, y_i \in \mathbb{R}, i = 1, 2,$

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \leq \bar{m}_1|x_1 - y_1| + \bar{m}_2|x_2 - y_2|$$

and

$$|g(t, x_1, x_2) - g(t, y_1, y_2)| \leq \bar{n}_1|x_1 - y_1| + \bar{n}_2|x_2 - y_2|.$$

In addition, it is assumed that

$$\delta_1(\bar{m}_1 + \bar{m}_2) + \delta_2(\bar{n}_1 + \bar{n}_2) < 1,$$

where δ_1 *and* δ_2 *are given by (6.36) and (6.37) respectively. Then the system (6.30) has a unique solution on* $[0, T]$.

Example 6.4 Consider the following system of coupled Riemann-Liouville fractional differential equations with uncoupled Hadamard type fractional integral boundary conditions

$$\begin{cases} {}_{RL}D^{7/6}x(t) = \dfrac{e^{-t}}{(5+t)^2}\dfrac{|x(t)|}{|x(t)|+1} + \dfrac{1}{(e^t+3)^2}\dfrac{|y(t)|}{|y(t)|+1} + \dfrac{\pi}{2}, \quad t \in [0, 3], \\ {}_{RL}D^{\sqrt{5}/2}y(t) = \dfrac{4|x(t)|}{33(t+1)^2} + \dfrac{2\sin y(t)}{17(e^t+1)} + \sqrt{3}, \quad t \in [0, 3], \\ x(0) = 0, \quad x(3) = \dfrac{1}{6}{}_HI^{\sqrt{2}}x(1/2) - \dfrac{1}{5}{}_HI^{3/4}x(1) + \dfrac{2}{9}{}_HI^{\sqrt{5}}x(3/2), \\ y(0) = 0, \quad y(3) = \dfrac{3}{4}{}_HI^{2/3}y(1/2) + \dfrac{1}{2}{}_HI^{\sqrt{3}}y(3/2) + \dfrac{\pi}{2}{}_HI^{5/4}y(5/3). \end{cases} \tag{6.38}$$

Here $q = 7/6, p = \sqrt{5}/2, n = 3, m = 3, T = 3, \alpha_1 = 1/6, \alpha_2 = -1/5, \alpha_3 = 2/9, \beta_1 = 3/4, \beta_2 = 1/2, \beta_3 = \pi/2, \rho_1 = \sqrt{2}, \rho_2 = 3/4, \rho_3 = \sqrt{5}, \gamma_1 = 2/3, \gamma_2 = \sqrt{3}, \gamma_3 = 5/4, \eta_1 = 1/2, \eta_2 = 1, \eta_3 = 3/2, \theta_1 = 1/2, \theta_2 = 3/2, \theta_3 = 5/3$, $f(t, x, y) = (e^{-t}|x|)/(((5+t)^2)(|x|+1)) + (|y|)/(((e^t+3)^2)(|y|+1)) + (\pi/2)$ and $g(t, x, y) = (4|x|)/(33((5+t)^2)) + (2\sin y(t))/(17(e^t+1)) + \sqrt{3}$. Clearly $|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq ((1/25)|x_1 - x_2| + (1/16)|y_1 - y_2|)$ and $|g(t, x_1, y_1) - g(t, x_2, y_2)| \leq ((4/33)|x_1 - x_2| + (1/17)|y_1 - y_2|)$. Using the given values, we find that $\Lambda \approx -12.96942934 \neq 0$, $\Phi \approx -47.08574657 \neq 0$, $\bar{m}_1 = 1/25, \bar{m}_2 = 1/16, \bar{n}_1 = 4/33, \bar{n}_2 = 1/17, \delta_1 \simeq 3.678923396, \delta_2 \simeq 3.402792438$. Also,

$$\delta_1(\bar{m}_1 + \bar{m}_2) + \delta_2(\bar{n}_1 + \bar{n}_2) \approx 0.9897135986 < 1.$$

Thus all the conditions of Theorem 6.5 are satisfied. Therefore, by Theorem 6.5, there exists a unique solution for the problem (6.38) on $[0, 3]$.

The second result, dealing with the existence of solutions for the problem (6.30), is analogous to Theorem 6.4 and is stated below.

Theorem 6.6 *Assume that there exist real constants* $\kappa_i,\ \nu_i \geq 0$ $(i = 1, 2)$ *and* $\kappa_0 > 0, \nu_0 > 0$ *such that* $\forall x_i \in \mathbb{R}$ $(i = 1, 2)$,

$$|f(t, x_1, x_2)| \leq \kappa_0 + \kappa_1|x_1| + \kappa_2|x_2|,$$
$$|g(t, x_1, x_2)| \leq \nu_0 + \nu_1|x_1| + \nu_2|x_2|.$$

Further suppose that

$$\delta_1\kappa_1 + \delta_2\nu_1 < 1 \ \text{ and } \ \delta_1\kappa_2 + \delta_2\nu_2 < 1,$$

where δ_1 *and* δ_2 *are given by (6.36) and (6.37) respectively. Then the system (6.30) has at least one solution.*

Proof Setting

$$\delta_0 = \min\{1 - (\delta_1\kappa_1 + \delta_2\nu_1),\ 1 - (\delta_1\kappa_2 + \delta_2\nu_2)\},\ \ \kappa_i,\ \nu_i \geq 0\ (i = 1, 2),$$

the proof runs parallel to that of Theorem 6.4. So, we omit it. □

6.4 Mixed Problems of Coupled Systems of Riemann-Liouville Differential Equations and Multiple Hadamard Integral Conditions

The aim of this section is to investigate the existence and uniqueness of solutions for a coupled system of Riemann-Liouville fractional differential equations supplemented with nonlocal multiple Hadamard fractional integral conditions of the form:

$$\begin{cases} {}_{RL}D^p x(t) = f(t, x(t), y(t)), & t \in [0, T], \quad 1 < p \leq 2, \\ {}_{RL}D^q y(t) = g(t, x(t), y(t)), & t \in [0, T], \quad 1 < q \leq 2, \\ x(0) = 0, \quad \displaystyle\sum_{i=1}^{\rho_1} \mu_{iH} I^{\alpha_i} x(\eta_i) = \sum_{j=1}^{\phi_1} \delta_{jH} I^{\beta_j} y(\xi_j) + K_1, \\ y(0) = 0, \quad \displaystyle\sum_{k=1}^{\rho_2} \tau_{kH} I^{\sigma_k} x(\gamma_k) = \sum_{l=1}^{\phi_2} \omega_{lH} I^{\nu_l} y(\theta_l) + K_2, \end{cases} \tag{6.39}$$

where ${}_{RL}D^q,\ {}_{RL}D^p$ are the standard Riemann-Liouville fractional derivative of orders $q, p, f, g : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ are given continuous functions, ${}_HI^{\alpha_i},\ {}_HI^{\beta_j},\ {}_HI^{\sigma_k}$ and

${}_HI^{\nu_l}$ are the Hadamard fractional integral of orders $\alpha_i, \beta_j, \sigma_k, \nu_l > 0$, $K_1, K_2 \in \mathbb{R}$ are given constants, $\eta_i, \xi_j, \gamma_k, \theta_l \in (0,T)$, and $\mu_i, \delta_j, \tau_k, \omega_l \in \mathbb{R}$, for $\rho_1, \rho_2, \phi_1, \phi_2 \in \mathbb{N}$, $i = 1,2,\ldots,\rho_1$, $j = 1,2,\ldots,\phi_1$, $k = 1,2,\ldots,\rho_2$, $l = 1,2,\ldots,\phi_2$ are real constants such that

$$\left(\sum_{i=1}^{\rho_1} \frac{\mu_i \eta_i^{p-1}}{(p-1)^{\alpha_i}}\right)\left(\sum_{l=1}^{\phi_2} \frac{\omega_l \theta_l^{q-1}}{(q-1)^{\nu_l}}\right) \neq \left(\sum_{j=1}^{\phi_1} \frac{\delta_j \xi_j^{q-1}}{(q-1)^{\beta_j}}\right)\left(\sum_{k=1}^{\rho_2} \frac{\tau_k \gamma_k^{p-1}}{(p-1)^{\sigma_k}}\right).$$

Lemma 6.4 *Given $\phi, \psi \in C([0,T],\mathbb{R})$, the problem*

$$\begin{cases} {}_{RL}D^p x(t) = \phi(t), \quad t \in [0,T], \quad 1 < p \le 2, \\ {}_{RL}D^q y(t) = \psi(t), \quad t \in [0,T], \quad 1 < q \le 2, \\ x(0) = 0, \quad \displaystyle\sum_{i=1}^{\rho_1} \mu_{i\,H} I^{\alpha_i} x(\eta_i) = \sum_{j=1}^{\phi_1} \delta_{j\,H} I^{\beta_j} y(\xi_j) + K_1, \\ y(0) = 0, \quad \displaystyle\sum_{k=1}^{\rho_2} \tau_{k\,H} I^{\sigma_k} x(\gamma_k) = \sum_{l=1}^{\phi_2} \omega_{l\,H} I^{\nu_l} y(\theta_l) + K_2, \end{cases} \tag{6.40}$$

is equivalent to the integral equations:

$$\begin{aligned} x(t) &= {}_{RL}I^p \phi(t) \\ &\quad + \frac{t^{p-1}}{\Omega'}\left\{ \sum_{l=1}^{\phi_2} \frac{\omega_l \theta_l^{q-1}}{(q-1)^{\nu_l}} \left(\sum_{l=1}^{\phi_1} \delta_{j\,H} I^{\beta_j}{}_{RL}I^q \psi(\xi_j) - \sum_{i=1}^{\rho_1} \mu_{i\,H} I^{\alpha_i}{}_{RL}I^p \phi(\eta_i) + K_1 \right) \right. \\ &\quad \left. - \sum_{l=1}^{\phi_1} \frac{\delta_j \xi_j^{q-1}}{(q-1)^{\beta_j}} \left(\sum_{l=1}^{\phi_2} \omega_{l\,H} I^{\nu_l}{}_{RL}I^q \psi(\theta_l) - \sum_{i=1}^{\rho_2} \tau_{k\,H} I^{\sigma_k}{}_{RL}I^p \phi(\gamma_k) + K_2 \right) \right\}, \end{aligned} \tag{6.41}$$

and

$$\begin{aligned} y(t) &= {}_{RL}I^q \psi(t) \\ &\quad + \frac{t^{q-1}}{\Omega'}\left\{ \sum_{i=1}^{\rho_2} \frac{\tau_k \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \left(\sum_{l=1}^{\phi_1} \delta_{j\,H} I^{\beta_j}{}_{RL}I^q \psi(\xi_j) - \sum_{i=1}^{\rho_1} \mu_{i\,H} I^{\alpha_i}{}_{RL}I^p \phi(\eta_i) + K_1 \right) \right. \\ &\quad \left. - \sum_{i=1}^{\rho_1} \frac{\mu_i \eta_i^{p-1}}{(p-1)^{\alpha_i}} \left(\sum_{l=1}^{\phi_2} \omega_{l\,H} I^{\nu_l}{}_{RL}I^q \psi(\theta_l) - \sum_{i=1}^{\rho_2} \tau_{k\,H} I^{\sigma_k}{}_{RL}I^p \phi(\gamma_k) + K_2 \right) \right\}, \end{aligned} \tag{6.42}$$

where

$$\Omega_1 = \sum_{i=1}^{\rho_1} \frac{\mu_i \eta_i^{p-1}}{(p-1)^{\alpha_i}} \sum_{l=1}^{\phi_2} \frac{\omega_l \theta_l^{q-1}}{(q-1)^{\nu_l}} - \sum_{l=1}^{\phi_1} \frac{\delta_j \xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{i=1}^{\rho_2} \frac{\tau_k \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \neq 0. \tag{6.43}$$

Proof As argued before, the equations in (6.40) are equivalent to the integral equations

$$x(t) = {}_{RL}I^p\phi(t) + c_1 t^{p-1} + c_2 t^{p-2}, \tag{6.44}$$

$$y(t) = {}_{RL}I^q\psi(t) + d_1 t^{q-1} + d_2 t^{q-2}, \tag{6.45}$$

for $c_1, c_2, d_1, d_2 \in \mathbb{R}$. The conditions $x(0) = 0, y(0) = 0$ imply that $c_2 = 0, d_2 = 0$. Taking the Hadamard fractional integral of order $\alpha_i > 0$, $\sigma_k > 0$ of (6.44) and $\beta_j > 0$, $\nu_l > 0$ of (6.45) and using the property of the Hadamard fractional integrals given in Lemma 1.6, we get the system

$$\sum_{i=1}^{\rho_1}\mu_{iH}I^{\alpha_i}{}_{RL}I^p\phi(\eta_i) + c_1\sum_{i=1}^{\rho_1}\frac{\mu_i\eta_i^{p-1}}{(p-1)^{\alpha_i}} = \sum_{l=1}^{\phi_1}\delta_{jH}I^{\beta_j}{}_{RL}I^q\psi(\xi_j) + d_1\sum_{l=1}^{\phi_1}\frac{\delta_j\xi_j^{q-1}}{(q-1)^{\beta_j}} + K_1,$$

$$\sum_{i=1}^{\rho_2}\tau_{kH}I^{\sigma_k}{}_{RL}I^p\phi(\gamma_k) + c_1\sum_{i=1}^{\rho_2}\frac{\tau_k\gamma_k^{p-1}}{(p-1)^{\sigma_k}} = \sum_{l=1}^{\phi_2}\omega_{lH}I^{\nu_l}{}_{RL}I^q\psi(\theta_l) + d_1\sum_{l=1}^{\phi_2}\frac{\omega_l\theta_l^{q-1}}{(q-1)^{\nu_l}} + K_2.$$

Solving the above system, we have

$$\begin{aligned} c_1 = \frac{1}{\Omega_1}\Bigg\{&\sum_{l=1}^{\phi_2}\frac{\omega_l\theta_l^{q-1}}{(q-1)^{\nu_l}}\left(\sum_{l=1}^{\phi_1}\delta_{jH}I^{\beta_j}{}_{RL}I^q\psi(\xi_j) - \sum_{i=1}^{\rho_1}\mu_{iH}I^{\alpha_i}{}_{RL}I^p\phi(\eta_i) + K_1\right) \\ &-\sum_{l=1}^{\phi_1}\frac{\delta_j\xi_j^{q-1}}{(q-1)^{\beta_j}}\left(\sum_{l=1}^{\phi_2}\omega_{lH}I^{\nu_l}{}_{RL}I^q\psi(\theta_l) - \sum_{i=1}^{\rho_2}\tau_{kH}I^{\sigma_k}{}_{RL}I^p\phi(\gamma_k) + K_2\right)\Bigg\}\end{aligned}$$

and

$$\begin{aligned} d_1 = \frac{1}{\Omega_1}\Bigg\{&\sum_{i=1}^{\rho_2}\frac{\tau_k\gamma_k^{p-1}}{(p-1)^{\sigma_k}}\left(\sum_{l=1}^{\phi_1}\delta_{jH}I^{\beta_j}{}_{RL}I^q\psi(\xi_j) - \sum_{i=1}^{\rho_1}\mu_{iH}I^{\alpha_i}{}_{RL}I^p\phi(\eta_i) + K_1\right) \\ &-\sum_{i=1}^{\rho_1}\frac{\mu_i\eta_i^{p-1}}{(p-1)^{\alpha_i}}\left(\sum_{l=1}^{\phi_2}\omega_{lH}I^{\nu_l}{}_{RL}I^q\psi(\theta_l) - \sum_{i=1}^{\rho_2}\tau_{kH}I^{\sigma_k}{}_{RL}I^p\phi(\gamma_k) + K_2\right)\Bigg\}.\end{aligned}$$

Substituting the values of c_1, c_2, d_1 and d_2 in (6.44) and (6.45), we obtain (6.41) and (6.42). The converse follows by direct computation. This completes the proof. □

In view of Lemma 6.4, we define an operator $\widehat{\mathscr{T}} : X \times Y \to X \times Y$ by

$$\widehat{\mathscr{T}}(x, y)(t) = \begin{pmatrix}\widehat{\mathscr{T}}_1(x, y)(t) \\ \widehat{\mathscr{T}}_2(x, y)(t)\end{pmatrix}, \tag{6.46}$$

where

$$
\begin{aligned}
&\widehat{\mathscr{T}}_1(x,y)(t)\\
&= {}_{RL}I^p f(s,x(s),y(s))(t) + \frac{t^{p-1}}{\Omega_1}\Bigg\{ \sum_{l=1}^{\phi_2} \frac{\omega_l \theta_l^{q-1}}{(q-1)^{\nu_l}} \Bigg(\sum_{l=1}^{\phi_1} \delta_{jH} I^{\beta_j}{}_{RL} I^q g(s,x(s),y(s))(\xi_j)\\
&\quad - \sum_{i=1}^{\rho_1} \mu_{iH} I^{\alpha_i}{}_{RL} I^p f(s,x(s),y(s))(\eta_i) + K_1 \Bigg)\\
&\quad - \sum_{l=1}^{\phi_1} \frac{\delta_j \xi_j^{q-1}}{(q-1)^{\beta_j}} \Bigg(\sum_{l=1}^{\phi_2} \omega_{lH} I^{\nu_l}{}_{RL} I^q g(s,x(s),y(s))(\theta_l)\\
&\quad - \sum_{i=1}^{\rho_2} \tau_{kH} I^{\sigma_k}{}_{RL} I^p f(s,x(s),y(s))(\gamma_k) + K_2 \Bigg)\Bigg\},
\end{aligned}
$$

and

$$
\begin{aligned}
&\widehat{\mathscr{T}}_2(x,y)(t)\\
&= {}_{RL}I^q g(s,x(s),y(s))(t) + \frac{t^{q-1}}{\Omega_1}\Bigg\{ \sum_{i=1}^{\rho_2} \frac{\tau_k \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \Bigg(\sum_{l=1}^{\phi_1} \delta_{jH} I^{\beta_j}{}_{RL} I^q g(s,x(s),y(s))(\xi_j)\\
&\quad - \sum_{i=1}^{\rho_1} \mu_{iH} I^{\alpha_i}{}_{RL} I^p f(s,x(s),y(s))(\eta_i) + K_1 \Bigg)\\
&\quad - \sum_{i=1}^{\rho_1} \frac{\mu_i \eta_i^{p-1}}{(p-1)^{\alpha_i}} \Bigg(\sum_{l=1}^{\phi_2} \omega_{lH} I^{\nu_l}{}_{RL} I^q g(s,x(s),y(s))(\theta_l)\\
&\quad - \sum_{i=1}^{\rho_2} \tau_{kH} I^{\sigma_k}{}_{RL} I^p f(s,x(s),y(s))(\gamma_k) + K_2 \Bigg)\Bigg\}.
\end{aligned}
$$

For the sake of convenience, we set

$$
M_1' = \frac{1}{\Gamma(p+1)}\Bigg(T^p + \frac{T^{p-1}}{|\Omega_1|} \sum_{l=1}^{\phi_2} \frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{i=1}^{\rho_1} \frac{|\mu_i|\eta_i^p}{p^{\alpha_i}} + \frac{T^{p-1}}{|\Omega_1|} \sum_{l=1}^{\phi_1} \frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{i=1}^{\rho_2} \frac{|\tau_k|\gamma_k^p}{p^{\sigma_k}} \Bigg), \tag{6.47}
$$

$$
M_2' = \frac{T^{p-1}}{|\Omega_1|\Gamma(q+1)}\Bigg(\sum_{l=1}^{\phi_2} \frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{l=1}^{\phi_1} \frac{|\delta_j|\xi_j^q}{q^{\beta_j}} + \sum_{l=1}^{\phi_1} \frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{l=1}^{\phi_2} \frac{|\omega_l|\theta_l^q}{q^{\nu_l}} \Bigg), \tag{6.48}
$$

$$
M_3' = \frac{T^{p-1}}{|\Omega_1|}\Bigg(|K_1| \sum_{l=1}^{\phi_2} \frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} + |K_2| \sum_{l=1}^{\phi_1} \frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}} \Bigg), \tag{6.49}
$$

$$M_4' = \frac{1}{\Gamma(q+1)}\left(T^q + \frac{T^{q-1}}{|\Omega_1|}\sum_{i=1}^{\rho_2}\frac{|\tau_k|\gamma_k^{p-1}}{(p-1)^{\sigma_k}}\sum_{l=1}^{\phi_1}\frac{|\delta_j|\xi_j^q}{q^{\beta_j}} + \frac{T^{q-1}}{|\Omega_1|}\sum_{i=1}^{\rho_1}\frac{|\mu_i|\eta_i^{p-1}}{(p-1)^{\alpha_i}}\sum_{l=1}^{\phi_2}\frac{|\omega_l|\theta_l^q}{q^{\nu_l}}\right), \tag{6.50}$$

$$M_5' = \frac{T^{q-1}}{|\Omega_1|\Gamma(p+1)}\left(\sum_{i=1}^{\rho_2}\frac{|\tau_k|\gamma_k^{p-1}}{(p-1)^{\sigma_k}}\sum_{i=1}^{\rho_1}\frac{|\mu_i|\eta_i^p}{p^{\alpha_i}} + \sum_{i=1}^{\rho_1}\frac{|\mu_i|\eta_i^{p-1}}{(p-1)^{\alpha_i}}\sum_{i=1}^{\rho_2}\frac{|\tau_k|\gamma_k^p}{p^{\sigma_k}}\right), \tag{6.51}$$

$$M_6' = \frac{T^{q-1}}{|\Omega_1|}\left(|K_1|\sum_{i=1}^{\rho_2}\frac{|\tau_k|\gamma_k^{p-1}}{(p-1)^{\sigma_k}} + |K_2|\sum_{i=1}^{\rho_1}\frac{|\mu_i|\eta_i^{p-1}}{(p-1)^{\alpha_i}}\right). \tag{6.52}$$

The first result is concerned with the existence and uniqueness of solutions for the problem (6.39) and is based on Banach's contraction mapping principle.

Theorem 6.7 *Assume that (6.1.1) and the following condition hold:*

$$(M_1' + M_3')(m_1 + m_2) + (M_2' + M_4')(n_1 + n_2) < 1,$$

where $M_i', i = 1, 2, 4, 5$ are given by (6.47), (6.48), (6.50) and (6.51) respectively. Then the system (6.39) has a unique solution on $[0, T]$.

Proof Letting $\sup_{t\in[0,T]} f(t, 0, 0) = N_1 < \infty$ and $\sup_{t\in[0,T]} g(t, 0, 0) = N_2 < \infty$, we define

$$r \geq \frac{(M_1' + M_3')N_1 + (M_2' + M_4')N_2 + M_5' + M_6'}{1 - (M_1' + M_3')(m_1 + m_2) - (M_2' + M_4')(n_1 + n_2)}.$$

where M_5' and M_6' are defined by (6.49) and (6.52), respectively.

Let us first show that $\widehat{\mathscr{T}} B_r \subset B_r$, where $B_r = \{(x, y) \in X \times Y : \|(x, y)\| \leq r\}$ and $\widehat{\mathscr{T}}$ is defined by (6.46).

For $(x, y) \in B_r$, we have

$$\begin{aligned}
&|\widehat{\mathscr{T}}_1(x, y)(t)| \\
&\leq {}_{RL}I^p(|f(s, x(s), y(s)) - f(s, 0, 0)| + |f(s, 0, 0)|)(T) + \frac{T^{p-1}}{|\Omega_1|}\left[\sum_{l=1}^{\phi_2}\frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}}\right. \\
&\quad \times\left(\sum_{l=1}^{\phi_1}|\delta_j|_H I^{\beta_j}{}_{RL}I^q(|g(s, x(s), y(s)) - g(s, 0, 0)| + |g(s, 0, 0)|)(\xi_j)\right. \\
&\quad \left. + \sum_{i=1}^{\rho_1}|\mu_i|_H I^{\alpha_i}{}_{RL}I^p(|f(s, x(s), y(s)) - f(s, 0, 0)| + |f(s, 0, 0)|)(\eta_i) + |K_1|\right) \\
&\quad + \sum_{l=1}^{\phi_1}\frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}}\left(\sum_{l=1}^{\phi_2}|\omega_l|_H I^{\nu_l}{}_{RL}I^q(|g(s, x(s), y(s)) - g(s, 0, 0)| + |g(s, 0, 0)|)(\theta_l)\right. \\
&\quad \left.\left. + \sum_{i=1}^{\rho_2}|\tau_k|_H I^{\sigma_k}{}_{RL}I^p(|f(s, x(s), y(s)) - f(s, 0, 0)| + |f(s, 0, 0)|)(\gamma_k) + |K_2|\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&\le {}_{RL}I^p(m_1\|x\| + m_2\|y\| + N_1)(T) + \frac{T^{p-1}}{|\Omega_1|}\Bigg[\sum_{l=1}^{\phi_2}\frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} \\
&\quad \times\Bigg(\sum_{l=1}^{\phi_1}|\delta_j|_H I^{\beta_j}{}_{RL}I^q(n_1\|x\| + n_2\|y\| + N_2)(\xi_j) \\
&\quad + \sum_{i=1}^{\rho_1}|\mu_i|_H I^{\alpha_i}{}_{RL}I^p(m_1\|x\| + m_2\|y\| + N_1)(\eta_i) + |K_1|\Bigg) \\
&\quad + \sum_{l=1}^{\phi_1}\frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}}\Bigg(\sum_{l=1}^{\phi_2}|\omega_l|_H I^{\nu_l}{}_{RL}I^q(n_1\|x\| + n_2\|y\| + N_2)(\theta_l) \\
&\quad + \sum_{i=1}^{\rho_2}|\tau_k|_H I^{\sigma_k}{}_{RL}I^p(m_1\|x\| + m_2\|y\| + N_1)(\gamma_k) + |K_2|\Bigg)\Bigg] \\
&= (m_1\|x\| + m_2\|y\| + N_1)\Bigg\{{}_{RL}I^p(1)(T) + \frac{T^{p-1}}{|\Omega_1|}\sum_{l=1}^{\phi_2}\frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}}\sum_{i=1}^{\rho_1}|\mu_i|_H I^{\alpha_i}{}_{RL}I^p(1)(\eta_i) \\
&\quad + \frac{T^{p-1}}{|\Omega_1|}\sum_{l=1}^{\phi_1}\frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}}\sum_{i=1}^{\rho_2}|\tau_k|_H I^{\sigma_k}{}_{RL}I^p(1)(\gamma_k)\Bigg\} + (n_1\|x\| + n_2\|y\| + N_2)\Bigg\{\frac{T^{p-1}}{|\Omega_1|} \\
&\quad \times\sum_{l=1}^{\phi_2}\frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}}\sum_{l=1}^{\phi_1}|\delta_j|_H I^{\beta_j}{}_{RL}I^q(1)(\xi_j) + \frac{T^{p-1}}{|\Omega_1|}\sum_{l=1}^{\phi_1}\frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}}\sum_{l=1}^{\phi_2}|\omega_l|_H I^{\nu_l}{}_{RL}I^q(1)(\theta_l)\Bigg\} \\
&\quad + |K_1|\frac{T^{p-1}}{|\Omega_1|}\sum_{l=1}^{\phi_2}\frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} + |K_2|\frac{T^{p-1}}{|\Omega_1|}\sum_{l=1}^{\phi_1}\frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}} \\
&= (m_1\|x\| + m_2\|y\| + N_1)\Bigg\{\frac{T^p}{\Gamma(p+1)} + \frac{T^{p-1}}{|\Omega_1|\Gamma(p+1)}\sum_{l=1}^{\phi_2}\frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}}\sum_{i=1}^{\rho_1}\frac{|\mu_i|\eta_i^p}{p^{\alpha_i}} \\
&\quad + \frac{T^{p-1}}{|\Omega_1|\Gamma(p+1)}\sum_{l=1}^{\phi_1}\frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}}\sum_{i=1}^{\rho_2}\frac{|\tau_k|\gamma_k^p}{p^{\sigma_k}}\Bigg\} + (n_1\|x\| + n_2\|y\| + N_2)\Bigg\{\frac{T^{p-1}}{|\Omega_1|\Gamma(q+1)} \\
&\quad \times\sum_{l=1}^{\phi_2}\frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}}\sum_{l=1}^{\phi_1}\frac{|\delta_j|\xi_j^q}{q^{\beta_j}} + \frac{T^{p-1}}{|\Omega_1|\Gamma(q+1)}\sum_{l=1}^{\phi_1}\frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}}\sum_{l=1}^{\phi_2}\frac{|\omega_l|\theta_l^q}{q^{\nu_l}}\Bigg\} \\
&\quad + |K_1|\frac{T^{p-1}}{|\Omega_1|}\sum_{l=1}^{\phi_2}\frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} + |K_2|\frac{T^{p-1}}{|\Omega_1|}\sum_{l=1}^{\phi_1}\frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}} \\
&= (m_1\|x\| + m_2\|y\| + N_1)M_1' + (n_1\|x\| + n_2\|y\| + N_2)M_2' + M_3' \\
&= (M_1'm_1 + M_2'n_1)\|x\| + (M_1'm_2 + M_2'n_2)\|y\| + M_1'N_1 + M_2'N_2 + M_3' \\
&\le (M_1'm_1 + M_2'n_1 + M_1'm_2 + M_2'n_2)r + M_1'N_1 + M_2'N_2 + M_3'.
\end{aligned}
$$

Hence

$$\|\widehat{\mathscr{T}}_1(x,y)\| \le [M_1'(m_1+m_2)+M_2'(n_1+n_2)]r + M_1'N_1 + M_2'N_2 + M_5'.$$

In a similar manner, we can obtain

$$\|\widehat{\mathscr{T}}_2(x,y)\| \le [M_3'(m_1+m_2)+M_4'(n_1+n_2)]r + M_3'N_1 + M_4'N_2 + M_6'.$$

Consequently, $\|\widehat{\mathscr{T}}(x,y)\| \le r$.
Now for $(x_2,y_2),(x_1,y_1) \in X \times Y$, and for any $t \in [0,T]$, we get

$$\begin{aligned}
&|\widehat{\mathscr{T}}_1(x_2,y_2)(t) - \mathscr{T}_1(x_1,y_1)(t)| \\
&\le {}_{RL}I^p|f(s,x_2(s),y_2(s)) - f(s,x_1(s),y_1(s))|(T) + \frac{T^{p-1}}{|\Omega_1|}\Bigg[\sum_{l=1}^{\phi_2}\frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} \\
&\quad \times\Bigg(\sum_{l=1}^{\phi_1}|\delta_j|_H I^{\beta_j}{}_{RL}I^q(|g(s,x_2(s),y_2(s)) - g(s,x_1(s),y_1(s))|)(\xi_j) \\
&\quad + \sum_{i=1}^{\rho_1}|\mu_i|_H I^{\alpha_i}{}_{RL}I^p(|f(s,x_2(s),y_2(s)) - f(s,x_1(s),y_1(s))|)(\eta_i)\Bigg) \\
&\quad + \sum_{l=1}^{\phi_1}\frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}}\Bigg(\sum_{l=1}^{\phi_2}|\omega_l|_H I^{\nu_l}{}_{RL}I^q(|g(s,x_2(s),y_2(s)) - g(s,x_1(s),y_1(s))|)(\theta_l) \\
&\quad + \sum_{i=1}^{\rho_2}|\tau_k|_H I^{\sigma_k}{}_{RL}I^p(|f(s,x_2(s),y_2(s)) - f(s,x_1(s),y_1(s))|)(\gamma_k)\Bigg)\Bigg] \\
&\le (m_1\|x_2-x_1\| + m_2\|y_2-y_1\|)\Bigg\{\frac{T^p}{\Gamma(p+1)} + \frac{T^{p-1}}{|\Omega_1|\Gamma(p+1)}\sum_{l=1}^{\phi_2}\frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}}\sum_{i=1}^{\rho_1}\frac{|\mu_i|\eta_i^p}{p^{\alpha_i}} \\
&\quad + \frac{T^{p-1}}{|\Omega_1|\Gamma(p+1)}\sum_{l=1}^{\phi_1}\frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}}\sum_{i=1}^{\rho_2}\frac{|\tau_k|\gamma_k^p}{p^{\sigma_k}}\Bigg\} + (n_1\|x_2-x_1\| + n_2\|y_2-y_1\|) \\
&\quad \times\Bigg\{\frac{T^{p-1}}{|\Omega_1|\Gamma(q+1)}\sum_{l=1}^{\phi_2}\frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}}\sum_{l=1}^{\phi_1}\frac{|\delta_j|\xi_j^q}{q^{\beta_j}} + \frac{T^{p-1}}{|\Omega_1|\Gamma(q+1)}\sum_{l=1}^{\phi_1}\frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}}\sum_{l=1}^{\phi_2}\frac{|\omega_l|\theta_l^q}{q^{\nu_l}}\Bigg\} \\
&= (m_1\|x_2-x_1\| + m_2\|y_2-y_1\|)M_1' + (n_1\|x_2-x_1\| + n_2\|y_2-y_1\|)M_2' \\
&= (M_1'm_1 + M_2'n_1)\|x_2-x_1\| + (M_1'm_2 + M_2'n_2)\|y_2-y_1\|,
\end{aligned}$$

and consequently, we obtain

$$\|\widehat{\mathscr{T}}_1(x_2,y_2) - \widehat{\mathscr{T}}_1(x_1,y_1)\| \le (M_1'm_1 + M_2'n_1 + M_1'm_2 + M_2'n_2)[\|x_2-x_1\| + \|y_2-y_1\|]. \tag{6.53}$$

Similarly, one can find that

$$\|\widehat{\mathscr{T}}_2(x_2, y_2) - \widehat{\mathscr{T}}_2(x_1, y_1)\| \leq (M_4' n_1 + M_3' m_1 + M_4' n_2 + M_3' m_2)[\|x_2 - x_1\| + \|y_2 - y_1\|]. \tag{6.54}$$

Thus we infer from (6.53) and (6.54) that

$$\|\widehat{\mathscr{T}}(x_2, y_2) - \widehat{\mathscr{T}}(x_1, y_1)\| \leq [(M_1' + M_3')(m_1 + m_2) + (M_2' + M_4')(n_1 + n_2)](\|x_2 - x_1\| + \|y_2 - y_1\|).$$

Then, by the condition $(M_1' + M_3')(m_1 + m_2) + (M_2' + M_4')(n_1 + n_2) < 1$, it follows that $\widehat{\mathscr{T}}$ is a contraction operator. Hence, by Banach's fixed point theorem, we conclude that the operator $\widehat{\mathscr{T}}$ has a unique fixed point, which is the unique solution of system (6.39). This completes the proof. □

Example 6.5 Consider the following system of coupled Riemann-Liouville fractional differential equations with Hadamard type fractional integral boundary conditions

$$\begin{cases} {}_{RL}D^{4/3}x(t) = \dfrac{t}{(t+6)^2}\dfrac{|x(t)|}{(1+|x(t)|)} + \dfrac{e^{-t}}{(t^2+3)^3}\dfrac{|y(t)|}{(1+|y(t)|)} + \dfrac{3}{4}, \ t \in [0,2], \\ {}_{RL}D^{3/2}y(t) = \dfrac{1}{18}\sin x(t) + \dfrac{1}{2^{2t}+19}\cos y(t) + \dfrac{5}{4}, \ \ t \in [0,2], \\ x(0) = 0, \quad 2{}_HI^{2/3}x(3/5) + \pi{}_HI^{7/5}x(1) = \sqrt{2}{}_HI^{3/2}y(1/3) \\ \qquad\qquad + e^2{}_HI^{5/4}y(\sqrt{3}) + 4, \\ y(0) = 0, \quad -3{}_HI^{9/5}x(2/3) + 4{}_HI^{7/4}x(9/7) + \dfrac{2}{5}{}_HI^{1/3}x(\sqrt{2}) \\ \qquad\qquad = \dfrac{e}{2}{}_HI^{11/6}y(8/5) - 2{}_HI^{12/11}y(1/4) - 10. \end{cases} \tag{6.55}$$

Here $p = 4/3$, $q = 3/2$, $T = 2$, $K_1 = 4$, $K_2 = -10$, $\rho_1 = 2$, $\phi_1 = 2$, $\rho_2 = 3$, $\phi_2 = 2$, $\mu_1 = 2$, $\mu_2 = \pi$, $\alpha_1 = 2/3$, $\alpha_2 = 7/5$, $\eta_1 = 3/5$, $\eta_2 = 1$, $\delta_1 = \sqrt{2}$, $\delta_2 = e^2$, $\beta_1 = 3/2$, $\beta_2 = 5/4$, $\xi_1 = 1/3$, $\xi_2 = \sqrt{3}$, $\tau_1 = -3$, $\tau_2 = 4$, $\tau_3 = 2/5$, $\sigma_1 = 9/5$, $\sigma_2 = 7/4$, $\sigma_3 = 1/3$, $\gamma_1 = 2/3$, $\gamma_2 = 9/7$, $\gamma_3 = \sqrt{2}$, $\omega_1 = e/2$, $\omega_2 = -2$, $\nu_1 = 11/6$, $\nu_2 = 12/11$, $\theta_1 = 8/5$, $\theta_2 = 1/4$, $f(t,x,y) = (t|x|)/(((t+6)^2)(1+|x|)) + (e^{-t}|y|)/(((t^2+3)^3)(1+|y|)) + (3/4)$ and $g(t,x,y) = (\sin x/18) + (\cos y)/(2^{2t} + 19) + (5/4)$. Then $|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq ((1/18)|x_1 - x_2| + (1/27)|y_1 - y_2|)$ and $|g(t,x_1,y_1) - g(t,x_2,y_2)| \leq ((1/18)|x_1 - x_2| + (1/20)|y_1 - y_2|)$. By using computer program, we can find $\Omega_1 \approx -218.9954766 \neq 0$. With the given values, it is found that $m_1 = 1/18$, $m_2 = 1/27$, $n_1 = 1/18$, $n_2 = 1/20$, $M_1' \simeq 2.847852451$, $M_2' \simeq 0.5295490231$, $M_3' \simeq 1.276954854$, $M_4' \simeq 4.723846069$ and

$$(M_1' + M_3')(m_1 + m_2) + (M_2' + M_4')(n_1 + n_2) \simeq 0.9364516398 < 1.$$

Thus all the conditions of Theorem 6.7 are satisfied. Therefore, by the conclusion of Theorem 6.7, the problem (6.55) has a unique solution on $[0, 2]$.

In the next result, we prove the existence of solutions for the problem (6.39) by means of Leray-Schauder alternative.

Theorem 6.8 *Suppose that (6.1.1) holds. In addition it is assumed that*

$$(M_1' + M_3')k_1 + (M_2' + M_4')\lambda_1 < 1 \text{ and } (M_1' + M_3')k_2 + (M_2' + M_4')\lambda_2 < 1,$$

where M_1', M_2', M_3', M_4' are given by (6.47), (6.48), (6.49) and (6.50) respectively. Then there exists at least one solution for the system (6.39) on $[0, T]$.

Proof In the first step, we show that the operator $\widehat{\mathscr{T}} : X \times Y \to X \times Y$ defined by (6.46) is completely continuous. By continuity of functions f and g, the operator $\widehat{\mathscr{T}}$ is continuous.

Let $\Theta \subset X \times Y$ be bounded. Then there exist positive constants P_1' and P_2' such that

$$|f(t, x(t), y(t))| \leq P_1', \quad |g(t, x(t), y(t))| \leq P_2', \quad \forall (x, y) \in \Theta.$$

Then, for any $(x, y) \in \Theta$, we have

$$\begin{aligned}
\|\widehat{\mathscr{T}}_1(x, y)\| &\leq {}_{RL}I^p|f(s, x(s), y(s))|(T) + \frac{T^{p-1}}{|\Omega_1|}\Bigg[\sum_{l=1}^{\phi_2} \frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} \\
&\quad \times \Bigg(\sum_{l=1}^{\phi_1} |\delta_j|_H I^{\beta_j}{}_{RL}I^q|g(s, x(s), y(s))|(\xi_j) \\
&\quad + \sum_{i=1}^{\rho_1} |\mu_i|_H I^{\alpha_i}{}_{RL}I^p|f(s, x(s), y(s))|(\eta_i) + |K_1|\Bigg) \\
&\quad + \sum_{l=1}^{\phi_1} \frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}}\Bigg(\sum_{l=1}^{\phi_2} |\omega_l|_H I^{\nu_l}{}_{RL}I^q|g(s, x(s), y(s))|(\theta_l) \\
&\quad + \sum_{i=1}^{\rho_2} |\tau_k|_H I^{\sigma_k}{}_{RL}I^p|f(s, x(s), y(s))|(\gamma_k) + |K_2|\Bigg)\Bigg] \\
&\leq \Bigg(\frac{T^p}{\Gamma(p+1)} + \frac{T^{p-1}}{|\Omega_1|\Gamma(p+1)} \sum_{l=1}^{\phi_2} \frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{i=1}^{\rho_1} \frac{|\mu_i|\eta_i^p}{p^{\alpha_i}} \\
&\quad + \frac{T^{p-1}}{|\Omega_1|\Gamma(p+1)} \sum_{l=1}^{\phi_1} \frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{i=1}^{\rho_2} \frac{|\tau_k|\gamma_k^p}{p^{\sigma_k}}\Bigg)P_1 + \Bigg(\frac{T^{p-1}}{|\Omega_1|\Gamma(q+1)} \\
&\quad \times \sum_{l=1}^{\phi_2} \frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{l=1}^{\phi_1} \frac{|\delta_j|\xi_j^q}{q^{\beta_j}} + \frac{T^{p-1}}{|\Omega_1|\Gamma(q+1)} \sum_{l=1}^{\phi_1} \frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{l=1}^{\phi_2} \frac{|\omega_l|\theta_l^q}{q^{\nu_l}}\Bigg)P_2 \\
&\quad + |K_1|\frac{T^{p-1}}{|\Omega_1|} \sum_{l=1}^{\phi_2} \frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} + |K_2|\frac{T^{p-1}}{|\Omega_1|} \sum_{l=1}^{\phi_1} \frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}} \\
&= M_1'P_1' + M_2'P_2' + M_5'.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\|\widehat{\mathscr{T}}_2(x,y)\| \leq & \Bigg(\frac{T^q}{\Gamma(q+1)} + \frac{T^{q-1}}{|\Omega_1|\Gamma(q+1)}\sum_{i=1}^{\rho_2}\frac{|\tau_k|\gamma_k^{p-1}}{(p-1)^{\sigma_k}}\sum_{l=1}^{\phi_1}\frac{|\delta_j|\xi_j^q}{q^{\beta_j}} \\
& + \frac{T^{q-1}}{|\Omega_1|\Gamma(q+1)}\sum_{i=1}^{\rho_1}\frac{|\mu_i|\eta_i^{p-1}}{(p-1)^{\alpha_i}}\sum_{l=1}^{\phi_2}\frac{|\omega_l|\theta_l^q}{q^{\nu_l}}\Bigg)P_2 + \Bigg(\frac{T^{q-1}}{|\Omega_1|\Gamma(p+1)} \\
& \times \sum_{i=1}^{\rho_2}\frac{|\tau_k|\gamma_k^{p-1}}{(p-1)^{\sigma_k}}\sum_{i=1}^{\rho_1}\frac{|\mu_i|\eta_i^p}{p^{\alpha_i}} + \frac{T^{q-1}}{|\Omega_1|\Gamma(p+1)}\sum_{i=1}^{\rho_1}\frac{|\mu_i|\eta_i^{p-1}}{(p-1)^{\alpha_i}}\sum_{i=1}^{\rho_2}\frac{|\tau_k|\gamma_k^p}{p^{\sigma_k}}\Bigg)P_1 \\
& + |K_1|\frac{T^{q-1}}{|\Omega_1|}\sum_{i=1}^{\rho_2}\frac{|\tau_k|\gamma_k^{p-1}}{(p-1)^{\sigma_k}} + |K_2|\frac{T^{q-1}}{|\Omega_1|}\sum_{i=1}^{\rho_1}\frac{|\mu_i|\eta_i^{p-1}}{(p-1)^{\alpha_i}} \\
= & M_4'P_2' + M_5'P_1' + M_6'.
\end{aligned}$$

Thus, it follows from the above inequalities that the operator $\widehat{\mathscr{T}}$ is uniformly bounded.

Next, we show that $\mathscr{T}$ is equicontinuous. Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$. Then, we have

$$\begin{aligned}
& |\widehat{\mathscr{T}}_1(x(t_2),y(t_2)) - \widehat{\mathscr{T}}_1(x(t_1),y(t_1))| \\
\leq & \frac{1}{\Gamma(p)}\int_0^{t_1}[(t_2-s)^{p-1} - (t_1-s)^{p-1}]|f(s,x(s),y(s))|ds \\
& + \frac{1}{\Gamma(p)}\int_{t_1}^{t_2}(t_2-s)^{p-1}|f(s,x(s),y(s))|ds + \frac{t_2^{p-1}-t_1^{p-1}}{|\Omega_1|}\Bigg[\sum_{l=1}^{\phi_2}\frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} \\
& \times\Bigg(\sum_{l=1}^{\phi_1}|\delta_j|_HI^{\beta_j}{}_{RL}I^q|g(s,x(s),y(s))|(\xi_j) \\
& + \sum_{i=1}^{\rho_1}|\mu_i|_HI^{\alpha_i}{}_{RL}I^p|f(s,x(s),y(s))|(\eta_i) + |\lambda_1|\Bigg) \\
& + \sum_{l=1}^{\phi_1}\frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}}\Bigg(\sum_{l=1}^{\phi_2}|\omega_l|_HI^{\nu_l}{}_{RL}I^q|g(s,x(s),y(s))|(\theta_l) \\
& + \sum_{i=1}^{\rho_2}|\tau_k|_HI^{\sigma_k}{}_{RL}I^p|f(s,x(s),y(s))|(\gamma_k) + |\lambda_2|\Bigg)\Bigg] \\
\leq & \frac{P_1'}{\Gamma(p+1)}[2(t_2-t_1)^p + |t_2^p - t_1^p|] \\
& + \frac{t_2^{p-1}-t_1^{p-1}}{|\Omega_1|}\Bigg[\sum_{l=1}^{\phi_2}\frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}}\Bigg(P_2'\sum_{l=1}^{\phi_1}\frac{|\delta_j|\xi_j^q}{q^{\beta_j}\Gamma(q+1)}) + P_1'\sum_{i=1}^{\rho_1}\frac{|\mu_i|\eta_i^p}{p^{\alpha_i}\Gamma(p+1)} \\
& + |K_1|\Bigg) + \sum_{l=1}^{\phi_1}\frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}}\Bigg(P_2'\sum_{l=1}^{\phi_2}\frac{|\omega_l|\theta_l^q}{q^{\nu_l}\Gamma(q+1)} + P_1'\sum_{i=1}^{\rho_2}\frac{|\tau_k|\gamma_k^p}{p^{\sigma_k}\Gamma(p+1)} + |K_2|\Bigg)\Bigg].
\end{aligned}$$

Analogously, we can obtain

$$\begin{aligned} & |\widehat{\mathscr{T}}_2(x(t_2), y(t_2)) - \widehat{\mathscr{T}}_2(x(t_1), y(t_1))| \\ \leq & \frac{P_2'}{\Gamma(q+1)}[2(t_2-t_1)^q + |t_2^q - t_1^q|] \\ & + \frac{t_2^{q-1} - t_1^{q-1}}{|\Omega_1|}\Bigg[\sum_{i=1}^{\rho_2} \frac{|\tau_k|\gamma_k^{p-1}}{(p-1)^{\sigma_k}}\Bigg(P_2' \sum_{l=1}^{\phi_1} \frac{|\delta_j|\xi_j^q}{q^{\beta_j}\Gamma(q+1)}) + P_1' \sum_{i=1}^{\rho_1} \frac{|\mu_i|\eta_i^p}{p^{\alpha_i}\Gamma(p+1)} \\ & + |K_1|\Bigg) + \sum_{i=1}^{\rho_1} \frac{|\mu_i|\eta_i^{p-1}}{(p-1)^{\alpha_i}}\Bigg(P_2' \sum_{l=1}^{\phi_2} \frac{|\omega_l|\theta_l^q}{q^{\nu_l}\Gamma(q+1)} + P_1' \sum_{i=1}^{\rho_2} \frac{|\tau_k|\gamma_k^p}{p^{\sigma_k}\Gamma(p+1)} + |K_2|\Bigg)\Bigg]. \end{aligned}$$

Therefore, the operator $\widehat{\mathscr{T}}(x, y)$ is equicontinuous, and thus the operator $\widehat{\mathscr{T}}(x, y)$ is completely continuous.

Finally, it will be verified that the set $\bar{\mathscr{E}} = \{(x, y) \in X \times Y | (x, y) = \lambda \widehat{\mathscr{T}}(x, y), 0 \leq \lambda \leq 1\}$ is bounded. Let $(x, y) \in \bar{\mathscr{E}}$, then $(x, y) = \lambda \widehat{\mathscr{T}}(x, y)$. For any $t \in [0, T]$, we have

$$x(t) = \lambda \widehat{\mathscr{T}}_1(x, y)(t), \quad y(t) = \lambda \widehat{\mathscr{T}}_2(x, y)(t).$$

Then

$$\begin{aligned} |x(t)| \leq & (k_0 + k_1\|x\| + k_2\|y\|)\Bigg(\frac{T^p}{\Gamma(p+1)} + \frac{T^{p-1}}{|\Omega_1|\Gamma(p+1)} \sum_{l=1}^{\phi_2} \frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{i=1}^{\rho_1} \frac{|\mu_i|\eta_i^p}{p^{\alpha_i}} \\ & + \frac{T^{p-1}}{|\Omega_1|\Gamma(p+1)} \sum_{l=1}^{\phi_1} \frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{i=1}^{\rho_2} \frac{|\tau_k|\gamma_k^p}{p^{\sigma_k}}\Bigg) + (l_0' + l_1'\|x\| + l_2'\|y\|) \\ & \times\Bigg(\frac{T^{p-1}}{|\Omega_1|\Gamma(q+1)} \sum_{l=1}^{\phi_2} \frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{l=1}^{\phi_1} \frac{|\delta_j|\xi_j^q}{q^{\beta_j}} \\ & + \frac{T^{p-1}}{|\Omega_1|\Gamma(q+1)} \sum_{l=1}^{\phi_1} \frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{l=1}^{\phi_2} \frac{|\omega_l|\theta_l^q}{q^{\nu_l}}\Bigg) \\ & + |K_1|\frac{T^{p-1}}{|\Omega_1|} \sum_{l=1}^{\phi_2} \frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} + |K_2|\frac{T^{p-1}}{|\Omega_1|} \sum_{l=1}^{\phi_1} \frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}} \end{aligned}$$

and

$$\begin{aligned} |y(t)| \leq & (\lambda_0 + \lambda_1\|x\| + \lambda_2\|y\|)\Bigg(\frac{T^q}{\Gamma(q+1)} + \frac{T^{q-1}}{|\Omega_1|\Gamma(q+1)} \sum_{i=1}^{\rho_2} \frac{|\tau_k|\gamma_k^{p-1}}{(p-1)^{\sigma_k}} \sum_{l=1}^{\phi_1} \frac{|\delta_j|\xi_j^q}{q^{\beta_j}} \\ & + \frac{T^{q-1}}{|\Omega_1|\Gamma(q+1)} \sum_{i=1}^{\rho_1} \frac{|\mu_i|\eta_i^{p-1}}{(p-1)^{\alpha_i}} \sum_{l=1}^{\phi_2} \frac{|\omega_l|\theta_l^q}{q^{\nu_l}}\Bigg) + (k_0 + k_1\|x\| + k_2\|y\|) \end{aligned}$$

$$\times\left(\frac{T^{q-1}}{|\Omega_1|\Gamma(p+1)}\sum_{i=1}^{\rho_2}\frac{|\tau_k|\gamma_k^{p-1}}{(p-1)^{\sigma_k}}\sum_{i=1}^{\rho_1}\frac{|\mu_i|\eta_i^p}{p^{\alpha_i}}\right.$$
$$\left.+\frac{T^{q-1}}{|\Omega_1|\Gamma(p+1)}\sum_{i=1}^{\rho_1}\frac{|\mu_i|\eta_i^{p-1}}{(p-1)^{\alpha_i}}\sum_{i=1}^{\rho_2}\frac{|\tau_k|\gamma_k^p}{p^{\sigma_k}}\right)$$
$$+|K_1|\frac{T^{q-1}}{|\Omega_1|}\sum_{i=1}^{\rho_2}\frac{|\tau_k|\gamma_k^{p-1}}{(p-1)^{\sigma_k}}+|K_2|\frac{T^{q-1}}{|\Omega_1|}\sum_{i=1}^{\rho_1}\frac{|\mu_i|\eta_i^{p-1}}{(p-1)^{\alpha_i}}.$$

Hence, we have

$$\|x\| \le (k_0 + k_1\|x\| + k_2\|y\|)M_1' + (\lambda_0 + \lambda_1\|x\| + \lambda_2\|y\|)M_2' + M_5'$$

and

$$\|y\| \le (\lambda_0 + \lambda_1\|x\| + \lambda_2'\|y\|)M_4' + (k_0 + k_1\|x\| + k_2\|y\|)M_3' + M_6',$$

which imply that

$$\|x\| + \|y\| \le (M_1' + M_3')k_0 + (M_2' + M_4')\lambda_0 + [(M_1' + M_3')k_1 + (M_2' + M_4')\lambda_1]\|x\|$$
$$+[(M_1' + M_3')k_2 + (M_2' + M_4')\lambda_2]\|y\| + M_5' + M_6'.$$

Consequently,

$$\|(x, y)\| \le \frac{(M_1' + M_3')k_0 + (M_2' + M_4')\lambda_0 + M_5' + M_6'}{M_0'},$$

for any $t \in [0, T]$, where

$$M_0' = \min\{1 - (M_1' + M_3')k_1 - (M_2' + M_4')\lambda_1,\ 1 - (M_1' + M_3')k_2 - (M_2' + M_4')\lambda_2\},$$

$k_i,\ \lambda_i \ge 0\ (i = 1, 2)$, which establishes that $\bar{\mathscr{E}}$ is bounded. Thus, by Theorem 1.3, the operator $\widehat{\mathscr{T}}$ has at least one fixed point. In consequence, the problem (6.39) has at least one solution on $[0, T]$. The proof is complete. □

Example 6.6 Consider the following system of coupled Riemann-Liouville fractional differential equations with multiple Hadamard type fractional integral boundary conditions

$$\begin{cases} {}_{RL}D^{\pi/2}x(t) = \dfrac{2}{5} + \dfrac{1}{(t+6)^2}\tan^{-1}x(t) + \dfrac{1}{20e}y(t),\ t \in [0,3], \\ {}_{RL}D^{7/4}y(t) = \dfrac{\sqrt{\pi}}{2} + \dfrac{1}{42}\sin x(t) + \dfrac{1}{t+20}y(t)\cos x(t),\quad t \in [0,3], \\ x(0) = 0,\quad 3{}_HI^{1/4}x(5/2) + \sqrt{5}{}_HI^{\sqrt{2}}x(7/8) + \tan(4){}_HI^{\sqrt{3}}x(9/4) \\ \qquad\qquad = \dfrac{\sqrt{8}\pi}{3}{}_HI^{5/3}y(5/4) - 2{}_HI^{6/11}y(\pi/3) + 2, \\ y(0) = 0,\quad -\dfrac{2}{3}{}_HI^{2/3}x(\pi/2) + 3{}_HI^{6/5}x(5/3) + \dfrac{\sqrt{2}}{\pi}{}_HI^{1/3}x(\sqrt{2}) \\ \qquad\qquad + \dfrac{7}{9}{}_HI^{11/9}x(\sqrt{5}) = e{}_HI^{7/6}y(\pi/6) - \log(9){}_HI^{3/4}y(7/4) - 1. \end{cases} \tag{6.56}$$

Here $p = \pi/2$, $q = 7/4$, $T = 3$, $K_1 = 2$, $K_2 = -1$, $\rho_1 = 3$, $\phi_1 = 2$, $\rho_2 = 4$, $\phi_2 = 2$, $\mu_1 = 3$, $\mu_2 = \sqrt{5}$, $\mu_3 = \tan(4)$, $\alpha_1 = 1/4$, $\alpha_2 = \sqrt{2}$, $\alpha_3 = \sqrt{3}$, $\eta_1 = 5/2$, $\eta_2 = 7/8$, $\eta_3 = 9/4$, $\delta_1 = \sqrt{8}\pi/3$, $\delta_2 = -2$, $\beta_1 = 5/3$, $\beta_2 = 6/11$, $\xi_1 = 5/4$, $\xi_2 = \pi/3$, $\tau_1 = -2/3$, $\tau_2 = 3$, $\tau_3 = \sqrt{2}/\pi$, $\tau_4 = 7/9$, $\sigma_1 = 2/3$, $\sigma_2 = 6/5$, $\sigma_3 = 1/3$, $\sigma_4 = 11/9$, $\gamma_1 = \pi/2$, $\gamma_2 = 5/3$, $\gamma_3 = \sqrt{2}$, $\gamma_4 = \sqrt{5}$, $\omega_1 = e$, $\omega_2 = -\log(9)$, $\nu_1 = 7/6$, $\nu_2 = 3/4$, $\theta_1 = \pi/6$, $\theta_2 = 7/4$, $f(t,x,y) = (2/5) + (\tan^{-1}x)/((t+6)^2) + (y)/(20e)$ and $g(t,x,y) = (\sqrt{\pi}/2) + (\sin x)/(42) + (y\cos x)/(t+20)$. By using computer program, we get $\Omega_1 \approx -59.01857601 \neq 0$. Clearly $|f(t,x,y)| \leq k_0 + k_1|x| + k_2|y|$ and $|g(t,x,y)| \leq \lambda_0 + \lambda_1|x| + \lambda_2|y|$, with $k_0 = 2/5$, $k_1 = 1/36$, $k_2 = 1/(20e)$, $\lambda_0 = \sqrt{\pi}/2$, $\lambda_1 = 1/42$, $\lambda_2 = 1/20$. With the given data, we find that $M_1' \simeq 7.406711671$, $M_2' \simeq 1.110132269$, $M_3' \simeq 7.790182643$, $M_4' \simeq 6.802999724$. Furthermore, we have

$$(M_1' + M_3')k_1 + (M_2' + M_4')\lambda_1 \approx 0.6105438577 < 1,$$

and

$$(M_1' + M_3')k_2 + (M_2' + M_4')\lambda_2 \approx 0.6751878489 < 1.$$

Thus all the conditions of Theorem 6.8 hold true and consequently the conclusion of Theorem 6.8 applies to the problem (6.56) on [0, 3].

6.5 Notes and Remarks

In this chapter, we have discussed the existence and uniqueness of solutions for coupled systems of nonlinear Riemann-Liouville fractional differential equations equipped with nonlocal coupled and uncoupled Hadamard fractional integral boundary conditions. The contents of in this chapter are adapted from the papers [20, 156] and [129].

Chapter 7
Nonlinear Langevin Equation and Inclusions Involving Hadamard-Caputo Type Fractional Derivatives

7.1 Introduction

In this chapter, we investigate the existence of solutions for nonlinear Langevin equations and inclusions involving Hadamard-Caputo type fractional derivatives equipped with nonlocal fractional integral conditions. We also study a coupled system of nonlinear Langevin equations with uncoupled boundary conditions.

The Langevin equation (first formulated by Langevin in 1908) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [62]. For some recent development on the fractional Langevin equation, see, for example, [5, 7, 71, 83, 111, 112, 115, 116, 166].

7.2 Nonlinear Langevin Equation Case

In this section, we study the existence and uniqueness of solutions for nonlinear Langevin equation involving Hadamard-Caputo type fractional derivatives with nonlocal fractional integral conditions:

$$\begin{cases} D^{\alpha}(D^{\beta}+\lambda)x(t)=f(t,x(t)), \quad 1<t<e, \\ \displaystyle\sum_{i=1}^{m}\theta_i I^{\mu_i}x(\eta_i)=\sum_{j=1}^{n}\phi_j I^{\gamma_j}x(\omega_j), \\ \displaystyle\sum_{k=1}^{p}\varepsilon_k I^{\sigma_k}x(\psi_k)=\sum_{l=1}^{q}\nu_l I^{\tau_l}x(\varphi_l), \end{cases} \tag{7.1}$$

where D^{ρ} denotes the Caputo-type Hadamard fractional derivative of order $\rho,\ \rho=\{\alpha,\beta\}$ with $0<\alpha,\beta<1$, $f:[1,e]\times\mathbb{R}\to\mathbb{R}$ is a continuous function,

B. Ahmad et al., *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, DOI 10.1007/978-3-319-52141-1_7

I^{κ} is the Hadamard fractional integral of order $\kappa > 0$, $\kappa = \{\mu_i, \gamma_j, \sigma_k, \tau_l\}$, η_i, ω_j, ψ_k, $\varphi_l \in (1, e)$ and $\theta_i, \phi_j, \varepsilon_k, \nu_l \in \mathbb{R}$, for all $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, p$, $l = 1, 2, \ldots, q$.

The significance of investigating nonlocal problem (7.1) is that the nonlocal Hadamard fractional integral conditions do not contain boundary values of unknown function x, which is a novel idea for studying non-boundary problems. In particular, if $m = n = p = q = 1$, then the conditions in (7.1) take the form

$$\frac{\theta_1}{\Gamma(\mu_1)} \int_1^{\eta_1} \left(\log \frac{t}{s}\right)^{\mu_1 - 1} \frac{x(s)}{s} ds = \frac{\phi_1}{\Gamma(\gamma_1)} \int_1^{\omega_1} \left(\log \frac{t}{s}\right)^{\gamma_1 - 1} \frac{x(s)}{s} ds,$$

and

$$\frac{\varepsilon_1}{\Gamma(\sigma_1)} \int_1^{\psi_1} \left(\log \frac{t}{s}\right)^{\sigma_1 - 1} \frac{x(s)}{s} ds = \frac{\nu_1}{\Gamma(\tau_1)} \int_1^{\varphi_1} \left(\log \frac{t}{s}\right)^{\tau_1 - 1} \frac{x(s)}{s} ds.$$

Several new existence and uniqueness results are proved by using a variety of fixed point theorems (such as Banach contraction principle, Krasnoselskii fixed point theorem, Leray-Schauder nonlinear alternative and Leray-Schauder degree theory).

For convenience, we set

$$\begin{aligned}
\Omega_1 &= \sum_{i=1}^{m} \theta_i \frac{(\log \eta_i)^{\mu_i}}{\Gamma(\mu_i + 1)} - \sum_{j=1}^{n} \phi_j \frac{(\log \omega_j)^{\gamma_j}}{\Gamma(\gamma_j + 1)}, \\
\Omega_2 &= \sum_{i=1}^{m} \theta_i \frac{(\log \eta_i)^{\beta + \mu_i}}{\Gamma(\beta + \mu_i + 1)} - \sum_{j=1}^{n} \phi_j \frac{(\log \omega_j)^{\beta + \gamma_j}}{\Gamma(\beta + \gamma_j + 1)}, \\
\Omega_3 &= \sum_{k=1}^{p} \varepsilon_k \frac{(\log \psi_k)^{\sigma_k}}{\Gamma(\sigma_k + 1)} - \sum_{l=1}^{q} \nu_l \frac{(\log \varphi_l)^{\tau_l}}{\Gamma(\tau_l + 1)}, \\
\Omega_4 &= \sum_{k=1}^{p} \varepsilon_k \frac{(\log \psi_k)^{\beta + \sigma_k}}{\Gamma(\beta + \sigma_k + 1)} - \sum_{l=1}^{q} \nu_l \frac{(\log \varphi_l)^{\beta + \tau_l}}{\Gamma(\beta + \tau_l + 1)},
\end{aligned} \tag{7.2}$$

and

$$\Omega = \Omega_1 \Omega_4 - \Omega_2 \Omega_3. \tag{7.3}$$

Lemma 7.1 *Let $\Omega \neq 0$, $0 < \alpha, \beta \leq 1$, $\mu_i, \gamma_j, \sigma_k, \tau_l > 0$, η_i, ω_j, ψ_k, $\varphi_l \in (1, e)$ and $\theta_i, \phi_j, \varepsilon_k, \nu_l \in \mathbb{R}$, for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, p$, $l = 1, 2, \ldots, q$, and $h \in C([1, e], \mathbb{R})$. Then the following problem*

$$D^{\alpha}(D^{\beta} + \lambda)x(t) = h(t), \qquad t \in (1, e), \tag{7.4}$$

$$\sum_{i=1}^{m} \theta_i I^{\mu_i} x(\eta_i) = \sum_{j=1}^{n} \phi_j I^{\gamma_j} x(\omega_j), \quad \sum_{k=1}^{p} \varepsilon_k I^{\sigma_k} x(\psi_k) = \sum_{l=1}^{q} \nu_l I^{\tau_l} x(\varphi_l), \tag{7.5}$$

is equivalent to the integral equation

$$x(t) = \frac{1}{\Omega}\Bigg[\left(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}h(\omega_j) - \lambda I^{\beta+\gamma_j}x(\omega_j)\right]\right.$$
$$\left. - \sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}h(\eta_i) - \lambda I^{\beta+\mu_i}x(\eta_i)\right]\right) \tag{7.6}$$
$$+ \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right)\left(\sum_{l=1}^{q}\nu_l\left[I^{\alpha+\beta+\tau_l}h(\varphi_l) - \lambda I^{\beta+\tau_l}x(\varphi_l)\right]\right.$$
$$\left. - \sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\sigma_k}h(\psi_k) - \lambda I^{\beta+\sigma_k}x(\psi_k)\right]\right)\Bigg] + I^{\alpha+\beta}h(t) - \lambda I^\beta x(t).$$

Proof By Lemma 1.5, the equation (7.4) can be expressed as

$$(D^\beta + \lambda)x(t) = I^\alpha h(t) + c_0,$$

which implies that

$$x(t) = I^{\alpha+\beta}h(t) - \lambda I^\beta x(t) + c_0\frac{(\log t)^\beta}{\Gamma(\beta+1)} + c_1, \tag{7.7}$$

for some $c_0, c_1 \in \mathbb{R}$.

Applying the Hadamard fractional integral of order $\kappa > 0$ on (7.7), we have

$$I^\kappa x(t) = I^{\alpha+\beta+\kappa}h(t) - \lambda I^{\beta+\kappa}x(t) + c_0\frac{(\log t)^{\beta+\kappa}}{\Gamma(\beta+\kappa+1)} + c_1\frac{(\log t)^\kappa}{\Gamma(\kappa+1)}. \tag{7.8}$$

Substituting $\kappa = \mu_i, \gamma_j, \sigma_k, \tau_l$, $t = \eta_i, \omega_j, \psi_k, \varphi_l$ in (7.8), respectively, and using conditions (7.5), we get

$$\Omega_1 c_1 + \Omega_2 c_0 = \sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}h(\omega_j) - \lambda I^{\beta+\gamma_j}x(\omega_j)\right]$$
$$- \sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}h(\eta_i) - \lambda I^{\beta+\mu_i}x(\eta_i)\right],$$

and

$$\Omega_3 c_1 + \Omega_4 c_0 = \sum_{l=1}^{p}\nu_l\left[I^{\alpha+\beta+\tau_l}h(\varphi_l) - \lambda I^{\beta+\tau_l}x(\varphi_l)\right]$$
$$- \sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\sigma_k}h(\psi_k) - \lambda I^{\beta+\sigma_k}x(\psi_k)\right].$$

Solving the above system for c_0, c_1, we find that

$$\begin{aligned}
c_0 = \frac{1}{\Omega}\Bigg[&\Omega_1\Bigg(\sum_{l=1}^{p} \nu_l \left[I^{\alpha+\beta+\tau_l} h(\varphi_l) - \lambda I^{\beta+\tau_l} x(\varphi_l) \right] \\
&- \sum_{k=1}^{p} \varepsilon_k \left[I^{\alpha+\beta+\sigma_k} h(\psi_k) - \lambda I^{\beta+\sigma_k} x(\psi_k) \right] \Bigg) \\
&- \Omega_3 \Bigg(\sum_{j=1}^{n} \phi_j \left[I^{\alpha+\beta+\gamma_j} h(\omega_j) - \lambda I^{\beta+\gamma_j} x(\omega_j) \right] \\
&- \sum_{i=1}^{m} \theta_i \left[I^{\alpha+\beta+\mu_i} h(\eta_i) - \lambda I^{\beta+\mu_i} x(\eta_i) \right] \Bigg) \Bigg], \\
c_1 = \frac{1}{\Omega}\Bigg[&\Omega_4\Bigg(\sum_{j=1}^{n} \phi_j \left[I^{\alpha+\beta+\gamma_j} h(\omega_j) - \lambda I^{\beta+\gamma_j} x(\omega_j) \right] \\
&- \sum_{i=1}^{m} \theta_i \left[I^{\alpha+\beta+\mu_i} h(\eta_i) - \lambda I^{\beta+\mu_i} x(\eta_i) \right] \\
&- \Omega_2 \Bigg(\sum_{l=1}^{p} \nu_l \left[I^{\alpha+\beta+\tau_l} h(\varphi_l) - \lambda I^{\beta+\tau_l} x(\varphi_l) \right] \\
&- \sum_{k=1}^{p} \varepsilon_k \left[I^{\alpha+\beta+\sigma_k} h(\psi_k) - \lambda I^{\beta+\sigma_k} x(\psi_k) \right] \Bigg) \Bigg].
\end{aligned}$$

Substituting the values of c_0 and c_1 into (7.7), we get (7.6). The converse follows by direct computation. This completes the proof. □

Let $\mathscr{E}_1 = C([1,e],\mathbb{R})$ denotes the Banach space of all continuous functions from $[1,e]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup_{t\in[1,e]} |x(t)|$. Throughout this section, for convenience, we use the notations:

$$I^z f(s,x(s))(y) = \frac{1}{\Gamma(z)} \int_1^y \left(\log\frac{y}{s}\right)^{z-1} \frac{f(s,x(s))}{s} ds,$$

and

$$I^z x(s)(y) = \frac{1}{\Gamma(z)} \int_1^y \left(\log\frac{y}{s}\right)^{z-1} \frac{x(s)}{s} ds,$$

where $z > 0$ and $y \in \{t, \eta_i, \omega_j, \psi_k, \varphi_l\}$.

In view of Lemma 7.1, we introduce the operator $\mathscr{Q} : \mathscr{E}_1 \to \mathscr{E}_1$ by

$$\begin{aligned}(\mathscr{Q}x)(t) = \frac{1}{\Omega}\Bigg[&\left(\Omega_4 - \frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\Big[I^{\alpha+\beta+\gamma_j}f(s,x(s))(\omega_j)\right.\\ &\left.-\lambda I^{\beta+\gamma_j}x(s)(\omega_j)\Big] - \sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}f(s,x(s))(\eta_i) - \lambda I^{\beta+\mu_i}x(s)(\eta_i)\right]\right)\\ &+\left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right)\left(\sum_{l=1}^{q}\nu_l\left[I^{\alpha+\beta+\tau_l}f(s,x(s))(\varphi_l) - \lambda I^{\beta+\tau_l}x(s)(\varphi_l)\right]\right.\\ &\left.-\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\sigma_k}f(s,x(s))(\psi_k) - \lambda I^{\beta+\sigma_k}x(s)(\psi_k)\right]\right)\Bigg]\\ &+I^{\alpha+\beta}f(s,x(s))(t) - \lambda I^{\beta}x(s)(t).\end{aligned} \tag{7.9}$$

Thus the nonlocal problem (7.1) will have solutions if and only if the operator $\mathscr{Q}$ has fixed points.

In the sequel, we set

$$\begin{aligned}\Lambda(u) = \frac{1}{|\Omega|}\Bigg[&\left(|\Omega_4| + \frac{|\Omega_3|}{\Gamma(\beta+1)}\right)\left(\sum_{j=1}^{n}|\phi_j|\frac{(\log\omega_j)^{u+\beta+\gamma_j}}{\Gamma(u+\beta+\gamma_j+1)}\right.\\ &\left.+\sum_{i=1}^{m}|\theta_i|\frac{(\log\eta_i)^{u+\beta+\mu_i}}{\Gamma(u+\beta+\mu_i+1)}\right)\\ &+\left(\frac{|\Omega_1|}{\Gamma(\beta+1)} + |\Omega_2|\right)\left(\sum_{l=1}^{q}|\nu_l|\frac{(\log\varphi_l)^{u+\beta+\tau_l}}{\Gamma(u+\beta+\tau_l+1)}\right.\\ &\left.+\sum_{k=1}^{p}|\varepsilon_k|\frac{(\log\psi_k)^{u+\beta+\sigma_k}}{\Gamma(u+\beta+\sigma_k+1)}\right)\Bigg] + \frac{1}{\Gamma(u+\beta+1)},\end{aligned} \tag{7.10}$$

where $u \in \{0, \alpha\}$.

7.2.1 Existence and Uniqueness Result via Banach's Fixed Point Theorem

Theorem 7.1 *Let $f : [1,e] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that:*

(7.1.1) there exists a constant $L > 0$ such that $|f(t,x) - f(t,y)| \le L|x-y|$, for each $t \in [1,e]$ and $x, y \in \mathbb{R}$.

If

$$L\Lambda(\alpha) + |\lambda|\Lambda(0) < 1, \tag{7.11}$$

where $\Lambda(.)$ is defined by (7.10), then the nonlocal problem (7.1) has a unique solution on $[1,e]$.

Proof By transforming the nonlocal problem (7.1) into a fixed point problem, $x = \mathscr{Q}x$, where the operator $\mathscr{Q}$ is defined by (7.9), the existence of the fixed points of the operator $\mathscr{Q}$ will imply the existence of solutions for problem (7.1). Applying the Banach's contraction mapping principle, we shall show that $\mathscr{Q}$ has a unique fixed point.

Setting $\sup_{t\in[1,e]} |f(t,0)| = M < \infty$ and choosing $R \geq \dfrac{M\Lambda(\alpha)}{1 - L\Lambda(\alpha) - |\lambda|\Lambda(0)}$, we show that $\mathscr{Q}B_R \subset B_R$, where $B_R = \{x \in \mathscr{E}_1 : \|x\| \leq R\}$ and the operator $\mathscr{Q}$ is defined by (7.9). For any $x \in B_R$, we have

$$
\begin{aligned}
&|(\mathscr{Q}x)(t)|\\
\leq & \left| \frac{1}{\Omega}\Bigg[\left(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n} \phi_j \left[I^{\alpha+\beta+\gamma_j} f(s,x(s))(\omega_j) - \lambda I^{\beta+\gamma_j} x(s)(\omega_j)\right]\right.\right.\\
& \left. - \sum_{i=1}^{m} \theta_i \left[I^{\alpha+\beta+\mu_i} f(s,x(s))(\eta_i) - \lambda I^{\beta+\mu_i} x(s)(\eta_i)\right]\right)\\
& + \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right)\left(\sum_{l=1}^{q} \nu_l \left[I^{\alpha+\beta+\tau_l} f(s,x(s))(\varphi_l) - \lambda I^{\beta+\tau_l} x(s)(\varphi_l)\right]\right.\\
& \left.\left. - \sum_{k=1}^{p} \varepsilon_k \left[I^{\alpha+\beta+\sigma_k} f(s,x(s))(\psi_k) - \lambda I^{\beta+\sigma_k} x(s)(\psi_k)\right]\right)\Bigg]\\
& \left. + I^{\alpha+\beta} f(s,x(s))(t) - \lambda I^{\beta} x(s)(t) \right|\\
\leq & \frac{1}{|\Omega|}\Bigg[\left(|\Omega_4| + \frac{(\log t)^\beta}{\Gamma(\beta+1)}|\Omega_3|\right)\left(\sum_{j=1}^{n} |\phi_j| \Big[(L\|x\| + M)\left(I^{\alpha+\beta+\gamma_j}1\right)(\omega_j)\right.\\
& + |\lambda|\|x\|\left(I^{\beta+\gamma_j}1\right)(\omega_j)\Big] + \sum_{i=1}^{m} |\theta_i| \Big[(L\|x\| + M)\left(I^{\alpha+\beta+\mu_i}1\right)(\eta_i)\\
& \left. + |\lambda|\|x\|\left(I^{\beta+\mu_i}1\right)(\eta_i)\Big]\right) + \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)}|\Omega_1| + |\Omega_2|\right)\\
& \times\left(\sum_{l=1}^{q} |\nu_l| \left[(L\|x\| + M)\left(I^{\alpha+\beta+\tau_l}1\right)(\varphi_l) + |\lambda|\|x\|\left(I^{\beta+\tau_l}1\right)(\varphi_l)\right]\right.\\
& \left. + \sum_{k=1}^{p} |\varepsilon_k| \left[(L\|x\| + M)\left(I^{\alpha+\beta+\sigma_k}1\right)(\psi_k) + |\lambda|\|x\|\left(I^{\beta+\sigma_k}1\right)(\psi_k)\right]\right)\Bigg]\\
& + (L\|x\| + M)\left(I^{\alpha+\beta}1\right)(t) + |\lambda|\|x(s)\|\left(I^{\beta}1\right)(t)
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{|\Omega|}\Bigg[\left(|\Omega_4| + \frac{(\log t)^{\beta}}{\Gamma(\beta+1)}|\Omega_3|\right)\left(\sum_{j=1}^{n}\left|\phi_j\right|\Big[(LR+M)\frac{\left(\log \omega_j\right)^{\alpha+\beta+\gamma_j}}{\Gamma(\alpha+\beta+\gamma_j+1)}\right.\\
&\quad +|\lambda|R\frac{\left(\log \omega_j\right)^{\beta+\gamma_j}}{\Gamma(\beta+\gamma_j+1)}\Big] + \sum_{i=1}^{m}|\theta_i|\Big[(LR+M)\frac{(\log \eta_i)^{\alpha+\beta+\mu_i}}{\Gamma(\alpha+\beta+\mu_i+1)}\\
&\quad \left.+|\lambda|R\frac{(\log \eta_i)^{\beta+\mu_i}}{\Gamma(\beta+\mu_i+1)}\Big]\right) + \left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}|\Omega_1| + |\Omega_2|\right)\\
&\quad \times\left(\sum_{l=1}^{q}|\nu_l|\left[(LR+M)\frac{(\log \varphi_l)^{\alpha+\beta+\tau_l}}{\Gamma(\alpha+\beta+\tau_l+1)} + |\lambda|R\frac{(\log \varphi_l)^{\beta+\tau_l}}{\Gamma(\beta+\tau_l+1)}\right]\right.\\
&\quad \left.+\sum_{k=1}^{p}|\varepsilon_k|\left[(LR+M)\frac{(\log \psi_k)^{\alpha+\beta+\sigma_k}}{\Gamma(\alpha+\beta+\sigma_k+1)} + |\lambda|R\frac{(\log \psi_k)^{\beta+\sigma_k}}{\Gamma(\beta+\sigma_k+1)}\right]\right)\Bigg]\\
&\quad +(LR+M)\frac{(\log t)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + |\lambda|R\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\\
&\le (LR+M)\Lambda(\alpha) + |\lambda|R\Lambda(0)\\
&= [L\Lambda(\alpha) + |\lambda|\Lambda(0)]R + M\Lambda(\alpha) \le R.
\end{aligned}
$$

This implies that $\|\mathscr{Q}x\| \le R$ for $x \in B_R$. Therefore, $\mathscr{Q}$ maps bounded subsets of B_R into bounded subsets of B_R.

Next, we let $x, y \in \mathscr{E}_1$. Then, for $t \in [1, e]$, we have

$$
\begin{aligned}
&|(\mathscr{Q}x)(t) - (\mathscr{Q}y)(t)|\\
&\le \frac{1}{|\Omega|}\Bigg[\left|\Omega_4 - \frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_3\right|\left(\sum_{j=1}^{n}\left|\phi_j\right|\Big[I^{\alpha+\beta+\gamma_j}\left(|f(s,x(s)) - f(s,y(s))|\right)(\omega_j)\right.\\
&\quad +|\lambda|I^{\beta+\gamma_j}(|x(s)-y(s)|)(\omega_j)\Big] + \sum_{i=1}^{m}|\theta_i|\Big[I^{\alpha+\beta+\mu_i}\left(|f(s,x(s)) - f(s,y(s))|\right)(\eta_i)\\
&\quad \left.+|\lambda|I^{\beta+\mu_i}(|x(s)-y(s)|)(\eta_i)\Big]\right) + \left|\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right|\\
&\quad \times\left(\sum_{l=1}^{q}|\nu_l|\left[I^{\alpha+\beta+\tau_l}\left(|f(s,x(s)) - f(s,y(s))|\right)(\varphi_l) + |\lambda|I^{\beta+\tau_l}(|x(s)-y(s)|)(\varphi_l)\right]\right.\\
&\quad +\sum_{k=1}^{p}|\varepsilon_k|\left[I^{\alpha+\beta+\sigma_k}\left(|f(s,x(s)) - f(s,y(s))|\right)(\psi_k)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ |\lambda| I^{\beta+\sigma_k}(|x(s)-y(s)|)(\psi_k)\Big]\Bigg)\Bigg] \\
&+I^{\alpha+\beta}\left(|f(s,x(s))-f(s,y(s))|\right)(t) + |\lambda| I^{\beta}(|x(s)-y(s)|)(t) \\
\leq\ & \frac{1}{|\Omega|}\Bigg[\Bigg(|\Omega_4| + \frac{(\log t)^{\beta}}{\Gamma(\beta+1)}|\Omega_3|\Bigg)\Bigg(\sum_{j=1}^{n}|\phi_j|\Big[(L\|x-y\|)\frac{\left(\log\omega_j\right)^{\alpha+\beta+\gamma_j}}{\Gamma(\alpha+\beta+\gamma_j+1)} \\
&+|\lambda|\|x-y\|\frac{\left(\log\omega_j\right)^{\beta+\gamma_j}}{\Gamma(\beta+\gamma_j+1)}\Big] + \sum_{i=1}^{m}|\theta_i|\Big[(L\|x-y\|)\frac{(\log\eta_i)^{\alpha+\beta+\mu_i}}{\Gamma(\alpha+\beta+\mu_i+1)} \\
&+|\lambda|\|x-y\|\frac{(\log\eta_i)^{\beta+\mu_i}}{\Gamma(\beta+\mu_i+1)}\Big]\Bigg) + \Bigg(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}|\Omega_1| + |\Omega_2|\Bigg) \\
&\times\Bigg(\sum_{l=1}^{q}|\nu_l|\Bigg[(L\|x-y\|)\frac{(\log\varphi_l)^{\alpha+\beta+\tau_l}}{\Gamma(\alpha+\beta+\tau_l+1)} + |\lambda|\|x-y\|\frac{(\log\varphi_l)^{\beta+\tau_l}}{\Gamma(\beta+\tau_l+1)}\Bigg] \\
&+\sum_{k=1}^{p}|\varepsilon_k|\Bigg[(L\|x-y\|)\frac{(\log\psi_k)^{\alpha+\beta+\sigma_k}}{\Gamma(\alpha+\beta+\sigma_k+1)} + |\lambda|\|x-y\|\frac{(\log\psi_k)^{\beta+\sigma_k}}{\Gamma(\beta+\sigma_k+1)}\Bigg]\Bigg)\Bigg] \\
&+(L\|x-y\|)\frac{(\log t)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + |\lambda|\|x-y\|\frac{(\log t)^{\beta}}{\Gamma(\beta+1)} \\
\leq\ & L\Lambda(\alpha)\|x-y\| + |\lambda|\|x-y\|\Lambda(0) \\
=\ & [L\Lambda(\alpha) + |\lambda|\Lambda(0)]\,\|x-y\|,
\end{aligned}
$$

which implies that $\|\mathscr{Q}x-\mathscr{Q}y\| \leq [L\Lambda(\alpha)+|\lambda|\Lambda(0)]\,\|x-y\|$. As $L\Lambda(\alpha)+|\lambda|\Lambda(0) < 1$, $\mathscr{Q}$ is a contraction. Therefore, by the Banach's contraction mapping principle, $\mathscr{Q}$ has a fixed point which is the unique solution of nonlocal problem (7.1). The proof is completed. □

Example 7.1 Consider the following nonlinear Langevin equation of Hadamard-Caputo type fractional derivatives with nonlocal fractional integral conditions

$$
\begin{cases}
D^{1/2}\left(D^{2/3}+\dfrac{1}{9}\right)x(t) = \dfrac{\log t}{3t}\cdot\dfrac{|x(t)|}{2+|x(t)|} + \dfrac{3}{2}t, \qquad 1<t<e, \\
\dfrac{13}{4}I^{2/3}x\left(\dfrac{e+7}{7}\right) + \dfrac{13}{3}I^{3/4}x\left(\dfrac{2e+7}{7}\right) + \dfrac{13}{2}I^{4/5}x\left(\dfrac{3e+7}{7}\right) \\
\qquad = \dfrac{1}{3}I^{1/6}x\left(\dfrac{e+2}{4}\right) + \dfrac{1}{4}I^{1/2}x\left(\dfrac{e+1}{2}\right), \\
\dfrac{2}{3}I^{1/5}x\left(\dfrac{3e}{4}\right) + \dfrac{3}{7}I^{1/4}x\left(\dfrac{4e}{5}\right) = 3I^{1/5}x\left(\dfrac{2e}{5}\right).
\end{cases}
\tag{7.12}
$$

Here $\alpha = 1/2$, $\beta = 2/3$, $\lambda = 1/9$, $m = 3$, $n = 2$, $p = 2$, $q = 1$, $\theta_1 = 13/4$, $\theta_2 = 13/3$, $\theta_3 = 13/2$, $\mu_1 = 2/3$, $\mu_2 = 3/4$, $\mu_3 = 4/5$, $\eta_1 = (e+7)/7$, $\eta_2 = (2e+7)/7$, $\eta_3 = (3e+7)/7$, $\phi_1 = 1/3$, $\phi_2 = 1/4$, $\gamma_1 = 1/6$, $\gamma_2 = 1/2$, $\omega_1 = (e+2)/4$, $\omega_2 = (e+1)/2$, $\varepsilon_1 = 2/3$, $\varepsilon_2 = 3/7$, $\sigma_1 = 1/5$, $\sigma_2 = 1/4$, $\psi_1 = 3e/4$, $\psi_2 = 4e/5$, $\nu_1 = 3$, $\tau_1 = 1/5$, $\varphi_1 = 2e/5$, and $f(t,x) = (\log t|x(t)|)/(3t(2+|x(t)|)) + 3t/2$. Since $|f(t,x)-f(t,y)| \le (1/6)|x-y|$, (7.1.1) is satisfied with $L = 1/6$. We can show that $\Lambda(\alpha) \approx 2.527538367$, and $\Lambda(0) \approx 4.083246365$. Thus $L\Lambda(\alpha) + |\lambda|\Lambda(0) \approx 0.8749504351 < 1$. Hence, by Theorem 7.1, the nonlocal problem (7.12) has a unique solution on $[1,e]$.

7.2.2 Existence Result via Krasnoselskii's Fixed Point Theorem

Theorem 7.2 *Let $f : [1,e] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (7.1.1). In addition, we assume that:*

(7.2.1) $|f(t,x)| \le \rho(t), \quad \forall (t,x) \in [1,e] \times \mathbb{R}$, *and* $\rho \in C([1,e], \mathbb{R}^+)$.

If

$$|\lambda|\Lambda(0) < 1, \tag{7.13}$$

where $\Lambda(0)$ is defined by (7.10), then the nonlocal problem (7.1) has at least one solution on $[1,e]$.

Proof Setting $\sup_{t\in[1,e]} |\rho(t)| = \|\rho\|$ and choosing $R \ge \dfrac{\|\rho\|\Lambda(\alpha)}{1-\|\lambda\|\Lambda(0)}$, we consider $B_R = \{x \in \mathscr{E}_1 : \|x\| \le R\}$. Let us define the operators $\mathscr{Q}_1$ and $\mathscr{Q}_2$ on B_R by

$$\begin{aligned}
(\mathscr{Q}_1 x)(t) = \frac{1}{\Omega}\Bigg[&\left(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n} \phi_j \left[I^{\alpha+\beta+\gamma_j} f(s,x(s))(\omega_j)\right]\right.\\
&\left.-\sum_{i=1}^{m} \theta_i \left[I^{\alpha+\beta+\mu_i} f(s,x(s))(\eta_i)\right]\right) + \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right)\\
&\times\left(\sum_{l=1}^{q} \nu_l \left[I^{\alpha+\beta+\tau_l} f(s,x(s))(\varphi_l)\right] - \sum_{k=1}^{p} \varepsilon_k \left[I^{\alpha+\beta+\sigma_k} f(s,x(s))(\psi_k)\right]\right)\Bigg]\\
&+I^{\alpha+\beta} f(s,x(s))(t),\\
(\mathscr{Q}_2 x)(t) = \frac{1}{\Omega}\Bigg[&\left(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n} -\phi_j \left[\lambda I^{\beta+\gamma_j} x(s)(\omega_j)\right]\right.
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\theta_i\left[\lambda I^{\beta+\mu_i}x(s)(\eta_i)\right]\Bigg)+\left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_1-\Omega_2\right)\\
&\times\left(\sum_{l=1}^{q}-\nu_l\left[\lambda I^{\beta+\tau_l}x(s)(\varphi_l)\right]+\sum_{k=1}^{p}\varepsilon_k\left[\lambda I^{\beta+\sigma_k}x(s)(\psi_k)\right]\right)\Bigg]\\
&-\lambda I^{\beta}x(s)(t),\quad t\in[1,e].
\end{aligned}
$$

Note that $\mathcal{Q}=\mathcal{Q}_1+\mathcal{Q}_2$. For any $x,y\in B_R$, we have

$$
\begin{aligned}
&|\mathcal{Q}_1x(t)+\mathcal{Q}_2y(t)|\\
\le&\frac{1}{|\Omega|}\Bigg[\left|\Omega_4-\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_3\right|\Bigg(\sum_{j=1}^{n}|\phi_j|\Big[\|\rho\|\left(I^{\alpha+\beta+\gamma_j}1\right)(\omega_j)\\
&+|\lambda|R\left(I^{\beta+\gamma_j}1\right)(\omega_j)\Big]+\sum_{i=1}^{m}|\theta_i|\Big[\|\rho\|\left(I^{\alpha+\beta+\mu_i}1\right)(\eta_i)\\
&+|\lambda|R\left(I^{\beta+\mu_i}1\right)(\eta_i)\Big]\Bigg)+\left|\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_1-\Omega_2\right|\\
&\times\Bigg(\sum_{l=1}^{q}|\nu_l|\left[\|\rho\|\left(I^{\alpha+\beta+\tau_l}1\right)(\varphi_l)+|\lambda|R\left(I^{\beta+\tau_l}1\right)(\varphi_l)\right]\\
&+\sum_{k=1}^{p}|\varepsilon_k|\left[\|\rho\|\left(I^{\alpha+\beta+\sigma_k}1\right)(\psi_k)+|\lambda|R\left(I^{\beta+\sigma_k}1\right)(\psi_k)\right]\Bigg)\Bigg]\\
&+\|\rho\|\left(I^{\alpha+\beta}1\right)(t)+|\lambda|R|\left(I^{\beta}1\right)(t)\\
\le&\|\rho\|\Lambda(\alpha)+R\|\lambda\|\Lambda(0)\le R,
\end{aligned}
$$

which implies that $\|\mathcal{Q}_1x+\mathcal{Q}_2y\|\le R$. It follows that $\mathcal{Q}_1x+\mathcal{Q}_2y\in B_R$.

For $x,y\in\mathcal{E}_1$ and for each $t\in[1,e]$, we have

$$
\|\mathcal{Q}_2x-\mathcal{Q}_2y\|\le|\lambda|\Lambda(0)\|x-y\|.
$$

Hence, by (7.13), $\mathcal{Q}_2$ is a contraction mapping. Continuity of f implies that the operator $\mathcal{Q}_1$ is continuous. Also, $\mathcal{Q}_1$ is uniformly bounded on B_R as

$$
\|\mathcal{Q}_1x\|\le\|\rho\|\Lambda(\alpha).
$$

Now, we prove the compactness of the operator $\mathcal{Q}_1$.

We define $\sup_{(t,x)\in[(1,e)\times B_R]}|f(t,x)|=\overline{f}<\infty$. Let $t_1,t_2\in[1,e]$ with $t_1<t_2$ and $x\in B_R$. Then, we have

$$
\begin{aligned}
&|(\mathscr{Q}_1x)(t_2) - (\mathscr{Q}_1x)(t_1)| \\
&\le \Bigg|\frac{1}{\Omega}\Bigg[\Omega_3\left(\frac{(\log t_2)^\beta - (\log t_1)^\beta}{\Gamma(\beta+1)}\right)\Bigg(\sum_{j=1}^n \phi_j\left[I^{\alpha+\beta+\gamma_j}f(s,x(s))(\omega_j)\right] \\
&\quad -\sum_{i=1}^m \theta_i\left[I^{\alpha+\beta+\mu_i}f(s,x(s))(\eta_i)\right]\Bigg) + \Omega_1\left(\frac{(\log t_2)^\beta - (\log t_1)^\beta}{\Gamma(\beta+1)}\right) \\
&\quad \times\Bigg(\sum_{l=1}^q \nu_l\left[I^{\alpha+\beta+\tau_l}f(s,x(s))(\varphi_l)\right] - \sum_{k=1}^p \varepsilon_k\left[I^{\alpha+\beta+\sigma_k}f(s,x(s))(\psi_k)\right]\Bigg)\Bigg]\Bigg| \\
&\quad + \left|I^{\alpha+\beta}f(s,x(s))(t_2) - I^{\alpha+\beta}f(s,x(s))(t_1)\right| \\
&\le \frac{\bar{f}}{|\Omega|}\Bigg[|\Omega_3|\left|\frac{(\log t_2)^\beta - (\log t_1)^\beta}{\Gamma(\beta+1)}\right|\Bigg(\sum_{j=1}^n |\phi_j|\frac{(\log \omega_j)^{\alpha+\beta+\gamma_j}}{\Gamma(\alpha+\beta+\gamma_j+1)} \\
&\quad +\sum_{i=1}^m |\theta_i|\frac{(\log \eta_i)^{\alpha+\beta+\mu_i}}{\Gamma(\alpha+\beta+\mu_i+1)}\Bigg) + |\Omega_1|\left|\frac{(\log t_2)^\beta - (\log t_1)^\beta}{\Gamma(\beta+1)}\right| \\
&\quad \times\Bigg(\sum_{l=1}^q |\nu_l|\frac{(\log \varphi_l)^{\alpha+\beta+\tau_l}}{\Gamma(\alpha+\beta+\tau_l+1)} + \sum_{k=1}^p |\varepsilon_k|\frac{(\log \psi_k)^{\alpha+\beta+\sigma_k}}{\Gamma(\alpha+\beta+\sigma_k+1)}\Bigg)\Bigg] \\
&\quad + \frac{\bar{f}}{\Gamma(\alpha+\beta+1)}\left[\left|(\log t_2)^{\alpha+\beta} - (\log t_1)^{\alpha+\beta}\right| + 2(\log t_2/t_1)^{\alpha+\beta}\right],
\end{aligned}
$$

which is independent of x, and tends to zero as $t_2 - t_1 \to 0$. Thus, $\mathscr{Q}_1$ is equicontinuous. So $\mathscr{Q}_1$ is relatively compact on B_R. Hence, by the Arzelá-Ascoli Theorem, $\mathscr{Q}_1$ is compact on B_R. Thus all the assumptions of Theorem 1.2 are satisfied. So the conclusion of Theorem 1.2 implies that the nonlocal problem (7.1) has at least one solution on $[1, e]$. This completes the proof. □

Example 7.2 Consider the following nonlinear Langevin equation involving Hadamard-Caputo type fractional derivatives with nonlocal fractional integral conditions

$$
\begin{cases}
D^{2/3}\left(D^{2/5} + \dfrac{1}{12}\right)x(t) = \dfrac{t\log t}{t+\sin\pi t}\cdot\dfrac{\operatorname{arccot} x(t)}{1+|x(t)|} + t^2\sin\pi t,\ 1<t<e, \\
4I^{2/3}x\left(\dfrac{2e}{3}\right) + 3I^{3/5}x\left(\dfrac{2e+1}{3}\right) = \dfrac{2}{5}I^{2/3}x\left(\dfrac{e+1}{3}\right) \\
\qquad\qquad + \dfrac{1}{2}I^{1/2}x\left(\dfrac{e+2}{3}\right) + \dfrac{2}{5}I^{3/2}x\left(\dfrac{e+3}{3}\right), \\
\dfrac{5}{7}I^{2/5}x\left(\dfrac{3e}{7}\right) + \dfrac{6}{7}I^{3/4}x\left(\dfrac{4e}{7}\right) = \dfrac{1}{5}I^{1/5}x\left(\dfrac{3e}{5}\right) + \dfrac{2}{7}I^{2/5}x\left(\dfrac{4e}{5}\right).
\end{cases}
\tag{7.14}
$$

Here $\alpha = 2/3$, $\beta = 2/5$, $\lambda = 1/12$, $m = 2$, $n = 3$, $p = 2$, $q = 2$, $\theta_1 = 4$, $\theta_2 = 3$, $\mu_1 = 2/3$, $\mu_2 = 3/5$, $\eta_1 = 2e/3$, $\eta_2 = (2e + 1)/3$, $\phi_1 = 2/5$, $\phi_2 = 1/2$, $\phi_3 = 2/5$, $\gamma_1 = 2/3$, $\gamma_2 = 1/2$, $\gamma_3 = 3/2$, $\omega_1 = (e + 1)/3$, $\omega_2 = (e + 2)/3$, $\omega_3 = (e+3)/3$, $\varepsilon_1 = 5/7$, $\varepsilon_2 = 6/7$, $\sigma_1 = 2/5$, $\sigma_2 = 3/4$, $\psi_1 = 3e/7$, $\psi_2 = 4e/7$, $\nu_1 = 1/5$, $\nu_2 = 2/7$, $\tau_1 = 1/5$, $\tau_2 = 2/5$, $\varphi_1 = 3e/5$, $\varphi_2 = 4e/5$, and $f(t, x) = (t \log t \operatorname{arccot} x(t))/((t+\sin \pi t)(1+|x(t)|))+t^2 \sin \pi t$. Since $|f(t, x)| \le (t \log t)/(t+\sin \pi t)+t^2 \sin \pi t$, (7.2.1) is satisfied. We find that $\Lambda(0) \approx 10.69222877$, $|\lambda|\Lambda(0) \approx 0.8910190640 < 1$. Hence, by Theorem 7.2, the nonlocal problem (7.14) has at least one solution on $[1, e]$.

7.2.3 Existence Result via Leray-Schauder's Nonlinear Alternative

Theorem 7.3 *Let $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that:*

(7.3.1) there exists a continuous nondecreasing function $\Upsilon : [0, \infty) \to (0, \infty)$ and a function $p \in C([1, e], \mathbb{R}^+)$ such that

$$|f(t, u)| \le p(t)\Upsilon(\|u\|) \text{ for each } (t, u) \in [1, e] \times \mathbb{R};$$

(7.3.2) there exists a constant $M > 0$ such that

$$\frac{M}{\|p\|\Upsilon(M)\Lambda(\alpha) + |\lambda|M\Lambda(0)} > 1,$$

where $\Lambda(.)$ is defined by (7.10).

Then, the nonlocal problem (7.1) has at least one solution on $[1, e]$.

Proof Firstly, we shall show that $\mathscr{Q}$ *maps bounded sets (balls) into bounded sets in* $\mathscr{E}_1$. For a number $R > 0$, let $B_R = \{x \in \mathscr{E}_1 : \|x\| \le R\}$ be a bounded ball in $\mathscr{E}_1$. Then, for $t \in [1, e]$, we have

$$\begin{aligned}
&|(\mathscr{Q}x)(t)| \\
\le& \left|\frac{1}{\Omega}\left[\left(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^n \phi_j \left[I^{\alpha+\beta+\gamma_j} f(s, x(s))(\omega_j) - \lambda I^{\beta+\gamma_j} x(s)(\omega_j)\right]\right.\right.\right. \\
&\left. - \sum_{i=1}^m \theta_i \left[I^{\alpha+\beta+\mu_i} f(s, x(s))(\eta_i) - \lambda I^{\beta+\mu_i} x(s)(\eta_i)\right]\right) \\
&+ \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right)\left(\sum_{l=1}^q \nu_l \left[I^{\alpha+\beta+\tau_l} f(s, x(s))(\varphi_l) - \lambda I^{\beta+\tau_l} x(s)(\varphi_l)\right]\right.
\end{aligned}$$

$$
-\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\sigma_k}f(s,x(s))(\psi_k)-\lambda I^{\beta+\sigma_k}x(s)(\psi_k)\right]\Bigg)\Bigg]
$$

$$
+I^{\alpha+\beta}f(s,x(s))(t)-\lambda I^{\beta}x(s)(t)\Bigg|
$$

$$
\leq \frac{1}{|\Omega|}\Bigg[\Bigg(|\Omega_4|+\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}|\Omega_3|\Bigg)\Bigg(\sum_{j=1}^{n}|\phi_j|\left[I^{\alpha+\beta+\gamma_j}\|p\|\Upsilon(\|x\|)(\omega_j)\right.
$$

$$
\left.+|\lambda|I^{\beta+\gamma_j}\|x\|(\omega_j)\right]+\sum_{i=1}^{m}|\theta_i|\Big[I^{\alpha+\beta+\mu_i}\|p\|\Upsilon(\|x\|)(\eta_i)
$$

$$
+|\lambda|I^{\beta+\mu_i}\|x\|(\eta_i)\Big]\Bigg)+\Bigg(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}|\Omega_1|+|\Omega_2|\Bigg)
$$

$$
\times\Bigg(\sum_{l=1}^{q}|\nu_l|\left[I^{\alpha+\beta+\tau_l}\|p\|\Upsilon(\|x\|)(\varphi_l)+|\lambda|I^{\beta+\tau_l}\|x\|(\varphi_l)\right]
$$

$$
+\sum_{k=1}^{p}|\varepsilon_k|\left[I^{\alpha+\beta+\sigma_k}\|p\|\Upsilon(\|x\|)(\psi_k)+|\lambda|I^{\beta+\sigma_k}\|x\|(\psi_k)\right]\Bigg)\Bigg]
$$

$$
+I^{\alpha+\beta}\|p\|\Upsilon(\|x\|)(t)+|\lambda|I^{\beta}\|x\|(t)
$$

$$
\leq \|p\|\Upsilon(\|x\|)\,\Lambda(\alpha)+|\lambda|R\Lambda(0)
$$

$$
\leq \|p\|\Upsilon(R)\,\Lambda(\alpha)+|\lambda|R\Lambda(0),
$$

and consequently,

$$
\|\mathcal{Q}x\|\leq \|p\|\Upsilon(R)\,\Lambda(\alpha)+|\lambda|R\Lambda(0).
$$

Next, it will be shown that $\mathcal{Q}$ *maps bounded sets into equicontinuous sets of* $\mathcal{E}_1$. Let $t_1, t_2 \in [1, e]$ with $t_1 < t_2$ and $x \in B_R$. Then, we have

$$
|(\mathcal{Q}x)(t_2)-(\mathcal{Q}x)(t_1)|
$$

$$
\leq \Bigg|\frac{1}{\Omega}\Bigg[\frac{(\log t_2)^{\beta}-\log t_1)^{\beta}}{\Gamma(\beta+1)}\Omega_3\Bigg(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}f(s,x(s))(\omega_j)-\lambda I^{\beta+\gamma_j}x(s)(\omega_j)\right]
$$

$$
-\sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}f(s,x(s))(\eta_i)-\lambda I^{\beta+\mu_i}x(s)(\eta_i)\right]\Bigg)
$$

$$
\begin{aligned}
&+\frac{(\log t_2)^\beta-(\log t_1)^\beta}{\Gamma(\beta+1)}\Omega_1\Bigg(\sum_{l=1}^{q}\nu_l\left[I^{\alpha+\beta+\tau_l}f(s,x(s))(\varphi_l)-\lambda I^{\beta+\tau_l}x(s)(\varphi_l)\right]\\
&-\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\sigma_k}f(s,x(s))(\psi_k)-\lambda I^{\beta+\sigma_k}x(s)(\psi_k)\right]\Bigg)\Bigg]\Bigg|\\
&+\left|I^{\alpha+\beta}f(s,x(s))(t_2)-\lambda I^{\beta}x(s)(t_2)-I^{\alpha+\beta}f(s,x(s))(t_1)+\lambda I^{\beta}x(s)(t_1)\right|\\
\le\ &\frac{1}{|\Omega|}\Bigg[\left|\frac{(\log t_2)^\beta-\log t_1)^\beta}{\Gamma(\beta+1)}\right||\Omega_3|\Bigg(\sum_{j=1}^{n}|\phi_j|\big[I^{\alpha+\beta+\gamma_j}\|p\|\Upsilon(\|x\|)(\omega_j)\\
&+|\lambda|I^{\beta+\gamma_j}\|x\|(\omega_j)\big]+\sum_{i=1}^{m}|\theta_i|\left[I^{\alpha+\beta+\mu_i}\|p\|\Upsilon(\|x\|)(\eta_i)+|\lambda|I^{\beta+\mu_i}\|x\|(\eta_i)\right]\Bigg)\\
&+\left|\frac{(\log t_2)^\beta-(\log t_1)^\beta}{\Gamma(\beta+1)}\right||\Omega_1|\Bigg(\sum_{l=1}^{q}|\nu_l|\big[I^{\alpha+\beta+\tau_l}\|p\|\Upsilon(\|x\|)(\varphi_l)\\
&+|\lambda|I^{\beta+\tau_l}\|x\|(\varphi_l)\big]+\sum_{k=1}^{p}|\varepsilon_k|\left[I^{\alpha+\beta+\sigma_k}\|p\|\Upsilon(\|x\|)(\psi_k)+|\lambda|I^{\beta+\sigma_k}\|x\|(\psi_k)\right]\Bigg)\Bigg]\\
&+\frac{\|p\|\Upsilon(\|x\|)}{\Gamma(\alpha+\beta+1)}\left|(\log t_2)^{\alpha+\beta}-(\log t_1)^{\alpha+\beta}\right|\\
&+\frac{|\lambda|\|x\|}{\Gamma(\beta+1)}\left|(\log t_1)^\beta-(\log t_2)^\beta+2\left(\log\frac{t_2}{t_1}\right)^\beta\right|\\
\le\ &\frac{1}{|\Omega|}\Bigg[\left|\frac{(\log t_1)^\beta-\log t_2)^\beta}{\Gamma(\beta+1)}\right||\Omega_3|\Bigg(\sum_{j=1}^{n}|\phi_j|\Bigg[\frac{\|p\|\Upsilon(R)(\log\omega_j)^{\alpha+\beta+\gamma_j}}{\Gamma(\alpha+\beta+\gamma_j+1)}\\
&+\frac{|\lambda|R(\log\omega_j)^{\beta+\gamma_j}}{\Gamma(\beta+\gamma_j+1)}\Bigg]+\sum_{i=1}^{m}|\theta_i|\Bigg[\frac{\|p\|\Upsilon(R)(\log\eta_i)^{\alpha+\beta+\mu_i}}{\Gamma(\alpha+\beta+\mu_i+1)}\\
&+\frac{|\lambda|R(\log\eta_i)^{\beta+\mu_i}}{\Gamma(\beta+\mu_i+1)}\Bigg]\Bigg)\\
&+\left|\frac{(\log t_2)^\beta-(\log t_1)^\beta}{\Gamma(\beta+1)}\right||\Omega_1|\Bigg(\sum_{l=1}^{q}|\nu_l|\Bigg[\frac{\|p\|\Upsilon(R)(\log\varphi_l)^{\alpha+\beta+\tau_l}}{\Gamma(\alpha+\beta+\tau_l+1)}\\
&+\frac{|\lambda|R(\log\varphi_l)^{\beta+\tau_l}}{\Gamma(\beta+\tau_l+1)}\Bigg]+\sum_{k=1}^{p}|\varepsilon_k|\Bigg[\frac{\|p\|\Upsilon(R)(\log\psi_k)^{\alpha+\beta+\sigma_k}}{\Gamma(\alpha+\beta+\sigma_k+1)}\\
&+\frac{|\lambda|R(\log\psi_k)^{\beta+\sigma_k}}{\Gamma(\beta+\sigma_k+1)}\Bigg]\Bigg)\Bigg]
\end{aligned}
$$

$$+\frac{\|p\|\Upsilon(R)}{\Gamma(\alpha+\beta+1)}+2(\log t_2/t_1)^{\alpha+\beta}$$
$$+\frac{|\lambda|R}{\Gamma(\beta+1)}\left|(\log t_1)^{\beta}-(\log t_2)^{\beta}+2\left(\log\frac{t_2}{t_1}\right)^{\beta}\right|.$$

As $t_2 - t_1 \to 0$, the right-hand side of the above inequality tends to zero, independently of $x \in B_R$. Therefore, by the Arzelá-Ascoli Theorem, the operator $\mathcal{Q} : \mathcal{E}_1 \to \mathcal{E}_1$ is completely continuous.

Finally, we show that there exists an open set $U \subset \mathcal{E}_1$ with $x \neq \theta\mathcal{Q}x$ for $\theta \in (0, 1)$ and $x \in \partial U$.

Let x be a solution. Then, for $t \in [1, e]$, as in the first step, we have

$$|x(t)| \leq \|p\|\Upsilon(\|x\|)\,\Lambda(\alpha) + |\lambda|\|x\|\Lambda(0)$$

which leads to

$$\frac{\|x\|}{\|p\|\Upsilon(\|x\|)\,\Lambda(\alpha) + |\lambda|\|x\|\Lambda(0)} \leq 1.$$

In view of (7.3.2), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in \mathcal{E}_1 \ : \ \|x\| < M\}.$$

We see that the operator $\mathcal{Q} : \overline{U} \to \mathcal{E}_1$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \theta\mathcal{Q}x$ for some $\theta \in (0, 1)$. Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that $\mathcal{Q}$ has a fixed point $x \in \overline{U}$, which is a solution of the boundary value problem (7.1). This completes the proof. □

Example 7.3 Consider the following nonlinear Langevin equation involving Hadamard-Caputo type fractional derivatives with nonlocal fractional integral conditions

$$\begin{cases} D^{1/2}\left(D^{4/5}+\frac{1}{7}\right)x(t)=\frac{\sin x(t)}{5\pi^2+\cos^2\pi x(t)}+\frac{2+\log t}{\pi^2}, \qquad 1<t<e, \\ 2I^{3/2}x\left(\frac{e+1}{3}\right)+3I^{4/3}x\left(\frac{e+2}{3}\right)+4I^{5/4}x\left(\frac{e+3}{3}\right) \\ \qquad +5I^{6/5}x\left(\frac{e+4}{3}\right)=\frac{2}{3}I^{2/3}x\left(\frac{e+3}{4}\right), \\ \frac{1}{3}I^{3/5}x\left(\frac{2e}{5}\right)+\frac{1}{6}I^{3/4}x\left(\frac{3e}{5}\right)=5I^{3/5}x\left(\frac{2e}{5}\right)+\frac{1}{2}I^{2/5}x\left(\frac{3e}{5}\right). \end{cases} \tag{7.15}$$

Here $\alpha = 1/2$, $\beta = 4/5$, $\lambda = 1/7$, $m = 4$, $n = 1$, $p = 2$, $q = 2$, $\theta_1 = 2$, $\theta_2 = 3$, $\theta_3 = 4$, $\theta_4 = 5$, $\mu_1 = 3/2$, $\mu_2 = 4/3$, $\mu_3 = 5/4$, $\mu_4 = 6/5$,

$\eta_1 = (e+1)/3$, $\eta_2 = (e+2)/3$, $\eta_3 = (e+3)/3$, $\eta_4 = (e+4)/3$, $\phi_1 = 2/3$, $\gamma_1 = 2/3$, $\omega_1 = (e+3)/4$, $\varepsilon_1 = 1/3$, $\varepsilon_2 = 1/6$, $\sigma_1 = 3/5$, $\sigma_2 = 3/4$, $\psi_1 = 2e/5$, $\psi_2 = 3e/5$, $\nu_1 = 5$, $\nu_2 = 1/2$, $\tau_1 = 3/5$, $\tau_2 = 2/5$, $\varphi_1 = 2e/5$, $\varphi_2 = 3e/5$, and $f(t,x) = (\sin x(t))/(5\pi^2 + \cos^2 \pi x(t)) + (2 + \log t)/\pi^2$. Then, we get that $\Lambda(\alpha) \approx 2.675517413$ and $\Lambda(0) \approx 5.058796431$. Clearly,

$$|f(t,x)| = \left| \frac{\sin x(t)}{5\pi^2 + \cos^2 \pi x(t)} + \frac{2 + \log t}{\pi^2} \right| \le \left(\frac{2 + \log t}{5\pi^2} \right) (|x(t)| + 5).$$

Choosing $p(t) = (2 + \log t)/(5\pi^2)$ and $\Upsilon(|x|) = |x| + 5$, we can show that $M > 7.092618387$. Hence, by Theorem 7.3, the nonlocal problem (7.15) has at least one solution on $[1,e]$.

7.2.4 Existence Result via Leray-Schauder's Degree Theory

Theorem 7.4 *Let $f : [1,e] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. In addition, we assume that:*

(7.4.1) there exist constants $0 \le \mu < [1 - |\lambda|\Lambda(0)]\,[\Lambda(\alpha)]^{-1}$ and $K > 0$ such that

$$|f(t,x)| \le \mu|x| + K \text{ for all } (t,x) \in [1,e] \times \mathbb{R},$$

where $\Lambda(.)$ is given by (7.10).

Then, the nonlocal problem (7.1) has at least one solution on $[1,e]$.

Proof Let us consider the fixed point problem

$$x = \mathcal{Q}x, \tag{7.16}$$

where the operator $\mathcal{Q} : \mathcal{E}_1 \to \mathcal{E}_1$ is defined by (7.9) and show that there exists a fixed point $x \in \mathcal{E}_1$ satisfying (7.16). It is sufficient to show that $\mathcal{Q} : \overline{B}_R \to \mathcal{E}_1$ satisfies

$$x \neq \kappa\mathcal{Q}x, \qquad \forall x \in \partial B_R, \qquad \forall \kappa \in [0,1], \tag{7.17}$$

where $B_R = \{x \in \mathcal{E}_1 : \|x\| < R, R > 0\}$. We define

$$H(\kappa, x) = \kappa\mathcal{Q}x, \qquad x \in \mathcal{E}_1, \qquad \kappa \in [0,1].$$

As shown in Theorem 7.3, the operator $\mathcal{Q}$ is continuous, uniformly bounded and equicontinuous. Then, by the Arzelá-Ascoli Theorem, a continuous map h_κ defined by $h_\kappa(x) = x - H(\kappa,x) = x - \kappa\mathcal{Q}x$ is completely continuous. If (7.17) is true, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree,

$$\begin{aligned}\deg(h_\kappa, B_R, 0) &= \deg(I - \kappa\mathscr{Q}, B_R, 0) = \deg(h_1, B_R, 0)\\ &= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \qquad 0 \in B_R, \end{aligned} \tag{7.18}$$

where I denotes the identity operator. By the nonzero property of Leray-Schauder degree, $h_1(x) = x - \mathscr{Q}x = 0$ for at least one $x \in B_R$. In order to establish (7.17), we assume that $x = \kappa\mathscr{Q}x$ for some $\kappa \in [0, 1]$. Then

$$\begin{aligned}
&|x(t)|\\
&\leq \Bigg|\frac{1}{\Omega}\Bigg[\left(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}f(s,x(s))(\omega_j) - \lambda I^{\beta+\gamma_j}x(s)(\omega_j)\right]\right.\\
&\quad \left.-\sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}f(s,x(s))(\eta_i) - \lambda I^{\beta+\mu_i}x(s)(\eta_i)\right]\right)\\
&\quad +\left(\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right)\left(\sum_{l=1}^{q}\nu_l\left[I^{\alpha+\beta+\tau_l}f(s,x(s))(\varphi_l) - \lambda I^{\beta+\tau_l}x(s)(\varphi_l)\right]\right.\\
&\quad \left.\left.-\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\sigma_k}f(s,x(s))(\psi_k) - \lambda I^{\beta+\sigma_k}x(s)(\psi_k)\right]\right)\right]\\
&\quad +I^{\alpha+\beta}f(s,x(s))(t) - \lambda I^\beta x(s)(t)\Bigg|\\
&\leq \frac{1}{|\Omega|}\Bigg[\left(|\Omega_4| + \frac{(\log t)^\beta}{\Gamma(\beta+1)}|\Omega_3|\right)\Bigg(\sum_{j=1}^{n}|\phi_j|\Big[I^{\alpha+\beta+\gamma_j}|f(s,x(s))|(\omega_j)\\
&\quad +|\lambda|I^{\beta+\gamma_j}|x(s)|(\omega_j)\Big] + \sum_{i=1}^{m}|\theta_i|\Big[I^{\alpha+\beta+\mu_i}|f(s,x(s))|(\eta_i)\\
&\quad +|\lambda|I^{\beta+\mu_i}|x(s)|(\eta_i)\Big]\Bigg) + \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)}|\Omega_1| + |\Omega_2|\right)\\
&\quad \times\Bigg(\sum_{l=1}^{q}|\nu_l|\left[I^{\alpha+\beta+\tau_l}|f(s,x(s))|(\varphi_l) + |\lambda|I^{\beta+\tau_l}|x(s)|(\varphi_l)\right]\\
&\quad +\sum_{k=1}^{p}|\varepsilon_k|\left[I^{\alpha+\beta+\sigma_k}|f(s,x(s))|(\psi_k) + |\lambda|I^{\beta+\sigma_k}|x(s)|(\psi_k)\right]\Bigg)\Bigg]\\
&\quad +I^{\alpha+\beta}|f(s,x(s))|(t) + |\lambda|I^\beta|x(s)|(t)
\end{aligned}$$

$$\leq (\mu\|x\| + K)\Lambda(\alpha) + |\lambda|\|x\|\Lambda(0)$$
$$= [\mu\Lambda(\alpha) + |\lambda|\Lambda(0)]\,\|x\| + K\Lambda(\alpha),$$

which, on solving for $\|x\| = \sup_{t\in[1,e]}|x(t)|$, yields

$$\|x\| \leq \frac{K\Lambda(\alpha)}{1-\mu\Lambda(\alpha)-|\lambda|\Lambda(0)}.$$

If $R = \dfrac{K\Lambda(\alpha)}{1-\mu\Lambda(\alpha)-|\lambda|\Lambda(0)} + 1$, the inequality (7.16) hold. This completes the proof. □

7.3 Langevin Inclusions Case

In this section, we study of existence of solutions for the following nonlinear Caputo-Hadamard fractional Langevin inclusion with nonlocal Hadamard fractional integral conditions:

$$\begin{cases} D^{\alpha}(D^{\beta}+\lambda)x(t) \in F(t,x(t)), \quad t \in [1,e], \\ \sum_{i=1}^{m} \theta_i I^{\mu_i} x(\eta_i) = \sum_{j=1}^{n} \phi_j I^{\gamma_j} x(\omega_j), \\ \sum_{k=1}^{p} \varepsilon_k I^{\sigma_k} x(\psi_k) = \sum_{l=1}^{q} \nu_l I^{\tau_l} x(\varphi_l), \end{cases} \tag{7.19}$$

where $F : [1,e]\times\mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map, $\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, while the rest of the quantities are the same as defined in problem (7.1).

Definition 7.1 A function $x \in \mathscr{C}^2([1,e],\mathbb{R})$ is called a solution of problem (7.19) if there exists a function $v \in L^1([1,e],\mathbb{R})$ with $v(t) \in F(t,x(t))$, a.e. on $[1,e]$ such that $D^{\alpha}(D^{\beta}+\lambda)x(t) = v(t)$, a.e. on $[1,e]$ and $\sum_{i=1}^{m}\theta_i I^{\mu_i}x(\eta_i) = \sum_{j=1}^{n}\phi_j I^{\gamma_j}x(\omega_j)$, $\sum_{k=1}^{p}\varepsilon_k I^{\sigma_k}x(\psi_k) = \sum_{l=1}^{q}\nu_l I^{\tau_l}x(\varphi_l)$.

7.3.1 The Lipschitz Case

In this section, we prove the existence of solutions for the problem (7.19) when the right hand side of the inclusion is not necessarily nonconvex valued by applying a fixed point theorem for multivalued maps due to Covitz and Nadler (Theorem 1.18).

Theorem 7.5 *Assume that:*

(7.5.1) $F : [1,e] \times \mathbb{R} \to \mathscr{P}_{cp}(\mathbb{R})$ *is such that* $F(\cdot, x) : [1,e] \to \mathscr{P}_{cp}(\mathbb{R})$ *is measurable for each* $x \in \mathbb{R}$;

(7.5.2) $H_d(F(t,x), F(t,\bar{x})) \leq m(t)|x-\bar{x}|$ *for almost all* $t \in [1,e]$ *and* $x, \bar{x} \in \mathbb{R}$ *with* $m \in C([1,e], \mathbb{R}^+)$ *and* $d(0, F(t,0)) \leq m(t)$ *for almost all* $t \in [1,e]$.

Then, the problem (7.19) has at least one solution on $[1,e]$ *if*

$$\|m\| \Lambda(\alpha) < 1,$$

where $\Lambda(.)$ *is defined in (7.10).*

Proof Define an operator $\mathscr{F} : \mathscr{E}_1 \to \mathscr{P}(\mathscr{E}_1)$ by

$$\mathscr{F}(x) = \left\{ \begin{array}{l} h \in \mathscr{E}_1 : \\ h(t) = \left\{ \begin{array}{l} I^{\alpha+\beta} v(s)(t) - \lambda I^{\beta} x(s)(t) \\ + \frac{1}{\Omega}\Bigg[\left(\Omega_4 - \frac{(\log t)^{\beta}}{\Gamma(\beta+1)} \Omega_3 \right) \Bigg(\sum_{j=1}^{n} \phi_j \Big[I^{\alpha+\beta+\gamma_j} v(s)(\omega_j) \\ -\lambda I^{\beta+\gamma_j} x(s)(\omega_j) \Big] - \sum_{i=1}^{m} \theta_i \left[I^{\alpha+\beta+\mu_i} v(s)(\eta_i) - \lambda I^{\beta+\mu_i} x(s)(\eta_i) \right] \Bigg) \\ + \left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)} \Omega_1 - \Omega_2 \right) \Bigg(\sum_{l=1}^{q} \nu_l \Big[I^{\alpha+\beta+\tau_l} v(s)(\varphi_l) \\ - \lambda I^{\beta+\tau_l} x(s)(\varphi_l) \Big] \\ - \sum_{k=1}^{p} \varepsilon_k \left[I^{\alpha+\beta+\sigma_k} v(s)(\psi_k) - \lambda I^{\beta+\sigma_k} x(s)(\psi_k) \right] \Bigg) \Bigg], \end{array} \right. \end{array} \right\}$$

for $v \in S_{F,x}$.

Observe that the set $S_{F,x}$ is nonempty for each $x \in \mathscr{E}_1$ by the assumption (7.5.1), so F has a measurable selection (see Theorem III.6 [57]). Now, we show that the operator $\mathscr{F}$ satisfies the assumptions of Theorem 1.18. To show that $\mathscr{F}(x) \in \mathscr{P}_{cl}(\mathscr{E}_1)$ for each $x \in \mathscr{E}_1$, let $\{u_n\}_{n \geq 0} \in \mathscr{F}(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $\mathscr{E}_1$. Then $u \in \mathscr{E}_1$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [1,e]$,

$$\begin{aligned} u_n(t) = & \, I^{\alpha+\beta} v_n(s)(t) - \lambda I^{\beta} x(s)(t) \\ & + \frac{1}{\Omega}\Bigg[\left(\Omega_4 - \frac{(\log t)^{\beta}}{\Gamma(\beta+1)} \Omega_3 \right) \Bigg(\sum_{j=1}^{n} \phi_j \Big[I^{\alpha+\beta+\gamma_j} v_n(s)(\omega_j) \\ & -\lambda I^{\beta+\gamma_j} x(s)(\omega_j) \Big] - \sum_{i=1}^{m} \theta_i \left[I^{\alpha+\beta+\mu_i} v_n(s)(\eta_i) - \lambda I^{\beta+\mu_i} x(s)(\eta_i) \right] \Bigg) \end{aligned}$$

$$+\left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_1-\Omega_2\right)\left(\sum_{l=1}^{q}\nu_l\left[I^{\alpha+\beta+\tau_l}v_n(s)(\varphi_l)-\lambda I^{\beta+\tau_l}x(s)(\varphi_l)\right]\right.$$
$$\left.\left.-\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\sigma_k}v_n(s)(\psi_k)-\lambda I^{\beta+\sigma_k}x(s)(\psi_k)\right]\right)\right].$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([1,e],\mathbb{R})$. Thus, $v\in S_{F,x}$ and for each $t\in[1,e]$,

$$v_n(t)\to v(t)=I^{\alpha+\beta}v(s)(t)-\lambda I^{\beta}x(s)(t)$$
$$+\frac{1}{\Omega}\left[\left(\Omega_4-\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\Big[I^{\alpha+\beta+\gamma_j}v(s)(\omega_j)\right.\right.$$
$$\left.-\lambda I^{\beta+\gamma_j}x(s)(\omega_j)\Big]-\sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}v(s)(\eta_i)-\lambda I^{\beta+\mu_i}x(s)(\eta_i)\right]\right)$$
$$+\left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_1-\Omega_2\right)\left(\sum_{l=1}^{q}\nu_l\left[I^{\alpha+\beta+\tau_l}v(s)(\varphi_l)-\lambda I^{\beta+\tau_l}x(s)(\varphi_l)\right]\right.$$
$$\left.\left.-\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\sigma_k}v(s)(\psi_k)-\lambda I^{\beta+\sigma_k}x(s)(\psi_k)\right]\right)\right].$$

Hence, $u\in\mathscr{F}$.

Next, we show that there exists $\gamma<1$ ($\gamma:=\|m\|\Lambda(\alpha)$) such that

$$H_d(\mathscr{F}(x),\mathscr{F}(\bar{x}))\le\gamma\|x-\bar{x}\| \ \text{ for each } \ x,\bar{x}\in\mathscr{E}_1.$$

Let $x,\bar{x}\in\mathscr{E}_1$ and $h_1\in\mathscr{F}(x)$. Then there exists $v_1(t)\in F(t,x(t))$ such that, for each $t\in[1,e]$,

$$h_1(t)=I^{\alpha+\beta}v_1(s)(t)-\lambda I^{\beta}x(s)(t)$$
$$+\frac{1}{\Omega}\left[\left(\Omega_4-\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\Big[I^{\alpha+\beta+\gamma_j}v_1(s)(\omega_j)\right.\right.$$
$$\left.-\lambda I^{\beta+\gamma_j}x(s)(\omega_j)\Big]-\sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}v_1(s)(\eta_i)-\lambda I^{\beta+\mu_i}x(s)(\eta_i)\right]\right)$$
$$+\left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_1-\Omega_2\right)\left(\sum_{l=1}^{q}\nu_l\left[I^{\alpha+\beta+\tau_l}v_1(s)(\varphi_l)-\lambda I^{\beta+\tau_l}x(s)(\varphi_l)\right]\right.$$
$$\left.\left.-\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\sigma_k}v_1(s)(\psi_k)-\lambda I^{\beta+\sigma_k}x(s)(\psi_k)\right]\right)\right].$$

By (7.5.2), we have

$$H_d(F(t,x), F(t,\bar{x})) \le m(t)|x(t) - \bar{x}(t)|.$$

So, there exists $w \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \le m(t)|x(t) - \bar{x}(t)|, \quad t \in [1, e].$$

Define $U : [1, e] \to \mathscr{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \le m(t)|x(t) - \bar{x}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{x}(t))$ is measurable (Proposition III.4 [57]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [1, e]$, we have $|v_1(t) - v_2(t)| \le m(t)|x(t) - \bar{x}(t)|$.

For each $t \in [1, e]$, let us define

$$\begin{aligned} h_2(t) = {} & I^{\alpha+\beta} v_2(s)(t) - \lambda I^{\beta} x(s)(t) \\ & + \frac{1}{\Omega}\Bigg[\left(\Omega_4 - \frac{(\log t)^{\beta}}{\Gamma(\beta+1)} \Omega_3 \right) \Bigg(\sum_{j=1}^{n} \phi_j \Big[I^{\alpha+\beta+\gamma_j} v_2(s)(\omega_j) \\ & - \lambda I^{\beta+\gamma_j} x(s)(\omega_j) \Big] - \sum_{i=1}^{m} \theta_i \left[I^{\alpha+\beta+\mu_i} v_2(s)(\eta_i) - \lambda I^{\beta+\mu_i} x(s)(\eta_i) \right] \Bigg) \\ & + \left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)} \Omega_1 - \Omega_2 \right) \Bigg(\sum_{l=1}^{q} \nu_l \left[I^{\alpha+\beta+\tau_l} v_2(s)(\varphi_l) - \lambda I^{\beta+\tau_l} x(s)(\varphi_l) \right] \\ & - \sum_{k=1}^{p} \varepsilon_k \left[I^{\alpha+\beta+\sigma_k} v_2(s)(\psi_k) - \lambda I^{\beta+\sigma_k} x(s)(\psi_k) \right] \Bigg) \Bigg]. \end{aligned}$$

Thus,

$$\begin{aligned} |h_1(t) - h_2(t)| \le {} & I^{\alpha+\beta} |v_1(s) - v_2(s)|(t) \\ & + \frac{1}{\Omega}\Bigg[\left(\Omega_4 - \frac{(\log t)^{\beta}}{\Gamma(\beta+1)} \Omega_3 \right) \Bigg(\sum_{j=1}^{n} \phi_j I^{\alpha+\beta+\gamma_j} |v_1(s) - v_2(s)|(t)(\omega_j) \\ & + \sum_{i=1}^{m} \theta_i I^{\alpha+\beta+\mu_i} |v_1(s) - v_2(s)|(t)\eta_i) \Bigg) \\ & + \left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)} \Omega_1 - \Omega_2 \right) \Bigg(\sum_{l=1}^{q} \nu_l I^{\alpha+\beta+\tau_l} |v_1(s) - v_2(s)|(t)(\varphi_l) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{p}\varepsilon_k I^{\alpha+\beta+\sigma_k}|v_1(s)-v_2(s)|(t)(\psi_k)\Bigg)\Bigg]\\
&\leq \|m\|\Bigg\{\frac{1}{\Gamma(\alpha+\beta+1)}\\
&+\frac{1}{|\Omega|}\Bigg[\Bigg(|\Omega_4|+\frac{|\Omega_3|}{\Gamma(\beta+1)}\Bigg)\Bigg(\sum_{j=1}^{n}|\phi_j|\frac{(\log\omega_j)^{\alpha+\beta+\gamma_j}}{\Gamma(\alpha+\beta+\gamma_j+1)}\\
&+\sum_{i=1}^{m}|\theta_i|\frac{(\log\eta_i)^{\alpha+\beta+\mu_i}}{\Gamma(\alpha+\beta+\mu_i+1)}\Bigg)\\
&+\Bigg(\frac{1}{\Gamma(\beta+1)}|\Omega_1|+|\Omega_2|\Bigg)\Bigg(\sum_{l=1}^{q}|\nu_l|\frac{(\log\varphi_l)^{\alpha+\beta+\tau_l}}{\Gamma(\alpha+\beta+\tau_l+1)}\\
&+\sum_{k=1}^{p}|\varepsilon_k|\frac{(\log\psi_k)^{\alpha+\beta+\sigma_k}}{\Gamma(\alpha+\beta+\sigma_k+1)}\Bigg)\Bigg]\Bigg\}\|x-\bar{x}\|\\
&\leq \|m\|\Lambda(\alpha)\|x-\bar{x}\|.
\end{aligned}
$$

Hence,

$$\|h_1-h_2\| \leq \|m\|\Lambda(\alpha)\|x-\bar{x}\|.$$

Analogously, interchanging the roles of x and $\bar{x}$, we obtain

$$H_d(\mathscr{F}(x),\mathscr{F}(\bar{x})) \leq \|m\|\Lambda(\alpha)\|x-\bar{x}\|.$$

In view of the given condition ($\|m\|\Lambda(\alpha) < 1$), we conclude that $\mathscr{F}$ is a contraction. Thus it follows by Theorem 1.18 that $\mathscr{F}$ has a fixed point x which is a solution of (7.19). This completes the proof. □

7.3.2 *The Carathéodory Case*

In this section, we consider the case when F has convex values and prove an existence result based on nonlinear alternative of Leray-Schauder type, assuming that F is Carathéodory.

Theorem 7.6 *Assume that (7.3.2) holds. In addition we suppose that:*

(7.6.1) $F : [1, e] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ *is* L^1*-Carathéodory and has nonempty compact and convex values;*

(7.6.2) there exist a continuous nondecreasing function $\Upsilon : [0, \infty) \to (0, \infty)$ *and a function* $p \in C([1, e], \mathbb{R}^+)$ *such that*

$$\|F(t,x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\Upsilon(\|x\|) \ \textit{for each}\ \ (t,x) \in [1,e] \times \mathbb{R}.$$

Then the problem (7.19) has at least one solution on $[1, e]$.

Proof Consider the operator $\mathscr{F} : \mathscr{E}_1 \to \mathscr{P}(\mathscr{E}_1)$ defined in the begin of the proof of Theorem 7.5, and show that $\mathscr{F}$ satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps. As a first step, we show that $\mathscr{F}$ *is convex for each* $x \in \mathscr{E}_1$. This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore, we omit the proof.

In the second step, we show that $\mathscr{F}$ *maps bounded sets (balls) into bounded sets in* $\mathscr{E}_1$. For a positive number ρ, let $B_\rho = \{x \in \mathscr{E}_1 : \|x\| \le \rho\}$ be a bounded ball in $\mathscr{E}_1$. Then, for each $h \in \mathscr{F}(x), x \in B_\rho$, there exists $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) &= I^{\alpha+\beta} v(s)(t) - \lambda I^{\beta} x(s)(t) \\ &\quad + \frac{1}{\Omega}\Bigg[\left(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\Big[I^{\alpha+\beta+\gamma_j} v(s)(\omega_j)\right. \\ &\quad \left. -\lambda I^{\beta+\gamma_j} x(s)(\omega_j)\Big] - \sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i} v(s)(\eta_i) - \lambda I^{\beta+\mu_i} x(s)(\eta_i)\right]\right) \\ &\quad + \left(\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right)\left(\sum_{l=1}^{q}\nu_l\left[I^{\alpha+\beta+\tau_l} v(s)(\varphi_l) - \lambda I^{\beta+\tau_l} x(s)(\varphi_l)\right]\right. \\ &\quad \left. - \sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\sigma_k} v(s)(\psi_k) - \lambda I^{\beta+\sigma_k} x(s)(\psi_k)\right]\right)\Bigg]. \end{aligned}$$

Then, for $t \in [1, e]$, we have

$$\begin{aligned} |h(t)| \le \Bigg| & I^{\alpha+\beta} v(s)(t) - \lambda I^{\beta} x(s)(t) \\ & + \frac{1}{\Omega}\Bigg[\left(\Omega_4 - \frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j} v(s)(\omega_j) - \lambda I^{\beta+\gamma_j} x(s)(\omega_j)\right]\right. \\ & \left. - \sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i} v(s(\eta_i) - \lambda I^{\beta+\mu_i} x(s)(\eta_i)\right]\right) \end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_1-\Omega_2\right)\left(\sum_{l=1}^{q}\nu_l\left[I^{\alpha+\beta+\tau_l}v(s(\varphi_l)-\lambda I^{\beta+\tau_l}x(s)(\varphi_l)\right]\right.\\
&\left.\left.\left.-\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\sigma_k}v(s(\psi_k)-\lambda I^{\beta+\sigma_k}x(s)(\psi_k)\right]\right)\right]\right|\\
\le\ & I^{\alpha+\beta}\|p\|\Upsilon\left(\|x\|\right)(t)+|\lambda|I^{\beta}\|x\|(t)\\
&+\frac{1}{|\Omega|}\left[\left(|\Omega_4|+\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}|\Omega_3|\right)\left(\sum_{j=1}^{n}\left|\phi_j\right|\left[I^{\alpha+\beta+\gamma_j}\|p\|\Upsilon\left(\|x\|\right)(\omega_j)\right.\right.\right.\\
&\left.+|\lambda|I^{\beta+\gamma_j}\|x\|(\omega_j)\right]+\sum_{i=1}^{m}|\theta_i|\Big[I^{\alpha+\beta+\mu_i}\|p\|\Upsilon\left(\|x\|\right)(\eta_i)\\
&\left.\left.+|\lambda|I^{\beta+\mu_i}\|x\|(\eta_i)\Big]\right)+\left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}|\Omega_1|+|\Omega_2|\right)\\
&\times\left(\sum_{l=1}^{q}|\nu_l|\left[I^{\alpha+\beta+\tau_l}\|p\|\Upsilon\left(\|x\|\right)(\varphi_l)+|\lambda|I^{\beta+\tau_l}\|x\|(\varphi_l)\right]\right.\\
&\left.\left.+\sum_{k=1}^{p}|\varepsilon_k|\left[I^{\alpha+\beta+\sigma_k}\|p\|\Upsilon\left(\|x\|\right)(\psi_k)+|\lambda|I^{\beta+\sigma_k}\|x\|(\psi_k)\right]\right)\right]\\
\le\ & \|p\|\Upsilon\left(\|x\|\right)\Lambda(\alpha)+|\lambda|\rho\Lambda(0)\\
\le\ & \|p\|\Upsilon\left(\rho\right)\Lambda(\alpha)+|\lambda|\rho\Lambda(0),
\end{aligned}
$$

which implies that

$$\|h\|\le\|p\|\Upsilon\left(\rho\right)\Lambda(\alpha)+|\lambda|\rho\Lambda(0).$$

Next, we will show that $\mathscr{F}$ *maps bounded sets into equicontinuous sets of* $\mathscr{E}_1$. Let $t_1, t_2 \in [1, e]$ with $t_1 < t_2$ and $x \in B_\rho$. Then, we have

$$
\begin{aligned}
&|h(t_2)-h(t_1)|\\
\le&\left|I^{\alpha+\beta}v(s)(t_2)-\lambda I^{\beta}x(s)(t_2)-I^{\alpha+\beta}v(s)(t_1)+\lambda I^{\beta}x(s)(t_1)\right.\\
&+\frac{1}{\Omega}\left[\frac{(\log t_2)^{\beta}-\log t_1)^{\beta}}{\Gamma(\beta+1)}\Omega_3\left(\sum_{j=1}^{n}\phi_j\left[I^{\alpha+\beta+\gamma_j}v(s)(\omega_j)-\lambda I^{\beta+\gamma_j}x(s)(\omega_j)\right]\right.\right.\\
&\left.-\sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}v(s)(\eta_i)-\lambda I^{\beta+\mu_i}x(s)(\eta_i)\right]\right)
\end{aligned}
$$

$$+\frac{(\log t_2)^\beta-(\log t_1)^\beta}{\Gamma(\beta+1)}\Omega_1\left(\sum_{l=1}^{q}\nu_l\left[I^{\alpha+\beta+\tau_l}v(s)(\varphi_l)-\lambda I^{\beta+\tau_l}x(s)(\varphi_l)\right]\right.$$

$$\left.\left.\left.-\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\sigma_k}v(s)(\psi_k)-\lambda I^{\beta+\sigma_k}x(s)(\psi_k)\right]\right)\right]\right|$$

$$\leq\frac{\|p\|\Upsilon(\|x\|)}{\Gamma(\alpha+\beta+1)}\left|(\log t_2)^{\alpha+\beta}-(\log t_1)^{\alpha+\beta}\right|$$

$$+\frac{|\lambda|\|x\|}{\Gamma(\beta+1)}\left|(\log t_1)^\beta-(\log t_2)^\beta+2\left(\log\frac{t_2}{t_1}\right)^\beta\right|$$

$$+\frac{1}{|\Omega|}\left[\left|\frac{(\log t_2)^\beta-\log t_1)^\beta}{\Gamma(\beta+1)}\right||\Omega_3|\left(\sum_{j=1}^{n}|\phi_j|\left[I^{\alpha+\beta+\gamma_j}\|p\|\Upsilon(\|x\|)(\omega_j)\right.\right.\right.$$

$$\left.\left.+|\lambda|I^{\beta+\gamma_j}\|x\|(\omega_j)\right]+\sum_{i=1}^{m}|\theta_i|\left[I^{\alpha+\beta+\mu_i}\|p\|\Upsilon(\|x\|)(\eta_i)+|\lambda|I^{\beta+\mu_i}\|x\|(\eta_i)\right]\right)$$

$$+\left|\frac{(\log t_2)^\beta-(\log t_1)^\beta}{\Gamma(\beta+1)}\right||\Omega_1|\left(\sum_{l=1}^{q}|\nu_l|\left[I^{\alpha+\beta+\tau_l}\|p\|\Upsilon(\|x\|)(\varphi_l)\right.\right.$$

$$\left.\left.\left.+|\lambda|I^{\beta+\tau_l}\|x\|(\varphi_l)\right]+\sum_{k=1}^{p}|\varepsilon_k|\left[I^{\alpha+\beta+\sigma_k}\|p\|\Upsilon(\|x\|)(\psi_k)+|\lambda|I^{\beta+\sigma_k}\|x\|(\psi_k)\right]\right)\right]$$

$$\leq\frac{\|p\|\Upsilon(\rho)}{\Gamma(\alpha+\beta+1)}\left|(\log t_2)^{\alpha+\beta}-(\log t_1)^{\alpha+\beta}\right|$$

$$+\frac{|\lambda|\rho}{\Gamma(\beta+1)}\left|(\log t_1)^\beta-(\log t_2)^\beta+2\left(\log\frac{t_2}{t_1}\right)^\beta\right|$$

$$+\frac{1}{|\Omega|}\left[\left|\frac{(\log t_1)^\beta-\log t_2)^\beta}{\Gamma(\beta+1)}\right||\Omega_3|\left(\sum_{j=1}^{n}|\phi_j|\left[\frac{\|p\|\Upsilon(\rho)(\log\omega_j)^{\alpha+\beta+\gamma_j}}{\Gamma(\alpha+\beta+\gamma_j+1)}\right.\right.\right.$$

$$\left.+\frac{|\lambda|\rho(\log\omega_j)^{\beta+\gamma_j}}{\Gamma(\beta+\gamma_j+1)}\right]$$

$$\left.+\sum_{i=1}^{m}|\theta_i|\left[\frac{\|p\|\Upsilon(\rho)(\log\eta_i)^{\alpha+\beta+\mu_i}}{\Gamma(\alpha+\beta+\mu_i+1)}+\frac{|\lambda|\rho(\log\eta_i)^{\beta+\mu_i}}{\Gamma(\beta+\mu_i+1)}\right]\right)$$

$$+\left|\frac{(\log t_2)^\beta-(\log t_1)^\beta}{\Gamma(\beta+1)}\right||\Omega_1|\left(\sum_{l=1}^{q}|\nu_l|\left[\frac{\|p\|\Upsilon(\rho)(\log\varphi_l)^{\alpha+\beta+\tau_l}}{\Gamma(\alpha+\beta+\tau_l+1)}\right.\right.$$

$$+\frac{|\lambda|\rho(\log\varphi_l)^{\beta+\tau_l}}{\Gamma(\beta+\tau_l+1)}\Bigg]$$
$$+\sum_{k=1}^{p}|\varepsilon_k|\left[\frac{\|p\|\Upsilon(\rho)(\log\psi_k)^{\alpha+\beta+\sigma_k}}{\Gamma(\alpha+\beta+\sigma_k+1)}+\frac{|\lambda|\rho(\log\psi_k)^{\beta+\sigma_k}}{\Gamma(\beta+\sigma_k+1)}\right]\Bigg)\Bigg].$$

As $t_2 - t_1 \to 0$, the right-hand side of the above inequality tends to zero, independently of $x \in B_\rho$. Therefore, by the Arzelá-Ascoli Theorem, the operator $\mathscr{F} : \mathscr{E}_1 \to \mathscr{P}(\mathscr{E}_1)$ is completely continuous.

By Lemma 1.1, $\mathscr{F}$ will be upper semi-continuous (u.s.c.) if, we prove that it has a closed graph since $\mathscr{F}$ is already shown to be completely continuous.

Thus, in our next step, we show that $\mathscr{F}$ *has a closed graph.* Let $x_n \to x_*, h_n \in \mathscr{F}(x_n)$ and $h_n \to h_*$. Then, we need to show that $h_* \in \mathscr{F}(x_*)$. Associated with $h_n \in \mathscr{F}(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in [1, e]$,

$$h_n(t) = I^{\alpha+\beta}v_n(s)(t) - \lambda I^{\beta}x(s)(t)$$
$$+\frac{1}{\Omega}\Bigg[\left(\Omega_4-\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_3\right)\Bigg(\sum_{j=1}^{n}\phi_j\Big[I^{\alpha+\beta+\gamma_j}v_n(s)(\omega_j)$$
$$-\lambda I^{\beta+\gamma_j}x(s)(\omega_j)\Big]-\sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}v_n(s)(\eta_i)-\lambda I^{\beta+\mu_i}x(s)(\eta_i)\right]\Bigg)$$
$$+\left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_1-\Omega_2\right)\Bigg(\sum_{l=1}^{q}\nu_l\left[I^{\alpha+\beta+\tau_l}v_n(s)(\varphi_l)-\lambda I^{\beta+\tau_l}x(s)(\varphi_l)\right]$$
$$-\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\sigma_k}v_n(s)(\psi_k)-\lambda I^{\beta+\sigma_k}x(s)(\psi_k)\right]\Bigg)\Bigg].$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [1, e]$,

$$h_*(t) = I^{\alpha+\beta}v_*(s)(t) - \lambda I^{\beta}x(s)(t)$$
$$+\frac{1}{\Omega}\Bigg[\left(\Omega_4-\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_3\right)\Bigg(\sum_{j=1}^{n}\phi_j\Big[I^{\alpha+\beta+\gamma_j}v_*(s)(\omega_j)$$
$$-\lambda I^{\beta+\gamma_j}x(s)(\omega_j)\Big]-\sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}v_*(s)(\eta_i)-\lambda I^{\beta+\mu_i}x(s)(\eta_i)\right]\Bigg)$$

$$+\left(\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_1-\Omega_2\right)\left(\sum_{l=1}^{q}\nu_l\left[I^{\alpha+\beta+\tau_l}v_*(s)(\varphi_l)-\lambda I^{\beta+\tau_l}x(s)(\varphi_l)\right]\right.$$
$$\left.\left.-\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\sigma_k}v_*(s)(\psi_k)-\lambda I^{\beta+\sigma_k}x(s)(\psi_k)\right]\right)\right].$$

Let us consider the linear operator $\Theta : L^1([1,e],\mathbb{R}) \to \mathscr{E}_1$ defined by

$$f \mapsto \Theta(v)(t) = I^{\alpha+\beta}v(s)(t)-\lambda I^\beta x(s)(t)$$
$$+\frac{1}{\Omega}\left[\left(\Omega_4-\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\Big[I^{\alpha+\beta+\gamma_j}v(s)(\omega_j)\right.\right.$$
$$\left.-\lambda I^{\beta+\gamma_j}x(s)(\omega_j)\Big]-\sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}v(s)(\eta_i)-\lambda I^{\beta+\mu_i}x(s)(\eta_i)\right]\right)$$
$$+\left(\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_1-\Omega_2\right)\left(\sum_{l=1}^{q}\nu_l\left[I^{\alpha+\beta+\tau_l}v(s)(\varphi_l)\right.\right.$$
$$\left.-\lambda I^{\beta+\tau_l}x(s)(\varphi_l)\right]$$
$$\left.\left.-\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\sigma_k}v(s)(\psi_k)-\lambda I^{\beta+\sigma_k}x(s)(\psi_k)\right]\right)\right].$$

Observe that

$$\|h_n(t)-h_*(t)\| = \left\| I^{\alpha+\beta}(v_n(s)-v_*(s))(t)\right.$$
$$+\frac{1}{\Omega}\left[\left(\Omega_4-\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j I^{\alpha+\beta+\gamma_j}(v_n(s)-v_*(s))(\omega_j)\right.\right.$$
$$\left.-\sum_{i=1}^{m}\theta_i I^{\alpha+\beta+\mu_i}(v_n(s)-v_*(s))(\eta_i)\right)$$
$$+\left(\frac{(\log t)^\beta}{\Gamma(\beta+1)}\Omega_1-\Omega_2\right)\left(\sum_{l=1}^{q}\nu_l I^{\alpha+\beta+\tau_l}(v_n(s)-v_*(s))(\varphi_l)\right.$$
$$\left.\left.\left.-\sum_{k=1}^{p}\varepsilon_k I^{\alpha+\beta+\sigma_k}(v_n(s)-v_*(s))(\psi_k)\right)\right]\right\| \to 0,$$

as $n \to \infty$.

Thus, it follows by Lemma 1.2 that $\Theta \circ S_{F,x}$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$\begin{aligned} h_*(t) &= I^{\alpha+\beta} v_*(s)(t) - \lambda I^{\beta} x(s)(t) \\ &\quad + \frac{1}{\Omega}\Bigg[\left(\Omega_4 - \frac{(\log t)^{\beta}}{\Gamma(\beta+1)} \Omega_3 \right) \left(\sum_{j=1}^{n} \phi_j \Big[I^{\alpha+\beta+\gamma_j} v_*(s)(\omega_j) \right. \\ &\quad \left. - \lambda I^{\beta+\gamma_j} x(s)(\omega_j) \Big] - \sum_{i=1}^{m} \theta_i \left[I^{\alpha+\beta+\mu_i} v_*(s)(\eta_i) - \lambda I^{\beta+\mu_i} x(s)(\eta_i) \right] \right) \\ &\quad + \left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)} \Omega_1 - \Omega_2 \right) \left(\sum_{l=1}^{q} \nu_l \left[I^{\alpha+\beta+\tau_l} v_*(s)(\varphi_l) - \lambda I^{\beta+\tau_l} x(s)(\varphi_l) \right] \right. \\ &\quad \left. - \sum_{k=1}^{p} \varepsilon_k \left[I^{\alpha+\beta+\sigma_k} v_*(s)(\psi_k) - \lambda I^{\beta+\sigma_k} x(s)(\psi_k) \right] \right) \Bigg], \end{aligned}$$

for some $v_* \in S_{F,x_*}$.

Finally, we show there exists an open set $U \subseteq \mathscr{E}_1$ with $x \notin \theta \mathscr{F}(x)$ for any $\theta \in (0,1)$ and all $x \in \partial U$. Let $\theta \in (0,1)$ and $x \in \theta \mathscr{F}(x)$. Then there exists $v \in L^1([1,e], \mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in [1,e]$, we have

$$\begin{aligned} x(t) &= \theta I^{\alpha+\beta} v(s)(t) - \lambda \theta I^{\beta} x(s)(t) \\ &\quad + \theta \frac{1}{\Omega}\Bigg[\left(\Omega_4 - \frac{(\log t)^{\beta}}{\Gamma(\beta+1)} \Omega_3 \right) \left(\sum_{j=1}^{n} \phi_j \Big[I^{\alpha+\beta+\gamma_j} v(s)(\omega_j) \right. \\ &\quad \left. - \lambda I^{\beta+\gamma_j} x(s)(\omega_j) \Big] - \sum_{i=1}^{m} \theta_i \left[I^{\alpha+\beta+\mu_i} v(s)(\eta_i) - \lambda I^{\beta+\mu_i} x(s)(\eta_i) \right] \right) \\ &\quad + \theta \left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)} \Omega_1 - \Omega_2 \right) \left(\sum_{l=1}^{q} \nu_l \left[I^{\alpha+\beta+\tau_l} v(s)(\varphi_l) - \lambda I^{\beta+\tau_l} x(s)(\varphi_l) \right] \right. \\ &\quad \left. - \sum_{k=1}^{p} \varepsilon_k \left[I^{\alpha+\beta+\sigma_k} v(s)(\psi_k) - \lambda I^{\beta+\sigma_k} x(s)(\psi_k) \right] \right) \Bigg]. \end{aligned}$$

As in the second step above, we have

$$|x(t)| \leq \|p\| \Upsilon(\|x\|) \Lambda(\alpha) + |\lambda| \|x\| \Lambda(0)$$

which leads to

$$\frac{\|x\|}{\|p\| \Upsilon(\|x\|) \Lambda(\alpha) + |\lambda| \|x\| \Lambda(0)} \leq 1.$$

In view of (7.6.3), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in \mathscr{E}_1 : \|x\| < M\}.$$

Note that the operator $\mathscr{F} : \overline{U} \to \mathscr{P}(\mathscr{E}_1)$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \theta\mathscr{F}(x)$ for some $\theta \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 1.15), we deduce that $\mathscr{F}$ has a fixed point $x \in \overline{U}$ which is a solution of the problem (7.19). This completes the proof. □

7.3.3 *The Lower Semicontinuous Case*

In the next result, F is not necessarily convex valued. We use the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo (Lemma 1.3) for lower semi-continuous maps with decomposable values, to establish the next existence result.

Theorem 7.7 *Assume that (7.3.2), (7.6.2) and the following condition holds:*

(7.7.1) $F : [1,e] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*

(a) $(t,x) \longmapsto F(t,x)$ *is* $\mathscr{L} \otimes \mathscr{B}$ *measurable,*
(b) $x \longmapsto F(t,x)$ *is lower semicontinuous for each* $t \in [1,e]$.

Then, the problem (7.19) has at least one solution on $[1,e]$.

Proof It follows from (7.6.2) and (7.7.1) that F is of l.s.c. type. Then, from Lemma 1.3, there exists a continuous function $f : \mathscr{E}_1 \to L^1([1,e],\mathbb{R})$ such that $f(x) \in \mathscr{F}(x)$ for all $x \in \mathscr{E}_1$.

Consider the problem

$$\begin{cases} D^{\alpha}(D^{\beta}+\lambda)x(t) = f(x(t)), \quad 1 < t < e, \\ \sum\limits_{i=1}^{m} \theta_i I^{\mu_i} x(\eta_i) = \sum\limits_{j=1}^{n} \phi_j I^{\gamma_j} x(\omega_j), \\ \sum\limits_{k=1}^{p} \varepsilon_k I^{\sigma_k} x(\psi_k) = \sum\limits_{l=1}^{q} \nu_l I^{\tau_l} x(\varphi_l), \end{cases} \tag{7.20}$$

Observe that if $x \in \mathscr{C}^2([1,e],\mathbb{R})$ is a solution of (7.20), then x is a solution to the problem (7.19). In order to transform the problem (7.20) into a fixed point problem, we define an operator $\overline{\mathscr{F}}$ as

$$
\begin{aligned}
\overline{\mathscr{F}}x(t) &= I^{\alpha+\beta}f(x(s))(t) - \lambda I^{\beta}x(s)(t) \\
&+\frac{1}{\Omega}\Bigg[\left(\Omega_4 - \frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_3\right)\left(\sum_{j=1}^{n}\phi_j\Big[I^{\alpha+\beta+\gamma_j}vf(x(s))(\omega_j)\right.\\
&\left.-\lambda I^{\beta+\gamma_j}x(s)(\omega_j)\Big] - \sum_{i=1}^{m}\theta_i\left[I^{\alpha+\beta+\mu_i}f(x(s))(\eta_i) - \lambda I^{\beta+\mu_i}x(s)(\eta_i)\right]\right)\\
&+\left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}\Omega_1 - \Omega_2\right)\left(\sum_{l=1}^{q}\nu_l\left[I^{\alpha+\beta+\tau_l}f(x(s))(\varphi_l) - \lambda I^{\beta+\tau_l}x(s)(\varphi_l)\right]\right.\\
&\left.-\sum_{k=1}^{p}\varepsilon_k\left[I^{\alpha+\beta+\sigma_k}f(x(s))(\psi_k) - \lambda I^{\beta+\sigma_k}x(s)(\psi_k)\right]\right)\Bigg].
\end{aligned}
$$

It can easily be shown that $\overline{\mathscr{F}}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 7.6. So, we omit it. This completes the proof. □

7.3.4 Examples

We illustrate our results with the help of some examples. Let us consider the following boundary value problem for Caputo-Hadamard fractional differential Langevin inclusion with nonlocal fractional integral boundary conditions

$$
\begin{cases}
D^{1/2}\left(D^{4/5} + \frac{1}{15}\right)x(t) \in F(t, x(t)), \qquad 1 < t < e,\\
2I^{3/2}x\left(\frac{e+1}{3}\right) + 3I^{4/3}x\left(\frac{e+2}{3}\right) + 4I^{5/4}x\left(\frac{e+3}{3}\right) + 5I^{6/5}x\left(\frac{e+4}{3}\right)\\
= \frac{2}{3}I^{2/3}x\left(\frac{e+3}{4}\right) + 4I^{2/3}x\left(\frac{2e}{3}\right) + 3I^{3/5}x\left(\frac{2e+1}{3}\right),\\
\frac{1}{3}I^{3/5}x\left(\frac{2e}{5}\right) + \frac{1}{6}I^{3/4}x\left(\frac{3e}{5}\right) = 5I^{3/5}x\left(\frac{2e}{5}\right) + \frac{1}{2}I^{2/5}x\left(\frac{3e}{5}\right).
\end{cases}
\tag{7.21}
$$

Here $\alpha = 1/2$, $\beta = 4/5$, $\lambda = 1/15$, $m = 4$, $n = 3$, $p = 2$, $q = 2$, $\theta_1 = 2$, $\theta_2 = 3$, $\theta_3 = 4$, $\theta_4 = 5$, $\mu_1 = 3/2$, $\mu_2 = 4/3$, $\mu_3 = 5/4$, $\mu_4 = 6/5$, $\eta_1 = (e+1)/3$, $\eta_2 = (e+2)/3$, $\eta_3 = (e+3)/3$, $\eta_4 = (e+4)/3$, $\phi_1 = 2/3$, $\phi_2 = 4$, $\phi_3 = 3$, $\gamma_1 = 2/3$, $\gamma_2 = 2/3$, $\gamma_3 = 3/5$, $\omega_1 = (e+3)/4$, $\omega_2 = (2e)/3$, $\omega_3 = (2e+1)/3$, $\varepsilon_1 = 1/3$, $\varepsilon_2 = 1/6$, $\sigma_1 = 3/5$, $\sigma_2 = 3/4$, $\psi_1 = 2e/5$, $\psi_2 = 3e/5$, $\nu_1 = 5$, $\nu_2 = 1/2$, $\tau_1 = 3/5$, $\tau_2 = 2/5$, $\varphi_1 = 2e/5$, $\varphi_2 = 3e/5$. Using the given data, we find that $\Omega_1 = 0.210907130$, $\Omega_2 = -0.537266438$, $\Omega_3 = -1.496430446$, $\Omega_4 = -0.269076088$, $\Omega = -0.860731921$, $\Lambda(1/2) \approx 7.786402339$, and $\Lambda(0) \approx 14.19541525$.

(a) Let $F : [1, e] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ be a multivalued map given by

$$x \to F(t,x) = \left[0,\ \frac{\sin x}{4(1+t)} + \frac{2}{3}\right]. \tag{7.22}$$

Then, we have

$$\sup\{|x| \ : \ x \in F(t,x)\} \le \frac{1}{4(1+t)} + \frac{2}{3},$$

and

$$H_d(F(t,x),\ F(t,\bar{x})) \le \frac{1}{4(1+t)}|x - \bar{x}|.$$

Let $m(t) = 1/(4(1+t))$. Then $H_d(F(t,x),\ F(t,\bar{x})) \le m(t)|x-\bar{x}|$, and $\|m\| = 1/8$. Further $\|m\|\Lambda(1/2) \approx 0.973300292 < 1$.

By Theorem 7.5, the problem (7.21) with $F(t,x)$ given by (7.22) has at least one solution on $[1, e]$.

(b) Let $F : [1, e] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ be a multivalued map given by

$$x \to F(t,x) = \left[\frac{|x|}{|x| + \log t + 1},\ \frac{e^x}{e^x + \cos^2 x} + \frac{3t^2}{2e^2} + \frac{2}{3}\right]. \tag{7.23}$$

For $f \in F$, we have

$$|f| \le \max\left(\frac{|x|}{|x| + \log t + 1},\ \frac{e^x}{e^x + \cos^2 x} + \frac{3t^2}{2e^2} + \frac{2}{3}\right) \le \frac{19}{6}, \quad x \in \mathbb{R}.$$

Thus,

$$\|F(t,x)\|_{\mathscr{P}} := \sup\{|y| \ : \ y \in F(t,x)\} \le \frac{19}{6} = p(t)\Upsilon(\|x\|), \quad x \in \mathbb{R},$$

with $p(t) = 1/6$, $\Upsilon(\|x\|) = 19$. Further, using the condition (7.6.3), we find that $M > 459.683223057$. Therefore, all the conditions of Theorem 7.6 are satisfied. So, the problem (7.21) with $F(t,x)$ given by (7.23) has at least one solution on [1,e].

7.4 Langevin Equations with Fractional Coupled Integral Conditions

In this section, we study the existence and uniqueness of solutions for a coupled system of Riemann-Liouville and Hadamard fractional Langevin equations with fractional coupled integral conditions of the form

$$\begin{cases} {}_{RL}D^{q_1}\left({}_{RL}D^{p_1}+\lambda_1\right)x(t)=f(t,x(t),y(t)), & a\le t\le T,\\ {}_{H}D^{q_2}\left({}_{H}D^{p_2}+\lambda_2\right)y(t)\quad=g(t,x(t),y(t)), & a\le t\le T,\\ x(a)=0, \qquad \sigma_1 x(\tau_1)=\sum_{i=1}^{m}\alpha_i {}_{H}I^{\rho_i}y(\eta_i),\\ y(a)=0, \qquad \sigma_2 y(\tau_2)=\sum_{j=1}^{n}\beta_j {}_{RL}I^{\gamma_j}x(\xi_j), \end{cases} \tag{7.24}$$

where ${}_{RL}D^q, {}_{H}D^p$ are the Riemann-Liouville and Hadamard fractional derivatives of orders q,p, respectively, with $q\in\{q_1,p_1\}, p\in\{q_2,p_2\}, 0<q_k,p_k\le 1, \lambda_k$ are given constants, $k=1,2$, ${}_{RL}I^{\gamma_j}, {}_{H}I^{\rho_i}$ are the Riemann-Liouville and Hadamard fractional integral of orders $\gamma_j,\rho_i>0$, respectively, $\eta_i,\xi_j\in(a,T)$ and $\alpha_i,\beta_j,\sigma_1,\sigma_2\in\mathbb{R}$ for all $i=1,2,\ldots,m, j=1,2,\ldots,n$, $\tau_1,\tau_2\in(a,T]$, $f,g:[a,T]\times\mathbb{R}^2\to\mathbb{R}$ are continuous functions.

In the following analysis, we set

$$\Omega_1'=\sum_{i=1}^{m}\frac{\alpha_i\Gamma(q_2)\left(\log\frac{\eta_i}{a}\right)^{q_2+p_2+\rho_i-1}}{\Gamma(q_2+p_2+\rho_i)},\quad \Omega_2'=\sum_{j=1}^{n}\frac{\beta_j\Gamma(q_1)(\xi_j-a)^{q_1+p_1+\gamma_j-1}}{\Gamma(q_1+p_1+\gamma_j)},$$
$$\Omega_3'=\frac{\sigma_2\Gamma(q_2)}{\Gamma(q_2+p_2)}\left(\log\frac{\tau_2}{a}\right)^{q_2+p_2-1},\quad \Omega_4'=\frac{\sigma_1\Gamma(q_1)}{\Gamma(q_1+p_1)}(\tau_1-a)^{q_1+p_1-1},$$

and

$$\Omega'=\Omega_1'\Omega_2'-\Omega_3'\Omega_4'.$$

Lemma 7.2 *Let $\Omega'\neq 0, 0<q_k,p_k\le 1$, $k=1,2,\rho_i,\gamma_j>0$, $\alpha_i,\beta_j,\sigma_1,\sigma_2\in\mathbb{R}$, $\eta_i,\xi_j\in(a,T)$, $i=1,2,\ldots,m, j=1,2,\ldots,n$, $\tau_1,\tau_2\in(a,T]$ and $\phi,\psi\in C([a,T],\mathbb{R}), a>0$. Then, the following problem*

$$\begin{cases} {}_{RL}D^{q_1}\left({}_{RL}D^{p_1}+\lambda_1\right)x(t)=\phi(t), & a\le t\le T,\\ {}_{H}D^{q_2}\left({}_{H}D^{p_2}+\lambda_2\right)y(t)\quad=\psi(t), & a\le t\le T,\\ x(a)=0, \qquad \sigma_1 x(\tau_1)=\sum_{i=1}^{m}\alpha_i {}_{H}I^{\rho_i}y(\eta_i),\\ y(a)=0, \qquad \sigma_2 y(\tau_2)=\sum_{j=1}^{n}\beta_j {}_{RL}I^{\gamma_j}x(\xi_j), \end{cases} \tag{7.25}$$

is equivalent to the integral equations

$$x(t)$$
$$= {}_{RL}I^{q_1+p_1}\phi(t)-\lambda_1 {}_{RL}I^{p_1}x(t)-\left[\frac{(t-a)^{q_1+p_1-1}\Gamma(q_1)}{\Omega'\Gamma(q_1+p_1)}\right]$$

$$\times\Bigg[\Bigg(\sum_{j=1}^{n}\beta_{jRL}I^{q_1+p_1+\gamma_j}\phi(\xi_j)-\lambda_1\sum_{j=1}^{n}\beta_{jRL}I^{q_1+\gamma_j}x(\xi_j) \tag{7.26}$$

$$+\lambda_2\sigma_{2H}I^{p_2}y(\tau_2)-\sigma_{2H}I^{q_2+p_2}\psi(\tau_2)\Bigg)\Omega_1'+\Bigg(\sum_{i=1}^{m}\alpha_{iH}I^{q_2+p_2+\rho_i}\psi(\eta_i)$$

$$-\lambda_2\sum_{i=1}^{m}\alpha_{iH}I^{q_2+\rho_i}y(\eta_i)+\lambda_1\sigma_{1RL}I^{p_1}x(\tau_1)-\sigma_{1RL}I^{q_1+p_1}\phi(\tau_1)\Bigg)\Omega_3'\Bigg],$$

and

$$\begin{aligned} y(t) &= {}_HI^{q_2+p_2}\psi(t)-\lambda_{2H}I^{p_2}y(t)-\left[\frac{\left(\log\frac{t}{a}\right)^{q_2+p_2-1}\Gamma(q_2)}{\Omega'\Gamma(q_2+p_2)}\right] \\ &\times\Bigg[\Bigg(\sum_{i=1}^{m}\alpha_{iH}I^{q_2+p_2+\rho_i}\psi(\eta_i)-\lambda_2\sum_{i=1}^{m}\alpha_{iH}I^{q_2+\rho_i}y(\eta_i) \\ &+\lambda_1\sigma_{1RL}I^{p_1}x(\tau_1)-\sigma_{1RL}I^{q_1+p_1}\phi(\tau_1)\Bigg)\Omega_2'+\Bigg(\sum_{j=1}^{n}\beta_{jRL}I^{q_1+p_1+\gamma_j}\phi(\xi_j) \\ &-\lambda_1\sum_{j=1}^{n}\beta_{jRL}I^{q_1+\gamma_j}x(\xi_j)+\lambda_2\sigma_{2H}I^{p_2}y(\tau_2)-\sigma_{2H}I^{q_2+p_2}\psi(\tau_2)\Bigg)\Omega_4'\Bigg]. \end{aligned} \tag{7.27}$$

Proof Using Lemmas 1.4 and 1.5, the first two equations in (7.25) can be expressed into equivalent integral equations as

$$x(t)={}_{RL}I^{q_1+p_1}\phi(t)-\lambda_{1RL}I^{p_1}x(t)-c_1\frac{\Gamma(q_1)(t-a)^{q_1+p_1-1}}{\Gamma(q_1+p_1)}-c_2(t-a)^{p_1-1}, \tag{7.28}$$

and

$$y(t)={}_HI^{q_2+p_2}\psi(t)-\lambda_{2H}I^{p_2}y(t)-d_1\frac{\Gamma(q_2)\left(\log\frac{t}{a}\right)^{q_2+p_2-1}}{\Gamma(q_2+p_2)}-d_2\left(\log\frac{t}{a}\right)^{p_2-1}, \tag{7.29}$$

for $c_1, c_2, d_1, d_2 \in \mathbb{R}$. The conditions $x(a) = y(a) = 0$ imply that $c_2 = d_2 = 0$.

Applying the Riemann-Liouville and Hadamard fractional integrals of order $\gamma_j, \rho_i > 0$ on (7.28)–(7.29), respectively, and using the property given in Lemma 1.6, we obtain

$$\sigma_{2H}I^{q_2+p_2}\psi(\tau_2)-\lambda_2\sigma_{2H}I^{p_2}y(\tau_2)-d_1\Omega_3'$$
$$=\sum_{j=1}^{n}\beta_{jRL}I^{q_1+p_1+\gamma_j}\phi(\xi_j)-\lambda_1\sum_{j=1}^{n}\beta_{jRL}I^{q_1+\gamma_j}x(\xi_j)-c_1\Omega_2',$$
$$\sigma_{1RL}I^{q_1+p_1}\phi(\tau_1)-\lambda_1\sigma_{1RL}I^{p_1}x(\tau_1)-c_1\Omega_4'$$
$$=\sum_{i=1}^{m}\alpha_{iH}I^{q_2+p_2+\rho_i}\psi(\eta_i)-\lambda_2\sum_{i=1}^{m}\alpha_{iH}I^{q_2+\rho_i}y(\eta_i)-d_1\Omega_1'.$$

Solving the above system for constants c_1 and d_1, we get

$$c_1=\frac{\Omega_1'}{\Omega'}\Bigg(\sum_{j=1}^{n}\beta_{jRL}I^{q_1+p_1+\gamma_j}\phi(\xi_j)-\lambda_1\sum_{j=1}^{n}\beta_{jRL}I^{q_1+\gamma_j}x(\xi_j)+\lambda_2\sigma_{2H}I^{p_2}y(\tau_2)$$
$$-\sigma_{2H}I^{q_2+p_2}\psi(\tau_2)\Bigg)+\frac{\Omega_3'}{\Omega'}\Bigg(\sum_{i=1}^{m}\alpha_{iH}I^{q_2+p_2+\rho_i}\psi(\eta_i)-\lambda_2\sum_{i=1}^{m}\alpha_{iH}I^{q_2+\rho_i}y(\eta_i)$$
$$+\lambda_1\sigma_{1RL}I^{p_1}x(\tau_1)-\sigma_{1RL}I^{q_1+p_1}\phi(\tau_1)\Bigg),$$

$$d_1=\frac{\Omega_2'}{\Omega'}\Bigg(\sum_{i=1}^{m}\alpha_{iH}I^{q_2+p_2+\rho_i}\psi(\eta_i)-\lambda_2\sum_{i=1}^{m}\alpha_{iH}I^{q_2+\rho_i}y(\eta_i)+\lambda_1\sigma_{1RL}I^{p_1}x(\tau_1)$$
$$-\sigma_{1RL}I^{q_1+p_1}\phi(\tau_1)\Bigg)+\frac{\Omega_4'}{\Omega'}\Bigg(\sum_{j=1}^{n}\beta_{jRL}I^{q_1+p_1+\gamma_j}\phi(\xi_j)-\lambda_1\sum_{j=1}^{n}\beta_{jRL}I^{q_1+\gamma_j}x(\xi_j)$$
$$+\lambda_2\sigma_{2H}I^{p_2}y(\tau_2)-\sigma_{2H}I^{q_2+p_2}\psi(\tau_2)\Bigg).$$

Substituting the values of c_1, c_2, d_1 and d_2 in (7.28) and (7.29), we obtain the solution (7.26) and (7.27). The converse follows by direct computation. This completes the proof □

For the sake of convenience, we use the following notations throughout this section:

$$_{RL}I^{w}h(s,x(s),y(s))(v)=\frac{1}{\Gamma(w)}\int_0^v(v-s)^{w-1}h(s,x(s),y(s))ds,$$

and

$$_{H}I^{u}h(s,x(s),y(s))(v)=\frac{1}{\Gamma(u)}\int_a^v\left(\log\frac{v}{s}\right)^{u-1}h(s,x(s),y(s))ds,$$

where $u\in\{q_2,p_2,\gamma_j\}$, $w=\{q_1,p_1,\rho_i\}$, $v\in\{t,\tau_1,\tau_2,\eta_i,\xi_j\}$ and $h=\{f,g\}$, $i=1,2,\ldots,m, j=1,2,\ldots,n$.

Let us introduce the space $X = \{x(t)|x(t) \in C([a,T],\mathbb{R})\}$ endowed with the norm $\|x\| = \sup\{|x(t)|, t \in [a,T]\}$. Obviously $(X, \|\cdot\|)$ is a Banach space. Likewise, $Y = \{y(t)|y(t) \in C([a,T])\}$ equipped with the norm $\|y\| = \sup\{|y(t)|, t \in [a,T]\}$ is a Banach space. Thus the product space $(X \times Y, \|(x,y)\|)$ is a Banach space with norm $\|(x,y)\| = \|x\| + \|y\|$.

In view of Lemma 7.2, we define an operator $\mathscr{Q} : X \times Y \to X \times Y$ by

$$\mathscr{Q}(x,y)(t) = \begin{pmatrix} \mathscr{Q}_1(x,y)(t) \\ \mathscr{Q}_2(x,y)(t) \end{pmatrix},$$

where

$$\begin{aligned}
&\mathscr{Q}_1(x,y)(t) \\
&= {}_{RL}I^{q_1+p_1} f(s,x(s),y(s))(t) - \lambda_1 {}_{RL}I^{p_1} x(t) \\
&\quad - \frac{(t-a)^{q_1+p_1-1}\Gamma(q_1)}{\Omega'\Gamma(q_1+p_1)} \Bigg[\Bigg(\sum_{j=1}^{n} \beta_{jRL} I^{q_1+p_1+\gamma_j} f(s,x(s),y(s))(\xi_j) \\
&\quad - \lambda_1 \sum_{j=1}^{n} \beta_{jRL} I^{q_1+\gamma_j} x(\xi_j) + \lambda_2 \sigma_{2H} I^{p_2} y(\tau_2) - \sigma_{2H} I^{q_2+p_2} g(s,x(s),y(s))(\tau_2) \Bigg) \Omega_1' \\
&\quad + \Bigg(\sum_{i=1}^{m} \alpha_{iH} I^{q_2+p_2+\rho_i} g(s,x(s),y(s))(\eta_i) - \lambda_2 \sum_{i=1}^{m} \alpha_{iH} I^{q_2+\rho_i} y(\eta_i) + \lambda_1 \sigma_{1RL} I^{p_1} x(\tau_1) \\
&\quad - \sigma_{1RL} I^{q_1+p_1} f(s,x(s),y(s))(\tau_1) \Bigg) \Omega_3' \Bigg],
\end{aligned}$$

and

$$\begin{aligned}
&\mathscr{Q}_2(x,y)(t) \\
&= {}_{H}I^{q_2+p_2} g(s,x(s),y(s))(t) - \lambda_{2H} I^{p_2} y(t) \\
&\quad - \frac{\left(\log \frac{t}{a}\right)^{q_2+p_2-1} \Gamma(q_2)}{\Omega'\Gamma(q_2+p_2)} \Bigg[\Bigg(\sum_{i=1}^{m} \alpha_{iH} I^{q_2+p_2+\rho_i} g(s,x(s),y(s))(\eta_i) \\
&\quad - \lambda_2 \sum_{i=1}^{m} \alpha_{iH} I^{q_2+\rho_i} y(\eta_i) + \lambda_1 \sigma_{1RL} I^{p_1} x(\tau_1) - \sigma_{1RL} I^{q_1+p_1} f(s,x(s),y(s))(\tau_1) \Bigg) \Omega_2' \\
&\quad + \Bigg(\sum_{j=1}^{n} \beta_{jRL} I^{q_1+p_1+\gamma_j} f(s,x(s),y(s))(\xi_j) - \lambda_1 \sum_{j=1}^{n} \beta_{jRL} I^{q_1+\gamma_j} x(\xi_j) \\
&\quad + \lambda_2 \sigma_{2H} I^{p_2} y(\tau_2) - \sigma_{2H} I^{q_2+p_2} g(s,x(s),y(s))(\tau_2) \Bigg) \Omega_4' \Bigg].
\end{aligned}$$

For convenience, we introduce the notations:

$$A_1 = \frac{\Gamma(q_1)(T-a)^{q_1+p_1-1}}{\Gamma(q_1+p_1)}, \qquad A_2 = \frac{\Gamma(q_2)\left(\log\frac{T}{a}\right)^{q_2+p_2-1}}{\Gamma(q_2+p_2)},$$

$$A_3 = \frac{(T-a)^{p_1}}{\Gamma(p_1+1)}, \qquad A_4 = \frac{(T-a)^{q_1+p_1}}{\Gamma(q_1+p_1+1)},$$

$$A_5 = \frac{(\tau_1-a)^{p_1}}{\Gamma(p_1+1)}, \qquad A_6 = \frac{(\tau_1-a)^{q_1+p_1}}{\Gamma(q_1+p_1+1)},$$

$$A_7 = \frac{\left(\log\frac{T}{a}\right)^{p_2}}{\Gamma(p_2+1)}, \qquad A_8 = \frac{\left(\log\frac{T}{a}\right)^{q_2+p_2}}{\Gamma(q_2+p_2+1)},$$

$$A_9 = \frac{\left(\log\frac{\tau_1}{a}\right)^{p_1}}{\Gamma(p_1+1)}, \qquad A_{10} = \frac{\left(\log\frac{\tau_1}{a}\right)^{q_1+p_1}}{\Gamma(q_1+p_1+1)},$$

$$A_{11} = \frac{\left(\log\frac{\tau_2}{a}\right)^{p_2}}{\Gamma(p_2+1)}, \qquad A_{12} = \frac{\left(\log\frac{\tau_2}{a}\right)^{q_2+p_2}}{\Gamma(q_2+p_2+1)},$$

$$A_{13} = \sum_{i=1}^{m}\frac{|\alpha_i|\left(\log\frac{\eta_i}{a}\right)^{q_2+\rho_i}}{\Gamma(q_2+\rho_i+1)}, \qquad A_{14} = \sum_{i=1}^{m}\frac{|\alpha_i|\left(\log\frac{\eta_i}{a}\right)^{q_2+p_2+\rho_i}}{\Gamma(q_2+p_2+\rho_i+1)},$$

$$A_{15} = \sum_{j=1}^{n}\frac{|\beta_j|(\xi_j-a)^{q_1+\gamma_j}}{\Gamma(q_1+\gamma_j+1)}, \qquad A_{16} = \sum_{j=1}^{n}\frac{|\beta_j|(\xi_j-a)^{q_1+p_1+\gamma_j}}{\Gamma(q_1+p_1+\gamma_j+1)}.$$

Now we present our first result, which is concerned with the existence and uniqueness of solutions for the problem (7.24) and is based on Banach's fixed point theorem.

Theorem 7.8 *Assume that* $f, g : [a,T]\times\mathbb{R}^2 \to \mathbb{R}$ *are continuous functions and there exist positive constants* $m_i, n_i, i = 1, 2$ *such that for all* $t \in [a,T], a > 0$ *and* x_i, $y_i \in \mathbb{R}, i = 1, 2,$

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \le m_1|x_1-x_2| + m_2|y_1-y_2|$$

and

$$|g(t,x_1,x_2) - g(t,y_1,y_2)| \le n_1|x_1-x_2| + n_2|y_1-y_2|.$$

In addition, suppose that

$$B_1 + C_1 < 1,$$

where

$$B_1 = (m_1 + m_2)M_1 + (n_1 + n_2)M_2 + M_3 + M_4,$$
$$C_1 = (m_1 + m_2)M_6 + (n_1 + n_2)M_5 + M_7 + M_8,$$

and

$$M_1 = \frac{A_1}{|\Omega'|}\left(|\sigma_1||\Omega_3'|A_6 + |\Omega_1'|A_{16} + A_4\right),$$
$$M_2 = \frac{A_1}{|\Omega'|}\left(|\sigma_2||\Omega_1'|A_{12} + |\Omega_3'|A_{14}\right)$$
$$M_3 = \frac{|\lambda_1|A_1}{|\Omega'|}\left(|\sigma_1||\Omega_3'|A_5 + |\Omega_1'|A_{15}\right) + |\lambda_1|A_3,$$
$$M_4 = \frac{|\lambda_2|A_1}{|\Omega'|}\left(|\sigma_2||\Omega_1'|A_{11} + |\Omega_3'|A_{13}\right)$$
$$M_5 = \frac{A_2}{|\Omega'|}\left(|\sigma_2||\Omega_4'|A_{12} + |\Omega_2'|A_{14} + A_8\right),$$
$$M_6 = \frac{A_2}{|\Omega'|}\left(|\sigma_1||\Omega_2'|A_6 + |\Omega_4'|A_{16}\right),$$
$$M_7 = \frac{|\lambda_2|A_2}{|\Omega'|}\left(|\sigma_2'||\Omega_4'|A_{11} + |\Omega_2'|A_{13}\right) + |\lambda_2|A_7,$$
$$M_8 = \frac{|\lambda_1|A_2}{|\Omega'|}\left(|\sigma_1||\Omega_2'|A_5 + |\Omega_4'|A_{15}\right).$$

Then, the problem (7.24) *has a unique solution on* $[a, T]$.

Proof Define $\sup_{t\in[a,T]} f(t, 0, 0) = N_1 < \infty$ and $\sup_{t\in[a,T]} g(t, 0, 0) = N_2 < \infty$ and consider the set $B_r = \{(x, y) \in X \times Y : \|(x, y)\| \le r\}$, where

$$r \ge \frac{(M_1 + M_6)N_1 + (M_2 + M_5)N_2}{1 - B_1 - C_1}.$$

Let us first show that $\mathcal{Q}B_r \subset B_r$. For $(x, y) \in B_r$, we have

$$|\mathcal{Q}_1(x, y)(t)|$$
$$= \sup_{t\in[a,T]}\Bigg\{ {}_{RL}I^{q_1+p_1}|f(s, x(s), y(s))|(t) + |\lambda_1|{}_{RL}I^{p_1}|x(s)|(t) + \left[\frac{(t-a)^{q_1+p_1-1}\Gamma(q_1)}{|\Omega'|\Gamma(q_1 + p_1)}\right]$$
$$\times\Bigg[\Bigg(\sum_{j=1}^{n}|\beta_j|{}_{RL}I^{q_1+p_1+\gamma_j}|f(s, x(s), y(s))|(\xi_j) + |\lambda_1|\sum_{j=1}^{n}|\beta_j|{}_{RL}I^{q_1+\gamma_j}|x(s)|(\xi_j)$$

$$
\begin{aligned}
&+|\lambda_2||\sigma_2|_H I^{p_2}|y(s)|(\tau_2) + |\sigma_2|_H I^{q_2+p_2}|g(s,x(s),y(s))|(\tau_2)\Bigg)|\Omega_1'| \\
&+\Bigg(\sum_{i=1}^{m}|\alpha_i|_H I^{q_2+p_2+\rho_i}|g(s,x(s),y(s))|(\eta_i) + |\lambda_2|\sum_{i=1}^{m}|\alpha_i|_H I^{q_2+\rho_i}|y(s)|(\eta_i) \\
&+|\lambda_1||\sigma_1|_{RL}I^{p_1}|x(s)|(\tau_1) + |\sigma_1|_{RL}I^{q_1+p_1}|f(s,x(s),y(s))|(\tau_1)\Bigg)|\Omega_3'|\Bigg]\Bigg\} \\
\le\ & {}_{RL}I^{q_1+p_1}(|f(s,x(s),y(s)) - f(s,0,0)| + |f(s,0,0)|)(T) + |\lambda_1|_{RL}I^{p_1}|x(s)|(T) \\
&+\left[\frac{(T-a)^{q_1+p_1-1}\Gamma(q_1)}{|\Omega'|\Gamma(q_1+p_1)}\right]\Bigg[\Bigg(\sum_{j=1}^{n}|\beta_j|_{RL}I^{q_1+p_1+\gamma_j}(|f(s,x(s),y(s)) - f(s,0,0)| \\
&+|f(s,0,0)|)(\xi_j) + |\lambda_1|\sum_{j=1}^{n}|\beta_j|_{RL}I^{q_1+\gamma_j}|x(s)|(\xi_j) + |\lambda_2||\sigma_2|_H I^{p_2}|y(s)|(\tau_2) \\
&+|\sigma_2|_H I^{q_2+p_2}(|g(s,x(s),y(s)) - g(s,0,0)| + |g(s,0,0)|)(\tau_2)\Bigg)|\Omega_1'| \\
&+\Bigg(\sum_{i=1}^{m}|\alpha_i|_H I^{q_2+p_2+\rho_i}(|g(s,x(s),y(s)) - g(s,0,0)| + |g(s,0,0)|)(\eta_i) \\
&+|\lambda_2|\sum_{i=1}^{m}|\alpha_i|_H I^{q_2+\rho_i}|y(s)|(\eta_i) + |\lambda_1||\sigma_1|_{RL}I^{p_1}|x(s)|(\tau_1) \\
&+|\sigma_1|_{RL}I^{q_1+p_1}(|f(s,x(s),y(s)) - f(s,0,0)| + |f(s,0,0)|)(\tau_1)\Bigg)|\Omega_3'|\Bigg] \\
\le\ & (m_1\|x\| + m_2\|y\| + N_1)_{RL}I^{q_1+p_1}(1)(T) + |\lambda_1|\|x\|_{RL}I^{p_1}(1)(T) \\
&+\frac{(T-a)^{q_1+p_1-1}\Gamma(q_1)}{|\Omega'|\Gamma(q_1+p_1)}\Bigg[\Bigg((m_1\|x\| + m_2\|y\| + N_1)\sum_{j=1}^{n}|\beta_j|_{RL}I^{q_1+p_1+\gamma_j}(1)(\xi_j) \\
&+|\lambda_1|\|x\|\sum_{j=1}^{n}|\beta_j|_{RL}I^{q_1+\gamma_j}(1)(\xi_j) + |\lambda_2||\sigma_2|\|y\|_H I^{p_2}(1)(\tau_2) \\
&+|\sigma_2|(n_1\|x\| + n_2\|y\| + N_2)_H I^{q_2+p_2}(1)(\tau_2)\Bigg)|\Omega_1'| \\
&+\Bigg((n_1\|x\| + n_2\|y\| + N_2)\sum_{i=1}^{m}|\alpha_i|_H I^{q_2+p_2+\rho_i}(1)(\eta_i) \\
&+|\lambda_2|\|y\|\sum_{i=1}^{m}|\alpha_i|_H I^{q_2+\rho_i}(1)(\eta_i)
\end{aligned}
$$

$$
\begin{aligned}
&+|\lambda_1||\sigma_1|\|x\|_{RL}I^{p_1}(1)(\tau_1) + |\sigma_1|(m_1\|x\| + m_2\|y\| + N_1)_{RL}I^{q_1+p_1}(1)(\tau_1)\Big)|\Omega_3'|\Big] \\
&= \Bigg(\frac{\Gamma(q_1)(T-a)^{q_1+p_1-1}}{|\Omega'|\Gamma(q_1+p_1)}\Bigg(\frac{|\Omega_3'||\sigma_1|(\tau_1-a)^{q_1+p_1}}{\Gamma(q_1+p_1+1)} + \sum_{j=1}^{n}\frac{|\Omega_1'||\beta_j|(\xi_j-a)^{q_1+p_1+\gamma_j}}{\Gamma(q_1+p_1+\gamma_j+1)}\Bigg) \\
&+\frac{(T-a)^{q_1+p_1}}{\Gamma(q_1+p_1+1)}\Bigg)(m_1\|x\| + m_2\|y\| + N_1) + \frac{\Gamma(q_1)(T-a)^{q_1+p_1-1}}{|\Omega'|\Gamma(q_1+p_1)} \times \\
&\times\Bigg(\frac{|\sigma_2||\Omega_1'|\left(\log\frac{\tau_2}{a}\right)^{q_2+p_2}}{\Gamma(q_2+p_2+1)} \\
&+\sum_{i=1}^{m}\frac{|\Omega_3'||\alpha_i|\left(\log\frac{\eta_i}{a}\right)^{q_2+p_2+\rho_i}}{\Gamma(q_2+p_2+\rho_i+1)}\Bigg)(n_1\|x\| + n_2\|y\| + N_2) \\
&+\Bigg(\frac{|\lambda_1|(T-a)^{p_1}}{\Gamma(p_1+1)} + \frac{|\lambda_1|\Gamma(q_1)(T-a)^{q_1+p_1-1}}{|\Omega'|\Gamma(q_1+p_1)}\Bigg(\frac{|\Omega_3'||\sigma_1|(\tau_1-a)^{p_1}}{\Gamma(p_1+1)} \\
&+\sum_{j=1}^{n}\frac{|\Omega_1'||\beta_j|(\xi_j-a)^{q_1+\gamma_j}}{\Gamma(q_1+\gamma_j+1)}\Bigg)\Bigg)\|x\| \\
&+\frac{|\lambda_2|\Gamma(q_1)(T-a)^{q_1+p_1-1}}{|\Omega'|\Gamma(q_1+p_1)}\Bigg(\frac{|\Omega_1'||\sigma_2|\left(\log\frac{\tau_2}{a}\right)^{p_2}}{\Gamma(p_2+1)} \\
&+\sum_{i=1}^{m}\frac{|\Omega_3'||\alpha_i|\left(\log\frac{\eta_i}{a}\right)^{q_2+\rho_i}}{\Gamma(q_2+\rho_i+1)}\Bigg)\|y\| \\
&= \left(\frac{A_1}{|\Omega'|}\left(|\sigma_1||\Omega_3'|A_6 + |\Omega_1'|A_{16}\right) + A_4\right)(m_1\|x\| + m_2\|y\| + N_1) \\
&+\frac{A_1}{|\Omega'|}\left(|\sigma_2||\Omega_1'|A_{12} + |\Omega_3'|A_{14}\right)(n_1\|x\| + n_2\|y\| + N_2) \\
&+\left(\frac{|\lambda_1|A_1}{|\Omega'|}\left(|\sigma_1||\Omega_3'|A_5 + |\Omega_1'|A_{15}\right) + |\lambda_1|A_3\right)\|x\| \\
&+\frac{|\lambda_2|A_1}{|\Omega'|}\left(|\sigma_2||\Omega_1'|A_{11} + |\Omega_3'|A_{13}\right)\|y\| \\
&= M_1(m_1\|x\| + m_2\|y\| + N_1) + M_2(n_1\|x\| + n_2\|y\| + N_2) + M_3\|x\| + M_4\|y\| \\
&= (m_1M_1 + n_1M_2 + M_3)\|x\| + (m_2M_1 + n_2M_2 + M_4)\|y\| + M_1N_1 + M_2N_2 \\
&\le ((m_1+m_2)M_1 + (n_1+n_2)M_2 + M_3 + M_4)r + M_1N_1 + M_2N_2 \\
&= B_1r + M_1N_1 + M_2N_2.
\end{aligned}
$$

Hence

$$\|\mathscr{Q}_1(x,y)\| \le B_1 r + M_1 N_1 + M_2 N_2.$$

In the same way, we can obtain

$$\|\mathscr{Q}_2(x,y)\| \le C_1 r + M_6 N_1 + M_5 N_2.$$

Consequently, $\|\mathscr{Q}(x,y)\| \le r$.

Now for $(x_2,y_2),(x_1,y_1) \in X \times Y$, and for any $t \in [a,T]$, we get

$$\begin{aligned}
&|\mathscr{Q}_1(x_2,y_2)(t) - \mathscr{Q}_1(x_1,y_1)(t)| \\
\le\ & {}_{RL}I^{q_1+p_1}(|f(s,x_2(s),y_2(s)) - f(s,x_1(s),y_1(s))|)(T) \\
&+|\lambda_1|\,{}_{RL}I^{p_1}(|x_2(s)-x_1(s)|)(T) \\
&+\frac{(T-a)^{q_1+p_1-1}\Gamma(q_1)}{|\Omega'|\Gamma(q_1+p_1)}\Bigg[\Bigg(\sum_{j=1}^{n}|\beta_j|\,{}_{RL}I^{q_1+p_1+\gamma_j}(|f(s,x_2(s),y_2(s)) \\
&-f(s,x_1(s),y_1(s))|)(\xi_j) \\
&+|\lambda_1|\sum_{j=1}^{n}|\beta_j|\,{}_{RL}I^{q_1+\gamma_j}(|x_2(s)-x_1(s)|)(\xi_j) + |\lambda_2||\sigma_2|\,{}_{H}I^{p_2}(|y_2(s)-y_1(s)|)(\tau_2) \\
&+|\sigma_2|\,{}_{H}I^{q_2+p_2}(|g(s,x_2(s),y_2(s)) - g(s,x_1(s),y_1(s))|)(\tau_2)\Bigg)|\Omega_1'| \\
&+\Bigg(\sum_{i=1}^{m}|\alpha_i|\,{}_{H}I^{q_2+p_2+\rho_i}(|g(s,x_2(s),y_2(s)) - g(s,x_1(s),y_1(s))|)(\eta_i) \\
&+|\lambda_2|\sum_{i=1}^{m}|\alpha_i|\,{}_{H}I^{q_2+\rho_i}(|y_2(s)-y_1(s)|)(\eta_i) + |\lambda_1||\sigma_1|\,{}_{RL}I^{p_1}(|x_2(s)-x_1(s)|)(\tau_1) \\
&+|\sigma_1|\,{}_{RL}I^{q_1+p_1}(|f(s,x_2(s),y_2(s)) - f(s,x_1(s),y_1(s))|)(\tau_1)\Bigg)|\Omega_3'|\Bigg] \\
\le\ & (m_1\|x_2-x_1\| + m_2\|y_2-y_1\|)\Bigg[\frac{|\Omega_1'|\Gamma(q_1)(T-a)^{q_1+p_1-1}}{|\Omega'|\Gamma(q_1+p_1)} \\
&\times\sum_{j=1}^{n}\frac{|\beta_j|(\xi_j-a)^{q_1+p_1+\gamma_j}}{\Gamma(q_1+p_1+\gamma_j+1)} \\
&+\frac{|\sigma_1||\Omega_3'|(T-a)^{q_1+p_1-1}\Gamma(q_1)(\tau_1-a)^{q_1+p_1}}{|\Omega'|\Gamma(q_1+p_1)\Gamma(q_1+p_1+1)} + \frac{(T-a)^{q_1+p_1}}{\Gamma(q_1+p_1+1)}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+(n_1\|x_2-x_1\|+n_2\|y_2-y_1\|)\Bigg[\frac{|\sigma_2||\Omega_1'|\Gamma(q_1)(T-a)^{q_1+p_1-1}\left(\log\frac{\tau_2}{a}\right)^{q_2+p_2}}{|\Omega'|\Gamma(q_1+p_1)\Gamma(q_2+p_2+1)}\\
&+\frac{|\Omega_3'|\Gamma(q_1)(T-a)^{q_1+p_1-1}}{|\Omega'|\Gamma(q_1+p_1)}\sum_{i=1}^{m}\frac{|\alpha_i|\left(\log\frac{\eta_i}{a}\right)^{q_2+p_2+\rho_i}}{\Gamma(q_2+p_2+\rho_i+1)}\Bigg]\\
&+\|x_2-x_1\|\Bigg[|\lambda_1|\frac{(T-a)^{p_1}}{\Gamma(p_1+1)}\\
&+\frac{|\lambda_1||\sigma_1||\Omega_3'|\Gamma(q_1)(T-a)^{q_1+p_1-1}(\tau_1-a)^{p_1}}{|\Omega'|\Gamma(q_1+p_1)\Gamma(p_1+1)}+\frac{|\lambda_1||\Omega_1'|\Gamma(q_1)(T-a)^{q_1+p_1-1}}{|\Omega'|\Gamma(q_1+p_1)}\\
&\times\sum_{j=1}^{n}\frac{|\beta_j|(\xi_j-a)^{q_1+\gamma_j}}{\Gamma(q_1+\gamma_j+1)}\Bigg]+\|y_2-y_1\|\Bigg[\left(\frac{|\lambda_2||\Omega_3'|\Gamma(q_1)(T-a)^{q_1+p_1-1}}{|\Omega'|\Gamma(q_1+p_1)}\right)\times\\
&\times\sum_{i=1}^{m}\frac{|\alpha_i|\left(\log\frac{\eta_i}{a}\right)^{q_2+\rho_i}}{\Gamma(q_2+\rho_i+1)}+\frac{|\lambda_2||\sigma_2||\Omega_1'|\Gamma(q_1)(T-a)^{q_1+p_1-1}\left(\log\frac{\tau_2}{a}\right)^{p_2}}{|\Omega'|\Gamma(q_1+p_1)\Gamma(p_2+1)}\Bigg]\\
=\ &(m_1\|x_2-x_1\|+m_2\|y_2-y_1\|)\left[\frac{A_1}{|\Omega'|}\left(|\sigma_1||\Omega_3'|A_6+|\Omega_1'|A_{16}+A_4\right)\right]\\
&+(n_1\|x_2-x_1\|+n_2\|y_2-y_1\|)\left[\frac{A_1}{|\Omega'|}\left(|\sigma_2||\Omega_1'|A_{12}+|\Omega_3'|A_{14}\right)\right]\\
&+\|x_2-x_1\|\left[\frac{|\lambda_1|A_1}{|\Omega'|}\left(|\sigma_1||\Omega_3'|A_5+|\Omega_1'|A_{15}\right)+|\lambda_1|A_3\right]\\
&+\|y_2-y_1\|\left[\frac{|\lambda_2|A_1}{|\Omega'|}\left(|\sigma_2||\Omega_1'|A_{11}+|\Omega_3'|A_{13}\right)\right]\\
=\ &M_1(m_1\|x_2-x_1\|+m_2\|y_2-y_1\|)+M_2(n_1\|x_2-x_1\|+n_2\|y_2-y_1\|)\\
&+M_3\|x_2-x_1\|+M_4\|y_2-y_1\|\\
=\ &(m_1M_1+n_1M_2+M_3)\|x_2-x_1\|+(m_2M_1+n_2M_2+M_4)\|y_2-y_1\|,
\end{aligned}
$$

and consequently, we obtain

$$\|\mathscr{Q}_1(x_2,y_2)-\mathscr{Q}_1(x_1,y_1)\|\le B_1[\|x_2-x_1\|+\|y_2-y_1\|]. \tag{7.30}$$

Similarly,

$$\|\mathscr{Q}_2(x_2,y_2)-\mathscr{Q}_2(x_1,y_1)\|\le C_1[\|x_2-x_1\|+\|y_2-y_1\|]. \tag{7.31}$$

It follows from (7.30) and (7.31) that

$$\|\mathscr{Q}(x_2,y_2)-\mathscr{Q}(x_1,y_1)\|\le [B_1+C_1](\|x_2-x_1\|+\|y_2-y_1\|).$$

Since $(B_1 + C_1) < 1$, $\mathscr{Q}$ is a contraction. So, by Banach's fixed point theorem, the operator $\mathscr{Q}$ has a unique fixed point, which is the unique solution of the problem (7.24). This completes the proof. □

Example 7.4 Consider the system of Riemann-Liouville and Hadamard fractional Langevin equations supplemented with fractional coupled integral conditions

$$\begin{cases} {}_{RL}D^{2/5}\left({}_{RL}D^{3/4} + \frac{1}{7}\right)x(t) = \frac{|x|\sin^2(2\pi t)}{(5-t)^2}\left(\frac{|x|}{|x|+2} + 1\right) + \frac{|y|}{(6-t)^2} - \frac{1}{2}, \\ {}_{H}D^{3/2}\left({}_{H}D^{1/5} - \frac{1}{11}\right)y(t) \ = \frac{|x|}{(7+t)^2} + \frac{\cos^2(\pi t)}{(7-t)^2}\cdot\left(\frac{|y|}{|y|+3} + 1\right)|y| + 1, \\ x\left(\frac{1}{4}\right) = 0, \qquad \sqrt{2}x(1) = \frac{1}{2}{}_{H}I^{\sqrt{3}}y\left(\frac{1}{3}\right) - \frac{1}{3}{}_{H}I^{4/5}y\left(\frac{3}{2}\right), \\ y\left(\frac{1}{4}\right) = 0, \qquad \frac{1}{2}y\left(\frac{3}{2}\right) = \frac{1}{6}{}_{RL}I^{\pi/2}x\left(\frac{2}{5}\right) + \frac{1}{8}{}_{RL}I^{\sqrt{5}}x\left(\frac{5}{3}\right), \ \frac{1}{4} \le t \le 2. \end{cases} \tag{7.32}$$

Here $q_1 = 2/5$, $q_2 = 5/6$, $p_1 = 3/4$, $p_2 = 3/7$, $\lambda_1 = 1/7$, $\lambda_2 = -1/11$, $n = 2$, $m = 2$, $a = 1/4$, $T = 2$, $\sigma_1 = \sqrt{2}$, $\sigma_2 = 1/2$, $\tau_1 = 1$, $\tau_2 = 3/2$, $\eta_1 = 1/3$, $\eta_2 = 3/2$, $\xi_1 = 2/5$, $\xi_2 = 5/3$, $\alpha_1 = 1/2$, $\alpha_2 = -1/3$, $\beta_1 = 1/6$, $\beta_2 = 1/8$, $\rho_1 = \sqrt{3}$, $\rho_2 = 4/5$, $\gamma_1 = \pi/2$, $\gamma_2 = \sqrt{5}$, and $f(t,x,y) = ((\sin^2(2\pi t))/((5-t)^2))(|x|/(|x|+2)+1)(|x|) + (|y|/((6-t)^2)) - (1/2)$ and $g(t,x,y) = (|x|/((7+t)^2)) + (\cos^2(\pi t))/((7-t)^2)(|y|/(|y|+3)+1)|y| + 1$. Obviously

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \le \frac{2}{27}|x_1 - x_2| + \frac{4}{121}|y_1 - y_2|,$$

and

$$|g(t,x_1,x_2) - g(t,y_1,y_2)| \le \frac{4}{225}|x_1 - x_2| + \frac{16}{507}|y_1 - y_2|.$$

Using the given values, we find that $\Omega' \approx -2.490241444 \neq 0$. Further, the assumptions of Theorem 7.8 are satisfied with $m_1 = 2/27$, $m_2 = 4/121$, $n_1 = 4/225$, $n_2 = 16/507$, $M_1 \simeq 2.584592457$, $M_2 \simeq 1.020410145$, $M_3 \simeq 0.176932454$, $M_4 \simeq 0.134734178$, $M_5 \simeq 3.202563777$, $M_6 \simeq 0.210181316$, $M_7 \simeq 0.143988733$, $M_8 \simeq 0.030190073$, $B_1 \approx 0.638901916$, $C_1 \approx 0.354697504$ and $B_1 + C_1 \approx 0.99359942 < 1$.

Hence, by Theorem 7.8, the problem (7.32) has a unique solution on $[1/4, 2]$.

In the next result, we prove the existence of solutions for the problem (7.24) by applying Leray-Schauder alternative.

Let us set the constants

$$E_1 = (M_1 + M_5)P_1 + (M_2 + M_6)R_1 + M_3 + M_7,$$
$$E_2 = (M_1 + M_5)P_2 + (M_2 + M_6)R_2 + M_4 + M_8,$$

and

$$E^* = \max\{1 - E_1, 1 - E_2\}. \tag{7.33}$$

Theorem 7.9 *Assume that there exist real constants P_i, $R_i \geq 0$ $(i = 1, 2)$ and $P_0 > 0, R_0 > 0$ such that $\forall x_i \in \mathbb{R}$ $(i = 1, 2)$,*

$$|f(t, x_1, x_2)| \leq P_0 + P_1|x_1| + P_2|x_2|,$$
$$|g(t, x_1, x_2)| \leq R_0 + R_1|x_1| + R_2|x_2|.$$

Further, it is assumed that

$$E_1 < 1 \text{ and } E_2 < 1,$$

where E_1 and E_2 are given by (7.33). Then, there exists at least one solution for the problem (7.24) on $[a, T]$.

Proof First, we show that the operator $\mathscr{Q} : X\times Y \to X\times Y$ is completely continuous. Notice that the operator $\mathscr{Q}$ is continuous in view of the continuity of f and g.

Let $U \subset X \times Y$ be bounded. Then there exist positive constants L_1 and L_2 such that

$$|f(t, x(t), y(t))| \leq L_1, \quad |g(t, x(t), y(t))| \leq L_2, \ \ \forall (x, y) \in U.$$

Then, for any $(x, y) \in U$, we have

$$\begin{aligned}
&\|\mathscr{Q}_1(x, y)\| \\
&\leq {}_{RL}I^{q_1+p_1}|f(s, x(s), y(s))|(T) + |\lambda_1|_{RL}I^{p_1}|x(s)|(T) + \frac{(T-a)^{q_1+p_1-1}\Gamma(q_1)}{|\Omega'|\Gamma(q_1+p_1)} \\
&\quad\times\Bigg[\Bigg(\sum_{j=1}^{n}|\beta_j|_{RL}I^{q_1+p_1+\gamma_j}|f(s, x(s), y(s))|(\xi_j) + |\lambda_1|\sum_{j=1}^{n}|\beta_j|_{RL}I^{q_1+\gamma_j}|x(s)|(\xi_j) \\
&\quad+|\lambda_2||\sigma_2|_H I^{p_2}|y(s)|(\tau_2) + |\sigma_2|_H I^{q_2+p_2}|g(s, x(s), y(s))|(\tau_2)\Bigg)|\Omega_1'| \\
&\quad+\Bigg(\sum_{i=1}^{m}|\alpha_i|_H I^{q_2+p_2+\rho_i}|g(s, x(s), y(s))|(\eta_i) + |\lambda_2|\sum_{i=1}^{m}|\alpha_i|_H I^{q_2+\rho_i}|y(s)|(\eta_i)
\end{aligned}$$

$$
\begin{aligned}
&+|\lambda_1||\sigma_1|_{RL}I^{p_1}|x(s)|(\tau_1) + |\sigma_1|_{RL}I^{q_1+p_1}|f(s,x(s),y(s))|(\tau_1)\Big)|\Omega_3'|\Big] \\
\le& \left(\frac{A_1}{|\Omega'|}\left(|\sigma_1||\Omega_3'|A_6 + |\Omega_1'|A_{16}\right) + A_4\right)L_1 + \frac{A_1}{|\Omega'|}\left(|\sigma_2||\Omega_1'|A_{12} + |\Omega_3'|A_{14}\right)L_2 \\
&+\left(\frac{|\lambda_1|A_1}{|\Omega'|}\left(|\sigma_1||\Omega_3'|A_5 + |\Omega_1'|A_{15}\right) + |\lambda_1|A_3 + \frac{|\lambda_2|A_1}{|\Omega'|}\left(|\sigma_2||\Omega_1'|A_{11} + |\Omega_3'|A_{13}\right)\right)r \\
=& M_1L_1 + M_2L_2 + (M_3 + M_4)\,r.
\end{aligned}
$$

In the same way, we can obtain

$$
\begin{aligned}
&\|\mathcal{Q}_2(x,y)\| \\
\le& \left(\frac{A_2}{|\Omega'|}\left(|\sigma_2||\Omega_4'|A_{12} + |\Omega_2'|A_{14}\right) + A_8\right)L_1 + \frac{A_2}{|\Omega'|}\left(|\sigma_1||\Omega_2'|A_6 + |\Omega_4'|A_{16}\right)L_2 \\
&+\left(\frac{|\lambda_2|A_2}{|\Omega'|}\left(|\sigma_2||\Omega_4'|A_{11} + |\Omega_2'|A_{13}\right) + |\lambda_2|A_7 + \frac{|\lambda_1|A_2}{|\Omega'|}\left(|\sigma_1||\Omega_2'|A_5 + |\Omega_4'|A_{15}\right)\right)r \\
=& M_5L_1 + M_6L_2 + (M_7 + M_8)\,r.
\end{aligned}
$$

Thus, it follows from the above inequalities that the operator $\mathcal{Q}$ is uniformly bounded.

Next, we show that $\mathcal{Q}$ is equicontinuous. Let $t_1, t_2 \in [a, T]$ with $t_1 < t_2$. Then, we have

$$
\begin{aligned}
&|\mathcal{Q}_1(x,y)(t_2) - \mathcal{Q}_1(x,y)(t_1)| \\
\le& |{}_{RL}I^{q_1+p_1}f(s,x(s),y(s))(t_2) - {}_{RL}I^{q_1+p_1}f(s,x(s),y(s))(t_1)| \\
&+|\lambda_1||{}_{RL}I^{p_1}x(t_2) - {}_{RL}I^{p_1}x(t_1)| + \left|(t_2-a)^{q_1+p_1-1} - (t_1-a)^{q_1+p_1-1}\right| \\
&\times\left[\frac{\Gamma(q_1)}{|\Omega'|\Gamma(q_1+p_1)}\right]\left[\left(\sum_{j=1}^{n}|\beta_j|_{RL}I^{q_1+p_1+\gamma_j}|f(s,x(s),y(s))|(\xi_j)\right.\right. \\
&+|\lambda_1|\sum_{j=1}^{n}|\beta_j|_{RL}I^{q_1+\gamma_j}|x(s)|(\xi_j) + |\sigma_2|_H I^{q_2+p_2}|g(s,x(s),y(s))|(\tau_2) \\
&\left.+|\lambda_2||\sigma_2|_H I^{p_2}|y(s)|(\tau_2)\right)|\Omega_1'| + \left(\sum_{i=1}^{m}|\alpha_i|_H I^{q_2+p_2+\rho_i}|g(s,x(s),y(s))|(\eta_i)\right. \\
&+|\lambda_2|\sum_{i=1}^{m}|\alpha_i|_H I^{q_2+\rho_i}|y(s)|(\eta_i) + |\sigma_1|_{RL}I^{q_1+p_1}|f(s,x(s),y(s))|(\tau_1) \\
&\left.\left.+|\lambda_1||\sigma_1|_{RL}I^{p_1}|x(s)|(\tau_1)\right)|\Omega_3'|\right]
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{L_1}{\Gamma(q_1+p_1)}\left(\int_a^{t_1}\left[(t_2-s)^{q_1+p_1-1}-(t_1-s)^{q_1+p_1-1}\right]ds+\int_{t_1}^{t_2}(t_2-s)^{q_1+p_1-1}ds\right)\\
&+\frac{|\lambda_1|r}{\Gamma(p_1)}\left(\int_a^{t_1}\left[(t_2-s)^{p_1-1}-(t_1-s)^{p_1-1}\right]ds+\int_{t_1}^{t_2}(t_1-s)^{p_1-1}ds\right)\\
&+\left|(t_2-s)^{q_1+p_1-1}-(t_1-s)^{q_1+p_1-1}\right|\left[\frac{\Gamma(q_1)}{|\Omega'|\Gamma(q_1+p_1)}\right]\\
&\times\Big[\left(|\lambda_1||\Omega_1'|A_{15}+|\lambda_2||\sigma_2||\Omega_1'|A_{11}+|\lambda_2||\Omega_3'|A_{13}+|\lambda_1||\sigma_1||\Omega_3'|A_5\right)r\\
&+\left(|\Omega_1'|A_{16}+|\sigma_1||\Omega_3'|A_6\right)L_1+\left(|\sigma_2||\Omega_1'|A_{12}+|\Omega_3'|A_{14}\right)L_2\Big].
\end{aligned}
$$

Analogously, we can obtain

$$
\begin{aligned}
&|\mathscr{Q}_2(x,y)(t_2)-\mathscr{Q}_2(x,y)(t_1)|\\
\le\ &|{}_HI^{q_2+p_2}g(s,x(s),y(s))(t_2)-{}_HI^{q_2+p_2}g(s,x(s),y(s))(t_1)|\\
&+|\lambda_2||{}_HI^{p_2}y(t_2)-{}_HI^{p_2}y(t_1)|+\left|\left(\log\frac{t_2}{a}\right)^{q_2+p_2-1}-\left(\log\frac{t_1}{a}\right)^{q_2+p_2-1}\right|\\
&\times\left[\frac{\Gamma(q_2)}{|\Omega'|\Gamma(q_2+p_2)}\right]\left[\left(\sum_{i=1}^m|\alpha_i|{}_HI^{q_2+p_2+\rho_i}|g(s,x(s),y(s))|(\eta_i)\right.\right.\\
&+|\lambda_2|\sum_{i=1}^m|\alpha_i|{}_HI^{q_2+\rho_i}|y(s)|(\eta_i)+|\sigma_1|{}_{RL}I^{q_1+p_1}|f(s,x(s),y(s))|(\tau_1)\\
&\left.+|\lambda_1||\sigma_1|{}_{RL}I^{p_1}|x(s)|(\tau_1)\right)|\Omega_2'|+\left(\sum_{j=1}^n|\beta_j|{}_{RL}I^{q_1+p_1+\gamma_j}|f(s,x(s),y(s))|(\xi_j)\right.\\
&+|\lambda_1|\sum_{j=1}^n|\beta_j|{}_{RL}I^{q_1+\gamma_j}|x(s)|(\xi_j)+|\sigma_2|{}_HI^{q_2+p_2}|g(s,x(s),y(s))|(\tau_2)\\
&\left.\left.+|\lambda_2||\sigma_2|{}_HI^{p_2}|y(s)|(\tau_2)\right)|\Omega_4'|\right]\\
\le\ &\frac{L_2}{\Gamma(q_2+p_2)}\left(\int_a^{t_1}\left[\left(\log\frac{t_2}{a}\right)^{q_2+p_2-1}-\left(\log\frac{t_1}{a}\right)^{q_2+p_2-1}\right]ds\right.\\
&\left.+\int_{t_1}^{t_2}\left(\log\frac{t_2}{a}\right)^{q_2+p_2-1}ds\right)+\frac{|\lambda_2|r}{\Gamma(p_2)}\left(\int_a^{t_1}\left[\left(\log\frac{t_2}{a}\right)^{q_2+p_2-1}-\left(\log\frac{t_1}{a}\right)^{q_2+p_2-1}\right]ds\right.\\
&\left.+\int_{t_1}^{t_2}\left(\log\frac{t_2}{a}\right)^{q_2+p_2-1}ds\right)+\left|\left(\log\frac{t_2}{a}\right)^{q_2+p_2-1}-\left(\log\frac{t_1}{a}\right)^{q_2+p_2-1}\right|\left[\frac{\Gamma(q_2)}{|\Omega'|\Gamma(q_2+p_2)}\right]\\
&\times\Big[\left(|\lambda_1||\sigma_1||\Omega_2'|A_5+|\lambda_1||\Omega_4'|A_{15}+|\lambda_2||\sigma_2||\Omega_4'|A_{11}+|\lambda_2||\Omega_2'|A_{13}\right)r\\
&+\left(|\sigma_1||\Omega_2'|A_6+|\Omega_4'|A_{16}\right)L_1+\left(|\Omega_2'|A_{14}+|\sigma_2||\Omega_4'|A_{12}\right)L_2\Big].
\end{aligned}
$$

Therefore, the operator $\mathscr{Q}(x,y)$ is equicontinuous, and hence it is completely continuous.

Finally, it will be verified that the set $\bar{\mathscr{E}} = \{(x,y) \in X \times Y | (x,y) = \kappa\mathscr{Q}(x,y), 0 \leq \kappa \leq 1\}$ is bounded. Let $(x,y) \in \bar{\mathscr{E}}$, $(x,y) = \kappa\mathscr{Q}(x,y)$. For any $t \in [a,T]$, we have

$$x(t) = \kappa\mathscr{Q}_1(x,y)(t), \quad y(t) = \kappa\mathscr{Q}_2(x,y)(t).$$

Then

$$\begin{aligned}
&|x(t)| \\
&= |\kappa\mathscr{Q}_1(x,y)(t)| \\
&\leq (P_0 + P_1\|x\| + P_2\|y\|)_{RL}I^{q_1+p_1}(1)(T) + \|x\||\lambda_1|_{RL}I^{p_1}(1)(T) \\
&\quad + \frac{(T-a)^{q_1+p_1-1}\Gamma(q_1)}{|\Omega'|\Gamma(q_1+p_1)}\Bigg[\Bigg((P_0 + P_1\|x\| + P_2\|y\|)\sum_{j=1}^{n}|\beta_j|_{RL}I^{q_1+p_1+\gamma_j}(1)(\xi_j) \\
&\quad + \|x\||\lambda_1|\sum_{j=1}^{n}|\beta_j|_{RL}I^{q_1+\gamma_j}(1)(\xi_j) + (R_0 + R_1\|x\| + R_2\|y\|)|\sigma_2|_H I^{q_2+p_2}(1)(\tau_2) \\
&\quad + \|y\||\lambda_2||\sigma_2|_H I^{p_2}(1)(\tau_2)\Bigg)|\Omega_1'| + \Bigg((R_0 + R_1\|x\| + R_2\|y\|)\sum_{i=1}^{m}|\alpha_i|_H I^{q_2+p_2+\rho_i}(1)(\eta_i) \\
&\quad + \|y\||\lambda_2|\sum_{i=1}^{m}|\alpha_i|_H I^{q_2+\rho_i}(1)(\eta_i) + (P_0 + P_1\|x\| + P_2\|y\|)|\sigma_1|_{RL}I^{q_1+p_1}(1)(\tau_1) \\
&\quad + \|x\||\lambda_1||\sigma_1|_{RL}I^{p_1}(1)(\tau_1)\Bigg)|\Omega_3'|\Bigg] \\
&\leq (P_0 + P_1\|x\| + P_2\|y\|)M_1 + (R_0 + R_1\|x\| + R_2\|y\|)M_2 + \|x\|M_3 + \|y\|M_4,
\end{aligned}$$

and

$$\begin{aligned}
&|y(t)| \\
&= |\kappa\mathscr{Q}_2(x,y)(t)| \\
&\leq (R_0 + R_1\|x\| + R_2\|y\|)_H I^{q_2+p_2}(1)(T) + \|y\||\lambda_2|_H I^{p_2}(1)(T) \\
&\quad + \frac{\left(\log\frac{T}{a}\right)^{q_2+p_2-1}\Gamma(q_2)}{|\Omega'|\Gamma(q_2+p_2)}\Bigg[\Bigg((R_0 + R_1\|x\| + R_2\|y\|)\sum_{i=1}^{m}|\alpha_i|_H I^{q_2+p_2+\rho_i}(1)(\eta_i) \\
&\quad + \|y\||\lambda_2|\sum_{i=1}^{m}|\alpha_i|_H I^{q_2+\rho_i}(1)(\eta_i) + (P_0 + P_1\|x\| + P_2\|y\|)|\sigma_1|_{RL}I^{q_1+p_1}(1)(\tau_1) \\
&\quad + \|x\||\lambda_1||\sigma_1|_{RL}I^{p_1}\|x\|(\tau_1)\Bigg)|\Omega_2'| + \Bigg((P_0 + P_1\|x\| + P_2\|y\|)\sum_{j=1}^{n}|\beta_j|_{RL}I^{q_1+p_1+\gamma_j}(1)(\xi_j)
\end{aligned}$$

$$+\|x\||\lambda_1|\sum_{j=1}^{n}|\beta_j|_{RL}I^{q_1+\gamma_j}(1)(\xi_j)+(R_0+R_1\|x\|+R_2\|y\|)|\sigma_2|_H I^{q_2+p_2}(1)(\tau_2)$$

$$+|\lambda_2||\sigma_2|_H I^{p_2}|y(s)|(\tau_2)\Bigg)|\Omega_4'|\Bigg]$$

$$\leq (P_0+P_1\|x\|+P_2\|y\|)M_5+(R_0+R_1\|x\|+R_2\|y\|)M_6+\|x\|M_7+\|y\|M_8.$$

In consequence, we get

$$\|x\|\leq (P_0+P_1\|x\|+P_2\|y\|)M_1+(R_0+R_1\|x\|+R_2\|y\|)M_2+\|x\|M_3+\|y\|M_4,$$

and

$$\|y\|\leq (P_0+P_1\|x\|+P_2\|y\|)M_5+(R_0+R_1\|x\|+R_2\|y\|)M_6+\|x\|M_7+\|y\|M_8,$$

which imply that

$$\begin{aligned}\|x\|+\|y\|\leq{}&(M_1+M_5)P_0+(M_2+M_6)R_0\\&+[(M_1+M_5)P_1+(M_2+M_6)R_1+M_3+M_7]\|x\|\\&+[(M_1+M_5)P_2+(M_2+M_6)R_2+M_4+M_8]\|y\|.\end{aligned}$$

Thus,

$$\|(x,y)\|\leq\frac{(M_1+M_5)P_0+(M_2+M_6)R_0}{E^*},$$

where E^* is defined by (7.33). This proves that $\bar{\mathscr{E}}$ is bounded. Thus, by Theorem 1.3, the operator $\mathscr{Q}$ has at least one fixed point. Hence the problem (7.24) has at least one solution. The proof is complete. □

Example 7.5 Consider the system of Langevin equations with fractional coupled integral conditions

$$\begin{cases}{}_{RL}D^{2/3}\left({}_{RL}D^{8/9}-\dfrac{1}{9}\right)x(t)=\dfrac{\sqrt{2}}{2}+\dfrac{\pi^2\cos^2(2\pi t)}{4(7\pi-t)^2}\cdot|x|\\ \qquad\qquad\qquad\qquad\qquad+\frac{\pi^2|y|}{(5\pi-t)^2}\cdot\left(\frac{|y|}{|y|+4}+1\right),\\ {}_HD^{7/8}\left({}_HD^{9/10}-\dfrac{1}{10}\right)y(t)=\dfrac{\sqrt{3}}{2}+\dfrac{\pi^2|x|}{(3\pi-t)^2}\cdot\left(\dfrac{|x|}{|x|+2}+1\right)+\dfrac{\pi^2\sin^2y(t)}{(4\pi-t)^2},\\ x\left(\dfrac{\pi}{2}\right)=0,\qquad \dfrac{1}{5}x(\pi)=\sqrt{2}{}_HI^{1/3}y(\pi)-\dfrac{1}{2}{}_HI^{1/4}y\left(\dfrac{\pi}{2}\right)+\dfrac{4}{5}{}_HI^{1/5}y(2\pi),\\ y\left(\dfrac{\pi}{2}\right)=0,\qquad \dfrac{\sqrt{2}}{2}y\left(\dfrac{3\pi}{2}\right)=3{}_{RL}I^{1/2}x\left(\dfrac{3\pi}{2}\right)-{}_{RL}I^{1/3}x(\pi),\\ \qquad\qquad \frac{\pi}{2}\leq t\leq 2\pi.\end{cases} \tag{7.34}$$

Here $q_1 = 2/3$, $q_2 = 7/8$, $p_1 = 8/9$, $p_2 = 9/10$, $\lambda_1 = -1/9$, $\lambda_2 = -1/10$, $n = 2$, $m = 3$, $a = \pi/2$, $T = 2\pi$, $\sigma_1 = 1/5$, $\sigma_2 = \sqrt{2}/2$, $\tau_1 = \pi$, $\tau_2 = 3\pi/2$, $\eta_1 = \pi$, $\eta_2 = \pi/2$, $\eta_3 = 2\pi$, $\xi_1 = 3\pi/2$, $\xi_2 = \pi$, $\alpha_1 = \sqrt{2}$, $\alpha_2 = -1/2$, $\alpha_3 = 4/5$, $\beta_1 = 3$, $\beta_2 = -1$, $\rho_1 = 1/3$, $\rho_2 = 1/4$, $\rho_3 = 1/5$, $\gamma_1 = 1/2$, $\gamma_2 = 1/3$, $f(t,x,y) = (\sqrt{2}/2) + (|x|\pi^2\cos^2(2\pi t))/(4(7\pi - t)^2) + (\pi^2|y|/(5\pi - t)^2)(|y|/(|y|+4)+1)$ and $g(t,x,y) = (\sqrt{3}/2) + (\pi^2|x|/(3\pi - t)^2)(|x|/(|x|+2)+1) + (\pi^2\sin^2 y(t))/(4\pi - t)^2)$. Obviously

$$|f(t,x_1,x_2)| \le \frac{\sqrt{2}}{2} + \frac{1}{169}|x_1| + \frac{5}{81}|x_2|,$$

and

$$|g(t,x_1,x_2)| \le \frac{\sqrt{3}}{2} + \frac{6}{25}|x_1| + \frac{4}{81}|x_2|.$$

With the given data, we find that $\Omega' \approx 24.06826232 \neq 0$. Also, the assumptions of Theorem 7.9 are satisfied with $P_0 = \sqrt{2}/2$, $P_1 = 1/169$, $P_2 = 5/81$, $R_0 = \sqrt{3}/2$, $R_1 = 6/25$, $R_2 = 4/81$, $M_1 \simeq 6.576589777$, $M_2 \simeq 0.310859135$, $M_3 \simeq 0.452982744$, $M_4 \simeq 0.577264904$, $M_5 \simeq 0.837414324$, $M_6 \simeq 0.606939903$, $M_7 \simeq 0.13791283$, $M_8 \simeq 0.051821087$, $E_1 \approx 0.85503718 < 1$, $E_2 \approx 0.61252556 < 1$ and $E^* = 0.387474439$. Thus all the conditions of Theorem 7.9 hold true and consequently by the conclusion of Theorem 7.9, the problem (7.34) has at least one solution on $[\pi/2, 2\pi]$.

7.5 Langevin Equations with Fractional Uncoupled Integral Conditions

In this section, we study the following system

$$\begin{cases} {}_{RL}D^{q_1}\,({}_{RL}D^{p_1} + \lambda_1)\,x(t) = f(t,x(t),y(t)), & a \le t \le T, \\ {}_{H}D^{q_2}\,({}_{H}D^{p_2} + \lambda_2)\,y(t) \quad = g(t,x(t),y(t)), & a \le t \le T, \\ x(a) = 0, \qquad \sigma_1 x(\tau_1) = \displaystyle\sum_{i=1}^{m} \alpha_{i\,RL}I^{\rho_i}x(\eta_i), \\ y(a) = 0, \qquad \sigma_2 y(\tau_2) = \displaystyle\sum_{j=1}^{n} \beta_{j\,H}I^{\gamma_j}y(\xi_j). \end{cases} \tag{7.35}$$

Note that the parameters in (7.35) are the same as considered in (7.24).

Now we present an auxiliary lemma to define the solutions for the problem (7.35). We do not provide the proof as it is similar to that of Lemma 7.2.

Lemma 7.3 (Auxiliary Lemma) *For $h \in C([a,T],\mathbb{R})$, the solution of the boundary value problem*

$$\begin{cases} {}_{RL}D^{q_1}\left({}_{RL}D^{p_1} + \lambda_1\right)x(t) = h(t), \quad 0 < q_1, p_1 \le 1, \\ x(a) = 0, \qquad \sigma_1 x(\tau_1) = \sum_{i=1}^{m} \alpha_{i\,RL}I^{\rho_i}x(\eta_i), \quad t \in [a,T], \end{cases} \tag{7.36}$$

is equivalent to the integral equation

$$\begin{aligned} x(t) &= {}_{RL}I^{q_1+p_1}h(t) - \lambda_{1\,RL}I^{p_1}x(t) - \frac{(t-a)^{q_1+p_1-1}\Gamma(q_1)}{\Psi_1\Gamma(q_1+p_1)}\Bigg(\sum_{i=1}^{m}\alpha_{i\,RL}I^{q_1+p_1+\rho_i}h(\eta_i) \\ &\quad -\lambda_1\sum_{i=1}^{m}\alpha_{i\,RL}I^{p_1+\rho_i}x(\eta_i) + \lambda_1\sigma_{1\,RL}I^{p_1}x(\tau_1) - \sigma_{1\,RL}I^{q_1+p_1}h(\tau_1)\Bigg), \end{aligned} \tag{7.37}$$

where

$$\Psi_1 := \sum_{i=1}^{m}\frac{\alpha_i\Gamma(q_1)(\eta_i-a)^{q_1+p_1+\rho_i-1}}{\Gamma(q_1+p_1+\rho_i)} - \frac{\sigma_1\Gamma(q_1)(\tau_1-a)^{q_1+p_1-1}}{\Gamma(q_1+p_1)} \neq 0. \tag{7.38}$$

7.5.1 Existence Results for Uncoupled Case

In view of Lemma 7.3, we define an operator $\mathcal{K} : X \times Y \to X \times Y$ by

$$\mathcal{K}(x,y)(t) = \begin{pmatrix} \mathcal{K}_1(x,y)(t) \\ \mathcal{K}_2(x,y)(t) \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{K}_1(x,y)(t) &= {}_{RL}I^{q_1+p_1}f(s,x(s),y(s))(t) - \lambda_{1\,RL}I^{p_1}x(t) - \left(\frac{(t-a)^{q_1+p_1-1}\Gamma(q_1)}{\Psi_1\Gamma(q_1+p_1)}\right) \\ &\quad \times\Bigg(\sum_{i=1}^{m}\alpha_{i\,RL}I^{q_1+p_1+\rho_i}f(s,x(s),y(s))(\eta_i) - \lambda_1\sum_{i=1}^{m}\alpha_{i\,RL}I^{p_1+\rho_i}x(\eta_i) \\ &\quad +\lambda_1\sigma_{1\,RL}I^{p_1}x(\tau_1) - \sigma_{1\,RL}I^{q_1+p_1}f(s,x(s),y(s))(\tau_1)\Bigg), \end{aligned}$$

and

$$\begin{aligned}\mathscr{K}_2(x,y)(t) &= {}_HI^{q_2+p_2}g(s,x(s),y(s))(t) - \lambda_2{}_HI^{p_2}y(t) - \left(\frac{\left(\log\frac{t}{a}\right)^{q_2+p_2-1}\Gamma(q_2)}{\Psi_2\Gamma(q_2+p_2)}\right)\\ &\quad\times\Bigg(\sum_{j=1}^{n}\beta_j{}_HI^{q_2+p_2+\gamma_j}g(s,x(s),y(s))(\xi_j) - \lambda_2\sum_{j=1}^{n}\beta_j{}_HI^{p_2+\gamma_j}y(\xi_j)\\ &\quad+\lambda_2\sigma_2{}_HI^{p_2}y(\tau_2) - \sigma_2{}_HI^{q_2+p_2}g(s,x(s),y(s))(\tau_2)\Bigg),\end{aligned}$$

where

$$\Psi_2 := \sum_{j=1}^{n}\frac{\beta_j\Gamma(q_2)\left(\log\frac{\xi_j}{a}\right)^{q_2+p_2+\gamma_j-1}}{\Gamma(q_2+p_2+\gamma_j)} - \frac{\sigma_2\Gamma(q_2)\left(\log\frac{\tau_2}{a}\right)^{q_2+p_2-1}}{\Gamma(q_2+p_2)} \neq 0. \tag{7.39}$$

For the sake of convenience, we set

$$A_{17} = \sum_{i=1}^{m}\frac{|\alpha_i|(\eta_i-a)^{q_1+\rho_i}}{\Gamma(q_1+\rho_i+1)}, \qquad A_{18} = \sum_{i=1}^{m}\frac{|\alpha_i|(\eta_i-a)^{q_1+p_1+\rho_i}}{\Gamma(q_1+p_1+\rho_i+1)},$$

$$A_{19} = \sum_{j=1}^{n}\frac{|\beta_j|\left(\log\frac{\xi_j}{a}\right)^{p_2+\gamma_j}}{\Gamma(p_2+\gamma_j+1)}, \qquad A_{20} = \sum_{j=1}^{n}\frac{|\beta_j|\left(\log\frac{\xi_j}{a}\right)^{q_2+p_2+\gamma_j}}{\Gamma(q_2+p_2+\gamma_j+1)},$$

and

$$M_9 = \frac{A_1}{|\Psi_1|}(|\sigma_1|A_6+A_{18})+A_4, \qquad M_{10} = \frac{|\lambda_1|A_1}{|\Psi_1|}(|\sigma_1|A_5+A_{17})+|\lambda_1|A_3,$$

$$M_{11} = \frac{A_2}{|\Psi_2|}(|\sigma_2|A_{12}+A_{20})+A_8, \qquad M_{12} = \frac{|\lambda_2|A_2}{|\Psi_2|}(|\sigma_2|A_{11}+A_{19})+|\lambda_2|A_7.$$

Now, we state the existence and uniqueness result for the problem (7.35). The proof of this result is similar to that of Theorem 7.8.

Theorem 7.10 *Assume that $f, g : [a,T]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist positive constants $\bar{m}_i, \bar{n}_i$, $i = 1,2$ such that for all $t \in [a,T]$ and $x_i, y_i \in \mathbb{R}$, $i = 1,2$,*

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \le \bar{m}_1|x_1-y_1| + \bar{m}_2|x_2-y_2|$$

and

$$|g(t,x_1,x_2) - g(t,y_1,y_2)| \le \bar{n}_1|x_1-y_1| + \bar{n}_2|x_2-y_2|.$$

Then, the problem (7.35) has a unique solution on $[a, T]$ *if* $\delta_1 + \delta_2 < 1$, *where*

$$\delta_1 = \bar{m}_1 M_9 + \bar{m}_2 M_9 + M_{10},$$
$$\delta_2 = \bar{n}_1 M_{11} + \bar{n}_2 M_{11} + M_{12}.$$

Example 7.6 Consider the system of Langevin equation with fractional integral conditions

$$\begin{cases} {}_{RL}D^{1/2}\left({}_{RL}D^{4/7} - \dfrac{1}{36}\right)x(t) = \dfrac{|x|}{5(t+1)^2}\cdot\left(\dfrac{|x|}{|x|+3}+1\right) + \dfrac{\sin y(t)}{4(t+3)^2} - 2, \\ {}_{H}D^{7/9}\left({}_{H}D^{1/3} + \dfrac{1}{25}\right)y(t) \quad = \dfrac{|x|\sin^2(\pi t)}{(5+t)^2} + \dfrac{|y|\sin^2(3\pi t)}{8(2+t)^2}\cdot\left(\dfrac{|y|}{|y|+3}+1\right) + \dfrac{1}{3}, \\ x\left(\dfrac{1}{10}\right) = 0, \qquad \dfrac{1}{5}x\left(\dfrac{1}{4}\right) = \dfrac{1}{2}{}_{RL}I^{1/4}x\left(\dfrac{1}{5}\right) - \dfrac{\sqrt{2}}{3}{}_{RL}I^{1/2}x\left(\dfrac{1}{6}\right), \\ y\left(\dfrac{1}{10}\right) = 0, \qquad \dfrac{1}{3}y\left(\dfrac{1}{3}\right) = \dfrac{\sqrt{3}}{6}{}_{H}I^{\sqrt{2}/5}y\left(\dfrac{1}{8}\right) - \dfrac{1}{3}{}_{H}I^{3/5}y\left(\dfrac{1}{9}\right), \quad \dfrac{1}{10} \le t \le \dfrac{1}{2}. \end{cases} \tag{7.40}$$

Here $q_1 = 1/2$, $q_2 = 7/9$, $p_1 = 4/7$, $p_2 = 1/3$, $\lambda_1 = -1/36$, $\lambda_2 = 1/25$, $n = 2$, $m = 2$, $a = 1/10$, $T = 1/2$, $\sigma_1 = 1/5$, $\sigma_2 = 1/3$, $\tau_1 = 1/4$, $\tau_2 = 1/3$, $\eta_1 = 1/5$, $\eta_2 = 1/6$, $\xi_1 = 1/8$, $\xi_2 = 1/9$, $\alpha_1 = 1/2$, $\alpha_2 = -\sqrt{2}/3$, $\beta_1 = \sqrt{3}/6$, $\beta_2 = -1/3$, $\rho_1 = 1/4$, $\rho_2 = 1/2$, $\gamma_1 = \sqrt{2}/5$, $\gamma_2 = 3/5$, $f(t,x,y) = (|x|/5(t+1)^2)(|x|/(|x|+3)+1) + (\sin y(t))/4(t+3)^2) - 2$ and $g(t,x,y) = (|x|\sin^2(\pi t))/((5+t)^2) + (|y|\sin^2(3\pi t))/(8(2+t)^2)(|y|/(|y|+3) + 1) + (1/3)$. Clearly

$$|f(t,x_1,x_2) - f(t,y_1,y_2)| \le \frac{16}{135}|x_1 - x_2| + \frac{1}{49}|y_1 - y_2|,$$

and

$$|g(t,x_1,x_2) - g(t,y_1,y_2)| \le \frac{4}{121}|x_1 - x_2| + \frac{2}{75}|y_1 - y_2|.$$

Then, the assumptions of Theorem 7.8 are satisfied with $\bar{m}_1 = 16/135$, $\bar{m}_2 = 1/49$, $\bar{n}_1 = 4/121$ and $\bar{n}_2 = 2/75$. Using the given values, we find that $\Psi_1 \approx -0.482501053 \neq 0$, $M_9 \simeq 2.160475289, M_{10} \simeq 0.220957062, M_{11} \simeq 3.472362774$, $M_{12} \simeq 0.152255571$, $\delta_1 \approx 0.48152272$, and $\delta_2 \approx 0.359640764$. Therefore, we have

$$\delta_1 + \delta_2 \approx 0.841163484 < 1.$$

Hence, by Theorem 7.10, the problem (7.40) has a unique solution on $[1/10, 1/2]$.

The second result, dealing with the existence of solutions for the problem (7.35), is analogous to Theorem 7.9 and is stated below.

Theorem 7.11 *Assume that there exist real constants* $u_i,\ v_i \geq 0$ $(i = 1, 2)$ *and* $u_0 > 0, v_0 > 0$ *such that* $\forall x_i \in \mathbb{R},\ (i = 1, 2)$, *we have*

$$|f(t, x_1, x_2)| \leq u_0 + u_1|x_1| + u_2|x_2|,$$
$$|g(t, x_1, x_2)| \leq v_0 + v_1|x_1| + v_2|x_2|.$$

Further, it is assumed that

$$l_1 < 1 \ \ and \ \ l_2 < 1,$$

where

$$l_1 = u_1 M_9 + v_1 M_{11} + M_{10} \qquad and \qquad l_2 = u_2 M_9 + v_2 M_{11} + M_{12}.$$

Then, the problem (7.35) *has at least one solution on* $[a, T]$.

Proof Setting

$$l_0 = \max\{1 - l_1, 1 - l_2\},$$

the proof is similar to that of Theorem 7.9. So, we omit it. □

Example 7.7 Consider the system of Langevin equation with fractional integral conditions

$$\begin{cases} {}_{RL}D^{4/11}\left({}_{RL}D^{9/11} + \dfrac{1}{13}\right)x(t) = \dfrac{1}{2} + \dfrac{|x|\sin(\pi t)}{(1+t)^2} + \dfrac{|y|}{20(1+t)^2}\left(\dfrac{|y|}{|y|+4} + 1\right), \\ {}_{H}D^{5/8}\left({}_{H}D^{7/8} + \dfrac{1}{15}\right)y(t) \quad = \dfrac{1}{3} + \dfrac{|x|}{(4+t)^2} \cdot \left(\dfrac{|x|}{|x|+5} + 1\right) + \dfrac{\cos^2 y(t)}{9(2+t)^2}, \\ x\left(\dfrac{\sqrt{2}}{10}\right) = 0, \quad \dfrac{1}{\sqrt{5}}x\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{1}{5}{}_{RL}I^{\sqrt{3}/2}x\left(\dfrac{\sqrt{2}}{9}\right) - \dfrac{1}{7}{}_{RL}I^{7/11}x\left(\dfrac{\sqrt{2}}{5}\right), \\ y\left(\dfrac{\sqrt{2}}{10}\right) = 0, \quad \dfrac{1}{\sqrt{7}}y\left(\dfrac{\sqrt{2}}{3}\right) = \dfrac{2}{9}{}_{H}I^{4/9}y\left(\dfrac{\sqrt{2}}{4}\right) - \dfrac{1}{6}{}_{H}I^{1/9}y\left(\dfrac{\sqrt{2}}{7}\right), \ \dfrac{\sqrt{2}}{10} \leq t \leq \sqrt{2}. \end{cases} \tag{7.41}$$

Here $q_1 = 4/11$, $q_2 = 5/8$, $p_1 = 9/11$, $p_2 = 7/8$, $\lambda_1 = 1/13$, $\lambda_2 = 1/15$, $n = 2$, $m = 2$, $a = \sqrt{2}/10$, $T = \sqrt{2}$, $\sigma_1 = 1/\sqrt{5}$, $\sigma_2 = 1/\sqrt{7}$, $\tau_1 = \sqrt{2}/2$, $\tau_2 = \sqrt{2}/3$, $\eta_1 = \sqrt{2}/9$, $\eta_2 = \sqrt{2}/5$, $\xi_1 = \sqrt{2}/4$, $\xi_2 = \sqrt{2}/7$, $\alpha_1 = 1/5$, $\alpha_2 = -1/7$, $\beta_1 = 2/9$, $\beta_2 = -1/6$, $\rho_1 = \sqrt{3}/2$, $\rho_2 = 7/11$, $\gamma_1 = 4/9$, $\gamma_2 = 1/9$, $(1/2) + (|x|\sin^(\pi t))/((1+t)^2) + (|y|/(20(1+t)^2))(|y|/(|y|+4) + 1)$ and $(1/3) + (|x|/((4+t)^2)(|x|/(|x|+5) + 1) + (\cos^2 y(t))/(9(2+t)^2)$. It is easy to obtain

$$|f(t, x_1, x_2)| \leq \frac{1}{2} + \frac{100}{(30+\sqrt{2})^2}|x_1| + \frac{25}{(10+\sqrt{2})^2}|x_2|,$$

$$|g(t, x_1, x_2)| \le \frac{1}{3} + \frac{120}{(40+\sqrt{2})^2}|x_1| + \frac{100}{9(20+\sqrt{2})^2}|x_2|,$$

and $\Psi_2 \approx -0.513415653 \neq 0$. The assumptions of Theorem 7.9 are satisfied with $u_0 = 1/2$, $u_1 = 100/(30+\sqrt{2})^2$, $u_2 = 25/(10+\sqrt{2})^2$, $v_0 = 1/3$, $v_1 = 120/(40+\sqrt{2})^2$, $v_2 = 100/(9(20+\sqrt{2})^2)$, $M_9 \simeq 1.733533545$, $M_{10} \simeq 0.160164397$, $M_{11} \simeq 5.002175405$, $M_{12} \simeq 0.366634632$, $l_1 \approx 0.685805692 < 1$, $l_2 \approx 0.820481732 < 1$ and $l_0 = 0.314194309$. Thus all the conditions of Theorem 7.11 holds true and consequently there exists at least one solution on $[\sqrt{2}/10, \sqrt{2}]$.

7.6 Notes and Remarks

In this chapter, we have developed the existence theory for nonlinear Langevin equations and inclusions involving Hadamard-Caputo type fractional derivatives with nonlocal fractional integral conditions. Moreover, the existence and uniqueness results for coupled systems of Riemann-Liouville and Hadamard type fractional Langevin equations with fractional coupled and uncoupled integral conditions are also obtained.

The results in this chapter are adapted from the papers [128, 155] and [149].

Chapter 8
Boundary Value Problems for Impulsive Multi-Order Hadamard Fractional Differential Equations

8.1 Introduction

Impulsive differential equations describe observed evolution processes of several real world phenomena in a natural manner, and exhibit several new phenomena such as noncontinuability and merging of solutions, rhythmical beating, etc. Dynamic processes associated with sudden changes in their states are governed by impulsive differential equations. The dynamical behavior of impulsive differential systems is much more complex than the one concerning dynamical systems without impulse effects. Many applied problems arising in control theory, population dynamics and medicines have stimulated the recent development in this field. Dynamic systems involving some continuous variable dynamic characteristics and certain reset maps that generate impulsive switching among them are termed as impulsive hybrid systems. Such systems are classified as impulsive differential systems [47, 105, 146], sampled data or digital control systems [103, 167] and impulsive switched systems [76, 79]. Hybrid systems find their applications in embedded control systems interacting with the physical situation. Time and event-based behaviors are more accurately described by hybrid models as such models help to face challenging design requirements in the design of control systems. Examples include automotive control [31, 42], mobile robotics [43], process industry [80], real-time software verification [32], transportation systems [117, 168], manufacturing [138].

Fractional differential equations with impulse effects have also received considerable attention as these equations are found to be of great importance to model the physical problems experiencing abrupt/sudden changes at different instants. Recently, a class of nonlinear fractional-order differential impulsive systems with Hadamard derivative was discussed in [170, 171].

In this chapter, we concentrate on the study of boundary value problems of impulsive multi-order Hadamard fractional differential equations equipped with Hadamard type integral boundary conditions.

B. Ahmad et al., *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, DOI 10.1007/978-3-319-52141-1_8

8.2 Boundary Value Problems for First Order Impulsive Multi-Order Hadamard Fractional Differential Equations

In this section, we are concerned with the existence of solutions for boundary value problems of impulsive Hadamard fractional differential equations of the form

$$\begin{cases} {}^C\mathscr{D}_{t_k}^{p_k}x(t) = f(t, x(t)), \quad t \in J_k \subset [t_0, T],\ t \neq t_k, \\ \Delta x(t_k) = \varphi_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\ \alpha x(t_0) + \beta x(T) = \sum_{i=0}^{m} \gamma_i \mathscr{J}_{t_i}^{q_i} x(t_{i+1}), \end{cases} \tag{8.1}$$

where ${}^C\mathscr{D}_{t_k}^{p_k}$ is the Hadamard fractional derivative of Caputo type of order $0 < p_k \leq 1$ on intervals $J_k := (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$ with $J_0 = [t_0, t_1]$, $0 < t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$ are the impulse points, $J := [t_0, T]$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\varphi_k \in C(\mathbb{R}, \mathbb{R})$, $\mathscr{J}_{t_i}^{q_i}$ denote the Hadamard fractional integral of order $q_i > 0$, $i = 0, 1, \ldots, m$. The jump conditions are defined by $\Delta x(t_k) = x(t_k^+) - x(t_k)$, $x(t_k^+) = \lim_{\varepsilon \to 0^+} x(t_k + \varepsilon)$, $k = 1, 2, 3, \ldots, m$.

Lemma 8.1 *Assume that* $\Phi = \alpha + \beta - \sum_{i=1}^{m} \frac{\gamma_i (\log (t_{i+1}/t_i))^{q_i}}{\Gamma(q_i + 1)} \neq 0$. *Then the solution of the problem* (8.1) *is equivalent to the following integral equation:*

$$\begin{aligned} x(t) = {} & \mathscr{J}_{t_k}^{p_k} f(t, x(t)) + \sum_{i=0}^{k-1} \left(\mathscr{J}_{t_i}^{p_i} f(t_{i+1}, x(t_{i+1})) + \varphi_{i+1}(x(t_{i+1})) \right) \\ & + \frac{1}{\Phi} \Bigg[\sum_{i=0}^{m} \gamma_i \mathscr{J}_{t_i}^{q_i + p_i} f(t_{i+1}, x(t_{i+1})) - \beta \mathscr{J}_{t_m}^{p_m} f(T, x(T)) \\ & - \beta \sum_{i=0}^{m-1} \left(\mathscr{J}_{t_i}^{p_i} f(t_{i+1}, x(t_{i+1})) + \varphi_{i+1}(x(t_{i+1})) \right) \\ & + \sum_{i=1}^{m} \left(\frac{\gamma_i (\log (t_{i+1}/t_i))^{q_i}}{\Gamma(q_i + 1)} \right) \left(\sum_{j=0}^{i-1} \left(\mathscr{J}_{t_j}^{p_j} f(t_{j+1}, x(t_{j+1})) + \varphi_{j+1}(x(t_{j+1})) \right) \right) \Bigg]. \end{aligned} \tag{8.2}$$

Proof The solution of first equation (8.1) on interval J_0 can be written as

$$x(t) = \mathscr{J}_{t_0}^{p_0} f(t, x(t)) + x_0,$$

where $x_0 \in \mathbb{R}$. For $t \in J_1$, by using the impulse condition $\Delta x(t_1) = \varphi_1(x(t_1))$, we obtain

$$\begin{aligned} x(t) &= \mathscr{I}_{t_1}^{p_1} f(t, x(t)) + x(t_1^+) \\ &= \mathscr{I}_{t_1}^{p_1} f(t, x(t)) + \mathscr{I}_{t_0}^{p_0} f(t_1, x(t_1)) + \varphi_1(x(t_1)) + x_0. \end{aligned}$$

Again for $t \in J_2$, we have

$$\begin{aligned} x(t) &= \mathscr{I}_{t_2}^{p_2} f(t, x(t)) + x(t_2^+) \\ &= \mathscr{I}_{t_2}^{p_2} f(t, x(t) + \mathscr{I}_{t_1}^{p_1} f(t_2, x(t_2)) + \varphi_2(x(t_2)) \\ &\quad + \mathscr{I}_{t_0}^{p_0} f(t_1, x(t_1)) + \varphi_1(x(t_1)) + x_0. \end{aligned}$$

Repeating above process, for $t \in J$, we obtain

$$x(t) = \mathscr{I}_{t_k}^{p_k} f(t, x(t)) + \sum_{i=0}^{k-1} \left(\mathscr{I}_{t_i}^{p_i} f(t_{i+1}, x(t_{i+1})) + \varphi_{i+1} x(t_{i+1}) \right) + x_0. \tag{8.3}$$

Applying the boundary condition of (8.1), it follows that

$$\begin{aligned} \alpha x(t_0) + \beta x(T) = (\alpha + \beta) x_0 + \beta \mathscr{I}_{t_m}^{p_m} f(T, x(T)) \\ + \beta \sum_{i=0}^{m-1} \left(\mathscr{I}_{t_i}^{p_i} f(t_{i+1}, x(t_{i+1})) + \varphi_{i+1}(x(t_{i+1})) \right) \end{aligned}$$

and

$$\begin{aligned} &\sum_{i=0}^{m} \gamma_i \mathscr{I}_{t_i}^{q_i} x(t_{i+1}) \\ &= \sum_{i=0}^{m} \gamma_i \mathscr{I}_{t_i}^{p_i + q_i} f(t_{i+1}, x(t_{i+1})) + x_0 \sum_{i=0}^{m} \frac{\gamma_i (\log (t_{i+1}/t_i))^{q_i}}{\Gamma(q_i + 1)} \\ &\quad + \sum_{i=1}^{m} \left(\frac{\gamma_i (\log (t_{i+1}/t_i))^{q_i}}{\Gamma(q_i + 1)} \right) \left(\sum_{j=0}^{i-1} \left(\mathscr{I}_{t_j}^{j-1} f(t_{j+1}, x(t_{j+1})) + \varphi_{j+1}(x(t_{j+1})) \right) \right), \end{aligned}$$

which leads to

$$\begin{aligned} x_0 = \frac{1}{\Phi} \Bigg[&\sum_{i=0}^{m} \gamma_i \mathscr{I}_{t_i}^{q_i + p_i} f(t_{i+1}, x(t_{i+1})) - \beta \mathscr{I}_{t_m}^{p_m} f(T, x(T)) \\ &- \beta \sum_{i=0}^{m-1} \left(\mathscr{I}_{t_i}^{p_i} f(t_{i+1}, x(t_{i+1})) + \varphi_{i+1}(x(t_{i+1})) \right) \end{aligned}$$

$$+\sum_{i=1}^{m}\left(\frac{\gamma_i(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)}\right)\left(\sum_{j=0}^{i-1}\left(\mathscr{J}_{t_j}^{j-1}f(t_{j+1},x(t_{j+1}))+\varphi_{j+1}(x(t_{j+1}))\right)\right)\Bigg].$$

Replacing the constant x_0 in the Eq. (8.3) by its above value, we obtain (8.2). The converse follows by direct computation. This completes the proof. □

Let $PC(J,\mathbb{R}) = \{x : J \to \mathbb{R};\ x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k),\ k = 1,2,\dots,m\}$. Obviously, $PC(J,\mathbb{R})$ is a Banach space with the norm $\|x\| = \sup\{|x(t)| : t \in J\}$. A function $x \in PC(J,\mathbb{R})$ is called a solution of problem (8.1) if it satisfies (8.1).

We define an operator $\mathscr{K} : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ as

$$\begin{aligned}
&\mathscr{K}x(t)\\
&= \mathscr{J}_{t_k}^{p_k}f(t,x(t)) + \sum_{i=0}^{k-1}\left(\mathscr{J}_{t_i}^{p_i}f(t_{i+1},x(t_{i+1}))+\varphi_{i+1}(x(t_{i+1}))\right)\\
&\quad+\frac{1}{\Phi}\Bigg[\sum_{i=0}^{m}\gamma_i\mathscr{J}_{t_i}^{q_i+p_i}f(t_{i+1},x(t_{i+1}))-\beta\mathscr{J}_{t_m}^{p_m}f(T,x(T))\\
&\quad-\beta\sum_{i=0}^{m-1}\left(\mathscr{J}_{t_i}^{p_i}f(t_{i+1},x(t_{i+1}))+\varphi_{i+1}(x(t_{i+1}))\right)\\
&\quad+\sum_{i=1}^{m}\left(\frac{\gamma_i(\log(t_{i+1}/t_i)^{q_i}}{\Gamma(q_i+1)}\right)\left(\sum_{j=0}^{i-1}\left(\mathscr{J}_{t_j}^{p_j}f(t_{j+1},x(t_{j+1}))+\varphi_{j+1}(x(t_{j+1}))\right)\right)\Bigg].
\end{aligned}$$

Clearly, the problem (8.1) transforms to a fixed point problem $x = \mathscr{K}x$.

Let us set the notations:

$$\begin{aligned}
\Lambda_1 &= \sum_{i=0}^{m}\frac{(\log(t_{i+1}/t_i))^{p_i}}{\Gamma(p_i+1)}\\
&\quad+\frac{1}{|\Phi|}\Bigg\{\sum_{i=0}^{m}\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i+p_i}}{\Gamma(q_i+p_i+1)}+|\beta|\sum_{i=0}^{m}\frac{(\log(t_{i+1}/t_i))^{p_i}}{\Gamma(p_i+1)}\\
&\quad+|\beta|\frac{(\log(T/t_m))^{p_m}}{\Gamma(p_m+1)}+\sum_{i=1}^{m}\sum_{j=0}^{i-1}\left(\frac{(\log(t_{j+1}/t_j))^{p_j}}{\Gamma(p_j+1)}\right)\left(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(p_m+1)}\right)\Bigg\},\\
\Lambda_2 &= m+\frac{1}{\Phi}\left[|\beta|m+\sum_{i=0}^{m}\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)}\right].
\end{aligned}$$

Theorem 8.1 *Let $f : J \times \mathbb{R} \to \mathbb{R}$ and $\varphi_k : \mathbb{R} \to \mathbb{R}, k = 1, 2, \ldots, m$ be continuous functions satisfying the following conditions:*

(8.1.1) $|f(t,x) - f(t,y)| \le L_1|x-y|, \quad \forall t \in J,\ L_1 > 0,\ x, y \in \mathbb{R}$;
(8.1.2) $|\varphi_k(u) - \varphi_k(v)| \le L_2|u-v|,\ L_2 > 0,$ *for all* $u, v \in \mathbb{R}, \quad \forall k = 1, 2, \ldots, m.$

If $L_1\Lambda_1 + L_2\Lambda_2 < 1$, *then the problem* (8.1) *has a unique solution on* J.

Proof We define a closed ball $B_r = \{x \in PC(J, \mathbb{R}) : \|x\| \le r\}$, where $r \ge (M_1\Lambda_1 + M_2\Lambda_2)(1 - L_1\Lambda_1 - L_2\Lambda_2)^{-1}$, with $M = \sup_{t\in J}|f(t,0)|,\ M_1 = \sup_{t\in J}|f(t_{i+1},\ 0)|$ and $M_2 = \sup_{t\in J}|\phi_{i+1}(0)|, i = 1, 2, \ldots, m-1.$

We will show that $\mathscr{K} : B_r \to B_r$. For any $x \in B_r$, we have

$$
\begin{aligned}
&|\mathscr{K}x(t)| \\
&\le \mathscr{J}_{t_k}^{p_k}|f(t,x(t))| + \sum_{i=0}^{k-1}\left(\mathscr{J}_{t_i}^{p_i}|f(t_{i+1},x(t_{i+1}))| + |\varphi_{i+1}(x(t_{i+1}))|\right) \\
&\quad + \frac{1}{|\Phi|}\Bigg[\sum_{i=0}^{m}|\gamma_i|\mathscr{J}_{t_i}^{q_i+p_i}|f(t_{i+1},x(t_{i+1}))| + |\beta|\mathscr{J}_{t_m}^{p_m}|f(T,x(T))| \\
&\quad + |\beta|\sum_{i=0}^{m-1}\left(\mathscr{J}_{t_i}^{p_i}|f(t_{i+1},x(t_{i+1}))| + |\varphi_{i+1}(x(t_{i+1}))|\right) \\
&\quad + \sum_{i=1}^{m}\left(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)}\right)\left(\sum_{j=0}^{i-1}\left(\mathscr{J}_{t_j}^{p_j}|f(t_{j+1},x(t_{j+1}))| + |\varphi_{j+1}(x(t_{j+1}))|\right)\right)\Bigg] \\
&\le \mathscr{J}_{t_k}^{p_k}(|f(t,x(t)) - f(t,0)| + |f(t,0)|) \\
&\quad + \sum_{i=0}^{k-1}\Big(\mathscr{J}_{t_i}^{p_i}(|f(t_{i+1},x(t_{i+1})) - f(t_{i+1},0)| + |f(t_{i+1},0)|) \\
&\quad + |\varphi_{i+1}(x(t_{i+1})) - \varphi_{i+1}(0)| + |\varphi_{i+1}(0)|\Big) \\
&\quad + \frac{1}{|\Phi|}\Bigg[\sum_{i=0}^{m}|\gamma_i|\mathscr{J}_{t_i}^{q_i+p_i}(|f(t_{i+1},x(t_{i+1})) - f(t_{i+1},0)| + |f(t_{i+1},0)|) \\
&\quad + |\beta|\mathscr{J}_{t_m}^{p_m}(|f(T,x(T)) - f(T,0)| + |f(T,0)|) \\
&\quad + |\beta|\sum_{i=0}^{m-1}\Big(\mathscr{J}_{t_i}^{p_i}(|f(t_{i+1},x(t_{i+1})) - f(t_{i+1},0)| + |f(t_{i+1},0)|) \\
&\quad + |\varphi_{i+1}(x(t_{i+1})) - \varphi_{i+1}(0)| + |\varphi_{i+1}(0)|\Big)
\end{aligned}
$$

$$+\sum_{i=1}^{m}\left(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)}\right)\left(\sum_{j=0}^{i-1}\left(\mathscr{J}_{t_j}^{p_j}(|f(t_{j+1},x(t_{j+1}))-f(t_{j+1},0)|\right.\right.$$

$$\left.\left.\left.+|f(t_{j+1},0)|)+|\varphi_{j+1}(x(t_{j+1}))-\varphi_{j+1}(0)|+|\varphi_{j+1}(0)|\right)\right)\right]$$

$$\leq (L_1r+M)\frac{(\log(t/t_k))^{p_k}}{\Gamma(p_k+1)}+\sum_{i=0}^{k-1}\left\{(L_1r+M_1)\frac{(\log(t_{i+1}/t_i))^{p_i}}{\Gamma(p_i+1)}+(L_2r+M_2)\right\}$$

$$+\frac{1}{|\Phi|}\left[\sum_{i=0}^{m}|\gamma_i|(L_1r+M_1)\frac{(\log(t_{i+1}/t_i))^{q_i+p_i}}{\Gamma(q_i+p_i+1)}+|\beta|(L_1r+M_1)\frac{(\log(T/t_m))^{p_m}}{\Gamma(p_m+1)}\right.$$

$$+|\beta|\sum_{i=0}^{m-1}\left\{(L_1r+M_1)\frac{(\log(t_{i+1}/t_i))^{p_i}}{\Gamma(p_i+1)}+(L_2r+M_2)\right\}$$

$$+\sum_{i=1}^{m}\left\{\left(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)}\right)\left(\sum_{j=0}^{i-1}(L_1r+M_1)\frac{(\log(t_{j+1}/t_j))^{p_j}}{\Gamma(p_j+1)}\right.\right.$$

$$\left.\left.\left.+(L_2r+M_2)\right)\right\}\right]$$

$$\leq (L_1\Lambda_1+L_2\Lambda_2)r+(M_1\Lambda_1+M_2\Lambda_2)\leq r.$$

Thus $\mathscr{K}B_r\subseteq B_r$. Next, we will show that $\mathscr{K}$ is a contraction mapping. For $x,y\in B_r$, we get

$$|\mathscr{K}x(t)-\mathscr{K}y(t)|$$

$$\leq \mathscr{J}_{t_k}^{p_k}|f(t,x(t))-f(t,y(t))|+\sum_{i=0}^{k-1}\left(\mathscr{J}_{t_i}^{p_i}|f(t_{i+1},x(t_{i+1}))-f(t_{i+1},y(x_{i+1}))|\right.$$

$$\left.+|\varphi_{i+1}(x(t_{i+1}))-\varphi_{i+1}(y(t_{i+1}))|\right)$$

$$+\frac{1}{|\Phi|}\left[\sum_{i=0}^{m}|\gamma_i|\mathscr{J}_{t_i}^{q_i+p_i}|f(t_{i+1},x(t_{i+1}))-f(t_{i+1},y(t_{i+1}))|\right.$$

$$+|\beta|\mathscr{J}_{t_m}^{p_m}|f(T,x(T))-f(T,y(T))|$$

$$+|\beta|\sum_{i=0}^{m-1}\left(\mathscr{J}_{t_i}^{p_i}|f(t_{i+1},x(t_{i+1}))-f(t_{i+1},y(t_{i+1}))|\right.$$

$$\left.+|\varphi_{i+1}(x(t_{i+1}))-\varphi_{i+1}(y(t_{i+1}))|\right)$$

$$+\sum_{i=1}^{m}\left(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)}\right)\left(\sum_{j=0}^{i-1}\left(\mathscr{J}_{t_j}^{p_j}|f(t_{j+1},x(t_{j+1}))-f(t_{j+1},y(t_{j+1}))|\right.\right.$$

$$\left.\left.\left.+|\varphi_{j+1}(x(t_{j+1}))-\varphi_{j+1}(y(t_{j+1}))|\right)\right)\right]$$

$$\leq (L_1\Lambda_1+L_2\Lambda_2)\|x-y\|.$$

Since $(L_1\Lambda_1+L_2\Lambda_2)<1$, the operator $\mathscr{K}$ is a contraction. Hence $\mathscr{K}$ has a unique fixed point on B_r. Therefore the problem (8.1) has a unique solution on J. □

Example 8.1 Consider the following boundary value problem for impulsive multi-order Hadamard fractional differential equation of the form

$$\begin{cases} {}^{C}\mathscr{D}_{t_k}^{\left(\frac{k+1}{k+2}\right)}x(t)=\dfrac{10-t^2}{8(t^2+24)}\left(\dfrac{(|x(t)|+2)^2}{|x(t)|+3}\right), & t\in\left[1,\dfrac{8e+1}{9}\right]\setminus\{t_k\},\\ \Delta x(t_k)=\dfrac{\sin|x(t_k)|}{5(11-k)}, \qquad t_k=\dfrac{ke+1}{k+1},\ k=1,2,\ldots,7, \\ \dfrac{3}{2}x(1)+\dfrac{4}{5}x\left(\dfrac{8e+1}{9}\right)=\displaystyle\sum_{i=0}^{7}(1-e^{-i})\,\mathscr{J}_{t_i}^{\left(\frac{i^2+5i+2}{i^2+4i+3}\right)}x(t_{i+1}). \end{cases} \tag{8.4}$$

Here $\alpha=3/2$, $\beta=4/5$, $m=7$, $p_k=(k+1)/(k+2)$, $\gamma_k=1-e^{-k}$, $q_k=(k^2+5k+2)/(k^2+4k+3)$ for $k=0,1,\ldots,7$. From the given information, we find that $\Phi\approx 2.0961081$, $\Lambda_1\approx 3.280445$ and $\Lambda_2\approx 13.552466$. The functions f and φ_k given by

$$f(t,x)=\frac{10-t^2}{8(t^2+24)}\left(\frac{(|x|+2)^2}{|x|+3}\right),\qquad \varphi_k(x)=\frac{\sin|x|}{5(11-k)},$$

satisfy the conditions:

$$|f(t,x)-f(t,y)|\leq\frac{2}{25}|x-y| \text{ and } |\varphi_k(x)-\varphi_k(y)|\leq\frac{1}{20}|x-y|,\quad \forall k=1,2,\ldots,7.$$

Thus, we get $L_1=2/25$, $L_2=1/20$ and $L_1\Lambda_1+L_2\Lambda_2\approx 0.940059<1$. Therefore the problem (8.1) has a unique solution on $[1,(8e+1)/9]$ due to Theorem 8.1.

Theorem 8.2 *Let f and φ_k, $k=1,2,\ldots,m$, be continuous functions. Assume that there are two positive real numbers N_1 and N_2 such that:*

(8.2.1) $|f(t,x)|\leq N_1$ *and* $|\varphi_k(x)|\leq N_2$, *for $t\in J$, $x\in\mathbb{R}$ and $k=1,2,\ldots,m$.*

Then, the problem (8.1) has at least one solution on J.

Proof Define a ball $B_\omega = \{x \in PC(J,\mathbb{R}) : \|x\| < \omega\}$. The proof is divided into three steps.

Step 1. We will show that the operator $\mathscr{K}$ (introduced after Lemma 8.1) is continuous. To prove this, we let $\{x_n\}$ be a sequence in $PC(J,\mathbb{R})$ such that $x_n \to x$ as $n \to \infty$. Then we have

$$\begin{aligned}
&|\mathscr{K}x_n(t) - \mathscr{K}x(t)|\\
\le\ & \mathscr{J}_{t_k}^{p_k}|f(t,x_n(t)) - f(t,x(t))|\\
&+\sum_{i=0}^{k-1}\Big(\mathscr{J}_{t_i}^{p_i}|f(t_{i+1},x_n(t_{i+1})) - f(t_{i+1},x(t_{i+1}))|\\
&+ |\varphi_{i+1}(x_n(t_{i+1})) - \varphi_{i+1}(x(t_{i+1}))|\Big)\\
&+\frac{1}{|\Phi|}\Bigg[\sum_{i=0}^{m}|\gamma_i|\mathscr{J}_{t_i}^{q_i+p_i}|f(t_{i+1},x_n(t_{i+1})) - f(t_{i+1},x(t_{i+1}))|\\
&+|\beta|\mathscr{J}_{t_m}^{p_m}|f(T,x_n(T)) - f(T,x(T))|\\
&+|\beta|\sum_{i=0}^{m-1}\Big(\mathscr{J}_{t_i}^{p_i}|f(t_{i+1},x_n(t_{i+1})) - f(t_{i+1},x(t_{i+1}))|\\
&+|\varphi_{i+1}(x_n(t_{i+1})) - \varphi_{i+1}(x(t_{i+1}))|\Big)\\
&+\sum_{i=1}^{m}\left(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)}\right)\left(\sum_{j=0}^{i-1}\Big(\mathscr{J}_{t_j}^{p_j}|f(t_{j+1},x_n(t_{j+1})) - f(t_{j+1},x(t_{j+1}))|\right.\\
&\left.+|\varphi_{j+1}(x_n(t_{j+1})) - \varphi_{j+1}(x(t_{j+1}))|\Big)\right)\Bigg].
\end{aligned}$$

Using the continuity of f and φ_k for $k = 1,2,\ldots,m$, we have $|f(t,x_n) - f(t,x)|$ and $|\varphi_k(x_n) - \varphi_k(x)|$ vanish as $n \to \infty$. Therefore $\|\mathscr{K}x_n - \mathscr{K}x\| \to 0$ which yields the continuity of the operator $\mathscr{K}$.

Step 2. The operator $\mathscr{K}$ maps bounded set into bounded set. For each $x \in \bar{B}_\omega$, we have

$$\begin{aligned}
&\|\mathscr{K}x\|\\
\le\ & \mathscr{J}_{t_k}^{p_k}|f(t,x(t))| + \sum_{i=0}^{k-1}\Big(\mathscr{J}_{t_i}^{p_i}|f(t_{i+1},x(t_{i+1}))| + |\varphi_{i+1}(x(t_{i+1}))|\Big)\\
&+\frac{1}{|\Phi|}\Bigg[\sum_{i=0}^{m}|\gamma_i|\mathscr{J}_{t_i}^{q_i+p_i}|f(t_{i+1},x(t_{i+1}))| + |\beta|\mathscr{J}_{t_m}^{p_m}f(T,x(T))|\\
&+|\beta|\sum_{i=0}^{m-1}\Big(\mathscr{J}_{t_i}^{p_i}|f(t_{i+1},x(t_{i+1}))| + |\varphi_{i+1}(x(t_{i+1}))|\Big)
\end{aligned}$$

$$+\sum_{i=1}^{m}\left(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)}\right)\left(\sum_{j=0}^{i-1}\left(\mathscr{J}_{t_j}^{p_j}|f(t_{j+1},x(t_{j+1}))|\right.\right.$$
$$\left.\left.\left.+|\varphi_{j+1}(x(t_{j+1}))|\right)\right)\right]$$
$$\leq \Lambda_1 N_1 + \Lambda_2 N_2,$$

which yields boundedness of $\mathscr{K}\bar{B}_\omega$.

Step 3. The operator $\mathscr{K}$ maps bounded set into equicontinuous set. Let $\tau_1, \tau_2 \in (t_k, t_{k+1})$ for each $k = 0, 1, 2, \ldots, m$. Then we have

$$|\mathscr{K}x(\tau_1) - \mathscr{K}x(\tau_2)| \leq \mathscr{J}_{t_k}^{t_k+1}|f(\tau_1, x(\tau_1)) - f(\tau_2, x(\tau_2))|.$$

Continuity of x and f imply that $\mathscr{K}x(\tau_1) \to \mathscr{K}x(\tau_2)$ as $\tau_1 \to \tau_2$. Consequently $\mathscr{K}$ is completely continuous by Azelá-Ascoli's Theorem.

Let $V = \{x \in B_\omega : \mu\mathscr{K}x = x \text{ for } \mu \in (0,1)\}$. For all $x \in V$, $x = \mu\mathscr{K}x$, we have

$$|x| \leq \mu|\mathscr{K}x| \leq \Lambda_1 N_1 + \Lambda_2 N_2.$$

Hence V is bounded. By Theorem 1.3, the problem (8.1) has at least one solution on J. □

Example 8.2 Consider the following boundary value problem for impulsive multi-order Hadamard fractional differential equations of the form

$$\begin{cases} {}^{C}\mathscr{D}_{t_k}^{\log(\sum_{i=0}^{k+1}(1/(i+1)!))}x(t) = \dfrac{(2-e^{-t})\log(|x(t)|+1)}{|x(t)|+2} - 2, \quad t \in \left[\dfrac{\pi}{2}, 2\pi\right]\setminus\{t_k\}, \\ \Delta x(t_k) = e^{-k/4}\cos(kx(t_k)) + e^{k/4}\sin(kx(t_k)), \qquad t_k = 2^{(2k-9)/9}\pi,\ k = 1, 2, \ldots, 8, \\ -e^{-\pi/2}x\left(\dfrac{\pi}{2}\right) + e^{-2\pi}x(2\pi) = \displaystyle\sum_{i=0}^{8}(-2)^i(i^2+1)\mathscr{J}_{t_i}^{\left(\frac{|5i-4|}{i+1}\right)}x(t_{i+1}). \end{cases} \tag{8.5}$$

Here $\alpha = -e^{-\pi/2}$, $\beta = e^{-2\pi}$, $m = 8$, $p_k = \log(\sum_{i=0}^{k+1}(1/(i+1)!))$, $\gamma_k = (-2)^k(k^2+1)$, $q_k = |5k-4|/(k+1)$ for $k = 0, 1, \ldots, 8$. We find that $\Phi \approx -1.422922 \neq 0$. The functions $f(t,x) = (2-e^{-t})\log(|x|+1)/(|x|+2) - 2$ and $\varphi_k(x) = e^{-k/4}\cos(kx) + e^{k/4}\sin(kx)$ are bounded as

$$|f(t,x)| \leq 4 \text{ and } |\varphi_k(x)| \leq \sqrt{e^{-4}+e^4}.$$

Hence the assumption (8.2.1) of Theorem 8.2 holds. Therefore the problem (8.2) has at least one solution on $[\pi/2, 2\pi]$.

Theorem 8.3 *Assume that:*

(8.3.1) $\lim\limits_{x\to 0}\dfrac{f(t,x)}{x}=0$ *and* $\lim\limits_{x\to 0}\dfrac{\varphi_k(x)}{x}=0$ *for* $k=1,2,\ldots,m.$

Then, the problem (8.1) *has at least one solution on* J.

Proof From (8.3.1), choosing $\epsilon = 1/(\Lambda_1+\Lambda_2)$, there exist constants $\delta_1,\delta_2\in\mathbb{R}^+$ such that

$$|f(t,x)|<\epsilon|x| \text{ where } |x|<\delta_1 \text{ and } |\varphi(x)|<\epsilon|x| \text{ where } |x|<\delta_2.$$

Next, we define an open ball $\Omega=\{u\in PC(J,\mathbb{R}) : \|u\|<\max\{\delta_1,\delta_2\}\}$. By Theorem 8.2, the operator $\mathscr{K}:\bar{\Omega}\to PC(J,\mathbb{R})$ is completely continuous. For any $x\in\partial\Omega$, we have

$$\begin{aligned}
&\|\mathscr{K}x\|\\
&\le \mathscr{J}_{t_k}^{p_k}|f(t,x(t))|+\sum_{i=0}^{k-1}\left(\mathscr{J}_{t_i}^{p_i}|f(t_{i+1},x(t_{i+1}))|+|\varphi_{i+1}(x(t_{i+1}))|\right)\\
&\quad+\frac{1}{|\Phi|}\Bigg[\sum_{i=0}^{m}|\gamma_i|\mathscr{J}_{t_i}^{q_i+p_i}|f(t_{i+1},x(t_{i+1}))|+|\beta|\mathscr{J}_{t_m}^{p_m}f(T,x(T))|\\
&\quad+|\beta|\sum_{i=0}^{m-1}\left(\mathscr{J}_{t_i}^{p_i}|f(t_{i+1},x(t_{i+1}))|+|\varphi_{i+1}(x(t_{i+1}))|\right)\\
&\quad+\sum_{i=1}^{m}\left(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)}\right)\left(\sum_{j=0}^{i-1}\left(\mathscr{J}_{t_j}^{p_j}|f(t_{j+1},x(t_{j+1}))|+|\varphi_{j+1}(x(t_{j+1}))|\right)\right)\Bigg]\\
&\le(\Lambda_1\epsilon+\Lambda_2\epsilon)\|x\|=\|x\|.
\end{aligned}$$

It follows from Theorem 1.5 case (*ii*) that the problem (8.1) has at least one solution on J. □

Example 8.3 Consider the following boundary value problem for impulsive multi-order Hadamard fractional differential equations:

$$\begin{cases}
{}^{C}\mathscr{D}_{t_k}^{\left(\frac{2k+2}{k^2+2k+2}\right)}x(t)=\dfrac{e^{tx(t)}(\sin x(t)-x(t))}{2t+1}, & t\in\left[\dfrac{4}{3},3\right]\setminus\{t_k\},\\
\Delta x(t_k)=\dfrac{kx^3(t_k)}{\log(|x(t_k)|+2)}, \quad t_k=\dfrac{k+8}{6}, & k=1,2,\ldots,9,\\
\sqrt{3}x\left(\dfrac{4}{3}\right)+\dfrac{3}{5}x(3)=\displaystyle\sum_{i=0}^{9}\left(\frac{i^2+1}{i^2+2}\right)\mathscr{J}_{t_i}^{\arctan i}x(t_{i+1}),
\end{cases}\tag{8.6}$$

Here $\alpha = \sqrt{3}$, $\beta = 3/5$, $m = 9$, $p_k = 2(k+1)/(k^2+1)$, $\gamma_k = (k^2+1)/(k^2+2)$, $q_k = \arctan(k)$ for $k = 0, 1, \ldots, 9$. We find that $\Phi \approx 2.003684 \neq 0$. The functions $f(t,x) = e^{tx}(\sin x - x)/(2t+1)$ and $\varphi_k(x) = kx^3/\log(|x|+2)$ satisfy

$$\lim_{x\to 0} \frac{f(t,x)}{x} = \lim_{x\to 0} \frac{e^{xt}}{2t+1}\left(\frac{\sin x}{x} - 1\right) = 0$$

and

$$\lim_{x\to 0} \frac{\varphi_k(x)}{x} = \lim_{x\to 0} \frac{kx^2}{\log(|x|+2)} = 0, \quad \forall k = 1, 2, \ldots, 9.$$

Thus the condition (8.3.1) of Theorem 8.3 holds. Therefore, we conclude that the problem (8.3) has at least one solution on $[4/3, 3]$.

Theorem 8.4 *Let f and φ_k for $k = 1, 2, \ldots, m$, be continuous functions satisfying the inequalities:*

(8.4.1) $|f(t,x)| \le a|x| + b$, $\forall (t,x) \in J \times \mathbb{R}$ *and* $|\varphi_k(x)| \le c|x| + d$, $\forall x \in \mathbb{R}, k = 1, \ldots, m$, *where constants* $a, c > 0$ *and* $b, d \ge 0$.

Then the problem (8.1) has at least one solution on J.

Proof Define a unit ball as $\mathscr{O} = \{x \in PC(J,\mathbb{R}) : \|x\| < 1\}$. It is straightforward to show that the operator $\mathscr{K} : \bar{\mathscr{O}} \to PC(J,\mathbb{R})$ is completely continuous. Suppose that there is $x^* \in \partial\mathscr{O}$. Then we choose $\lambda = (a+c)\Lambda_1 + (b+d)\Lambda_2 + 1$ such that $\mathscr{K}x^* = \lambda x^*$. By taking the norm of both sides: $\|\mathscr{K}x^*\| = \|\lambda x^*\|$, we obtain $\|\mathscr{K}\|\|x^*\| \ge \lambda\|x^*\|$. Then, we have

$$\begin{aligned}
\|\mathscr{K}\| &= \sup_{\|x\|=1} |\mathscr{K}x| \\
&= \sup_{\|x\|=1} \Bigg\{ \mathscr{J}_{t_k}^{p_k}|f(t,x(t))| + \sum_{i=0}^{k-1}\left(\mathscr{J}_{t_i}^{p_i}|f(t_{i+1},x(t_{i+1}))| + |\varphi_{i+1}(x(t_{i+1}))|\right) \\
&\quad + \frac{1}{|\Phi|}\Bigg[\sum_{i=0}^{m}|\gamma_i|\mathscr{J}_{t_i}^{q_i+p_i}|f(t_{i+1},x(t_{i+1}))| + |\beta|\mathscr{J}_{t_m}^{p_m}|f(T,x(T))| \\
&\quad + |\beta|\sum_{i=0}^{m-1}\left(\mathscr{J}_{t_i}^{p_i}|f(t_{i+1},x(t_{i+1}))| + |\varphi_{i+1}(x(t_{i+1}))|\right) \\
&\quad + \sum_{i=1}^{m}\left(\frac{|\gamma_i|(\log(t_{i+1}/t_i))^{q_i}}{\Gamma(q_i+1)}\right)\left(\sum_{j=0}^{i-1}\left(\mathscr{J}_{t_j}^{p_j}|f(t_{j+1},x(t_{j+1}))|\right.\right. \\
&\quad \left.\left. + \ |\varphi_{j+1}(x(t_{j+1}))|\right)\right)\Bigg]\Bigg\} \\
&\le (a+c)\Lambda_1 + (b+d)\Lambda_2 = \lambda - 1,
\end{aligned}$$

which contradicts $\|\mathscr{K}\| \geq \lambda$. Hence the assumptions of Theorem 1.4 hold. Therefore the problem (8.1) has at least one solution on J.

Example 8.4 Consider the following boundary value problem for impulsive multi-order Hadamard fractional differential equations:

$$\begin{cases} {}^C\mathscr{D}_{t_k}^{\sqrt{1-\sin^2(k+1)}}x(t) = e^{2t/3}\sin x(t) + tx(t)\cos x(t) + 2, \quad t \in \left[\frac{3}{2}, 3\right] \setminus \{t_k\}, \\ \Delta x(t_k) = kx(t_k) - \log\left(|x(t_k)| + \frac{3}{5}\right), \qquad t_k = 3\cdot 2^{(k-11)/11},\ k = 1, 2, \dots, 10, \\ \frac{4}{3}x\left(\frac{3}{2}\right) - \frac{3}{4}x(3) = \sum_{i=0}^{10}\frac{(-1)^i}{i+1}\mathscr{I}_{t_i}^{\left(\frac{3k+2}{2k+3}\right)}x(t_{i+1}). \end{cases} \tag{8.7}$$

Here $\alpha = 4/3$, $\beta = -3/4$, $m = 10$, $p_k = \sqrt{1-\sin^2(k+1)}$, $\gamma_k = (-1)^k/(k+1)$, $q_k = (3k+2)/(2k+3)$, for $k = 0, 1, \dots, 10$. Using the given data, we find that $\Phi \approx 0.605503 \neq 0$. The functions $f(t, x) = e^{2t/3}\sin x + tx\cos x + 2$ and $\varphi(x) = kx - \log(|x| + (3/5))$ satisfy the inequalities

$$|f(t, x)| \leq t|x| + (2 + e^{2t/3}) \leq 3|x| + (2 + e^2),$$

and

$$|\varphi_k(x)| \leq |x|(k+1) + \frac{3}{5} \leq 11|x| + \frac{3}{5}.$$

Therefore (8.4.1) holds. According to Theorem 8.4, the problem (8.7) has at least one solution on $[3/2, 3]$.

8.3 On Caputo-Hadamard Type Fractional Impulsive Boundary Value Problems with Nonlinear Fractional Integral Conditions

In this section, we investigate a nonlinear boundary value problem of impulsive hybrid multi-orders Caputo-Hadamard fractional differential equation with nonlinear integral boundary conditions given by

$$\begin{cases} {}^C\mathscr{D}_{t_k}^{\alpha_k}x(t) = f(t, x(t)), \quad t \in [t_0, T] \setminus \{t_1, t_2, \dots, t_m\}, \\ \Delta x(t_k) = \varphi_k(x(t_k)), \quad k = 1, 2, \dots, m, \\ \Delta\delta x(t_k) = \varphi_k^*(x(t_k)), \quad k = 1, 2, \dots, m, \\ x(t_0) = \mathscr{I}_{t_0}^{\mu}g(\xi, x(\xi)), \qquad x(T) = \mathscr{I}_{t_m}^{\nu}h(\eta, x(\eta)), \end{cases} \tag{8.8}$$

where ${}^{C}\mathscr{D}_{t_k}^{\alpha_k}$ is the Caputo-Hadamard fractional derivative of order $1 < \alpha_k \leq 2$ on intervals $J_k := (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$ with $J_0 = [t_0, t_1]$, $0 < t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$ are the impulse points, $J := [t_0, T]$, $f, g, h : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $\varphi_k, \varphi_k^* \in C(\mathbb{R}, \mathbb{R})$, $\mathscr{J}_{\phi}^{\varphi}$ is the Hadamard fractional integral of order $\varphi > 0$, $\varphi \in \{\mu, \nu\}$, $\phi \in \{t_0, t_m\}$, $\xi \in J_0$ and $\eta \in J_m$. The jump conditions are defined by $\Delta x(t_k) = x(t_k^+) - x(t_k)$, $\Delta \delta x(t_k) = \delta x(t_k^+) - \delta x(t_k)$, $x(t_k^+) = \lim_{\varepsilon \to 0^+} x(t_k + \varepsilon)$, $k = 1, 2, 3, \ldots, m$, and $\delta := t(d/dt)$ is the delta derivative.

Let $J^- = J \setminus \{t_1, t_2, \ldots, t_m\}$ and $PC(J, \mathbb{R}) = \{x : J \to \mathbb{R} :\ x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$, $k = 1, 2, \ldots, m\}$, and $PC^1(J, \mathbb{R}) = \{x \in PC(J, \mathbb{R}) :\ x'(t)$ is continuous everywhere except for some t_k at which $x'(t_k^+)$ and $x'(t_k^-)$ exist and $x'(t_k^-) = x'(t_k)$, $k = 1, 2, \ldots, m\}$. $PC(J, \mathbb{R})$ and $PC^1(J, \mathbb{R})$ are Banach spaces with the norms $\|x\|_{PC} = \sup\{|x(t)|;\ t \in J\}$ and $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$ respectively. Let $E = PC^1(J, \mathbb{R}) \cap C^2(J^-, \mathbb{R})$. A function $x \in E$ is called a solution of problem (8.8) if it satisfies (8.8).

Lemma 8.2 *The solution $x \in E$ of the problem (8.8) is equivalent to the integral equation:*

$$
\begin{aligned}
x(t) = {} & \mathscr{J}_{t_k}^{\alpha_k} f(t, x(t)) + \sum_{j=1}^{k} \left(\log \frac{t}{t_j}\right) \left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1} f(t_j, x(t_j)) + \varphi_j^*(x(t_j))\right) \\
& + \sum_{j=1}^{k} \left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}} f(t_j, x(t_j)) + \varphi_j(x(t_j))\right) + \frac{\log(t/t_0)}{\log(T/t_0)} \Bigg[\mathscr{J}_{t_m}^{\nu} h(\eta, x(\eta)) \\
& - \mathscr{J}_{t_0}^{\mu} g(\xi, x(\xi)) - \mathscr{J}_{t_m}^{\alpha_m} f(T, x(T)) \\
& - \sum_{j=1}^{m} \left(\log \frac{T}{t_j}\right) \left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1} f(t_j, x(t_j)) + \varphi_j^*(x(t_j))\right) \\
& - \sum_{j=1}^{m} \left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}} f(t_j, x(t_j)) + \varphi_j(x(t_j))\right) \Bigg] + \mathscr{J}_{t_0}^{\mu} g(\xi, x(\xi)).
\end{aligned}
\tag{8.9}
$$

Proof Applying the Hadamard fractional integral of order α_0 on the fractional differential equation in (8.8) and using Lemma 1.2, we get

$$
x(t) = \mathscr{J}_{t_0}^{\alpha_0} f(t, x(t)) + c_1 \log\left(\frac{t}{t_0}\right) + c_2,\ t \in J_0, \tag{8.10}
$$

where $c_1 = \delta x(t_0)$ and $c_2 = x(t_0)$. From (8.10), it follows that

$$
x(t_1) = \mathscr{J}_{t_0}^{\alpha_0} f(t_1, x(t_1)) + c_1 \log\left(\frac{t_1}{t_0}\right) + c_2,
$$

and

$$\delta x(t_1) = \mathscr{J}_{t_0}^{\alpha_0 - 1} f(t_1, x(t_1)) + c_1.$$

Now, applying the Hadamard fractional integral of order α_1 on the Hadamard equation in (8.8) for $t \in J_1$ and using the jump conditions at the point t_1, we get

$$\begin{aligned} x(t) &= \mathscr{J}_{t_1}^{\alpha_1} f(t, x(t)) + \delta x(t_1^+) \log\left(\frac{t}{t_1}\right) + x(t_1^+) \\ &= \mathscr{J}_{t_1}^{\alpha_1} f(t, x(t)) \\ &\quad + \left[\mathscr{J}_{t_0}^{\alpha_0 - 1} f(t_1, x(t_1)) + \varphi_1^*(x(t_1)) + c_1\right] \log\left(\frac{t}{t_1}\right) \\ &\quad + \left[\mathscr{J}_{t_0}^{\alpha_0} f(t_1, x(t_1)) + \varphi_1(x(t_1)) + c_1 \log\left(\frac{t_1}{t_0}\right) + c_2\right], \end{aligned} \tag{8.11}$$

which, on taking delta-derivative, yields

$$\delta x(t) = \mathscr{J}_{t_1}^{\alpha_1 - 1} f(t, x(t)) + \mathscr{J}_{t_0}^{\alpha_0 - 1} f(t_1, x(t_1)) + \varphi_1^*(x(t_1)) + c_1.$$

Next, for $t \in J_2$, by direct computation and using the property $\log a + \log b = \log ab$ for $a, b > 0$, we obtain

$$\begin{aligned} x(t) &= \mathscr{J}_{t_2}^{\alpha_2} f(t, x(t)) + \delta x(t_2^+) \log\left(\frac{t}{t_2}\right) + x(t_2^+) \\ &= \mathscr{J}_{t_2}^{\alpha_2} f(t, x(t)) + \log\left(\frac{t}{t_2}\right) \\ &\quad \times \left[\mathscr{J}_{t_0}^{\alpha_0 - 1} f(t_1, x(t_1)) + \varphi_1^*(x(t_1)) + \mathscr{J}_{t_1}^{\alpha_1 - 1} f(t_2, x(t_2)) + \varphi_2^*(x(t_2)) + c_1\right] \\ &\quad + \mathscr{J}_{t_1}^{\alpha_1} f(t_2, x(t_2)) + \varphi_2(x(t_2)) + \mathscr{J}_{t_0}^{\alpha_0} f(t_1, x(t_1)) + \varphi_1(x(t_1)) \\ &\quad + \log\left(\frac{t_2}{t_1}\right) \left[\mathscr{J}_{t_0}^{\alpha_0 - 1} f(t_1, x(t_1)) + \varphi_1^*(x(t_1))\right] + c_1 \log\left(\frac{t_2}{t_0}\right) + c_2 \\ &= \mathscr{J}_{t_2}^{\alpha_2} f(t, x(t)) + \left(\mathscr{J}_{t_1}^{\alpha_1 - 1} f(t_2, x(t_2)) + \varphi_2^*(x(t_2))\right) \log\left(\frac{t}{t_2}\right) \\ &\quad + \left(\mathscr{J}_{t_0}^{\alpha_0 - 1} f(t_1, x(t_1)) + \varphi_1^*(x(t_1))\right) \log\left(\frac{t}{t_1}\right) + c_1 \log\left(\frac{t}{t_0}\right) \\ &\quad + \mathscr{J}_{t_1}^{\alpha_1} f(t_2, x(t_2)) + \varphi_2(x(t_2)) + \mathscr{J}_{t_0}^{\alpha_0} f(t_1, x(t_1)) + \varphi_1(x(t_1)) + c_2. \end{aligned}$$

Repeating the above process, for each $t \in J_k$, we get

$$
\begin{aligned}
x(t) = {} & \mathscr{I}_{t_k}^{\alpha_k} f(t, x(t)) \\
& + \sum_{j=1}^{k} \left(\log \left(\frac{t}{t_j} \right) \right) \left(\mathscr{I}_{t_{j-1}}^{\alpha_{j-1}-1} f(t_j, x(t_j)) + \varphi_j^*(x(t_j)) \right) \\
& + \sum_{j=1}^{k} \left(\mathscr{I}_{t_{j-1}}^{\alpha_{j-1}} f(t_j, x(t_j)) + \varphi_j(x(t_j)) \right) + c_1 \log \left(\frac{t}{t_0} \right) + c_2.
\end{aligned} \tag{8.12}
$$

Using the given integral boundary conditions in (8.12), we find that

$$
\begin{aligned}
c_1 = {} & \frac{1}{\log(T/t_0)} \Bigg[\mathscr{I}_{t_m}^{\nu} h(\eta, x(\eta)) - \mathscr{I}_{t_0}^{\mu} g(\xi, x(\xi)) - \mathscr{I}_{t_m}^{\alpha_m} f(T, x(T)) \\
& - \sum_{j=1}^{m} \left(\log \left(\frac{T}{t_j} \right) \right) \left(\mathscr{I}_{t_{j-1}}^{\alpha_{j-1}-1} f(t_j, x(t_j)) + \varphi_j^*(x(t_j)) \right) \\
& - \sum_{j=1}^{m} \left(\mathscr{I}_{t_{j-1}}^{\alpha_{j-1}} f(t_j, x(t_j)) + \varphi_j(x(t_j)) \right) \Bigg], \\
c_2 = {} & \mathscr{I}_{t_0}^{\mu} g(\xi, x(\xi)).
\end{aligned}
$$

Substituting the above values of c_1 and c_2 in (8.12), we obtain the solution (8.9). Conversely, it can easily be shown by direct computation that the integral equation (8.9) satisfies the problem (8.8). This completes the proof. □

In view of Lemma 8.2, we define an operator $\mathscr{K} : E \to E$ by

$$
\begin{aligned}
\mathscr{K} x(t) = {} & \mathscr{I}_{t_k}^{\alpha_k} f(t, x(t)) + \sum_{j=1}^{k} \left(\log \frac{t}{t_j} \right) \left(\mathscr{I}_{t_{j-1}}^{\alpha_{j-1}-1} f(t_j, x(t_j)) + \varphi_j^*(x(t_j)) \right) \\
& + \sum_{j=1}^{k} \left(\mathscr{I}_{t_{j-1}}^{\alpha_{j-1}} f(t_j, x(t_j)) + \varphi_j(x(t_j)) \right) \\
& + \frac{\log(t/t_0)}{\log(T/t_0)} \Bigg[\mathscr{I}_{t_m}^{\nu} h(\eta, x(\eta)) \\
& - \mathscr{I}_{t_m}^{\alpha_m} f(T, x(T)) - \sum_{j=1}^{m} \left(\log \frac{T}{t_j} \right) \left(\mathscr{I}_{t_{j-1}}^{\alpha_{j-1}-1} f(t_j, x(t_j)) + \varphi_j^*(x(t_j)) \right) \\
& - \sum_{j=1}^{m} \left(\mathscr{I}_{t_{j-1}}^{\alpha_{j-1}} f(t_j, x(t_j)) + \varphi_j(x(t_j)) \right) \Bigg] + \frac{\log(T/t)}{\log(T/t_0)} \mathscr{I}_{t_0}^{\mu} g(\xi, x(\xi)).
\end{aligned} \tag{8.13}
$$

8.3.1 Existence Result via Krasnoselskii-Zabreiko's Fixed Point Theorem

In this section, we present our first existence result for the problem (8.8) which relies on Krasnoselskii-Zabreiko's fixed point theorem (Theorem 1.10).

Theorem 8.5 *Assume that:*

(8.5.1) the function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous such that $f(t,0) \neq 0$ for some $t \in J$ and that

$$\lim_{\|x\|\to\infty} \frac{f(t,x)}{x} = \lambda(t);$$

(8.5.2) the functions $g, h : J \times \mathbb{R} \to \mathbb{R}$ and $\varphi_j, \varphi_j^* : \mathbb{R} \to \mathbb{R}$, $j = 1, 2, \ldots, m$, are continuous and there exist positive constants A, B, C and D such that

$$|g(t,x(t))| \leq A|x(t)|, \quad |h(t,x(t))| \leq B|x(t)|,$$
$$|\varphi_j^*(x(t))| \leq C|x(t)|, \quad |\varphi_j(x(t))| \leq D|x(t)|, \forall j = 1, 2, \ldots, m.$$

Then, the problem (8.8) has at least one solution on J if

$$\lambda_{\max} := \max_{t\in J} |\lambda(t)| < \frac{1-\Lambda_2}{\Lambda_1}, \tag{8.14}$$

where

$$\Lambda_1 = \sup_{j=1,2,\ldots,m} \left\{ \frac{(\log(T/t_j))^{\alpha_j}}{\Gamma(\alpha_j+1)} \right\} + \frac{(\log(T/t_m))^{\alpha_m}}{\Gamma(\alpha_m+1)}$$
$$+2\sum_{j=1}^{m} \left(\frac{(\log(t_j/t_{j-1}))^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1}+1)} \cdot \log \frac{(T/t_j)^{\alpha_{j-1}}}{t_{j-1}/t_j} \right),$$

$$\Lambda_2 = A\frac{(\log(\eta/t_m))^{\nu}}{\Gamma(\nu+1)} + B\frac{(\log(\xi/t_0))^{\mu}}{\Gamma(\mu+1)} + 2C\log\left(\frac{T^m}{\prod_{j=1}^m t_j}\right) + 2mD.$$

Proof Let $\{x_n\}$ be a sequence converging to x. For $t \in J$, we have

$$|\mathscr{K}x_n(t) - \mathscr{K}x(t)|$$
$$\leq \mathscr{J}_{t_k}^{\alpha_k}|f(t,x_n(t)) - f(t,x(t))|$$
$$+\sum_{j=1}^{k}\left(\log\frac{t}{t_j}\right)\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1}|f(t_j,(x_n(t_j)) - f(t_j,x(t_j))| + |\varphi_j^*(x_n(t_j)) - \varphi_j^*(x(t_j))|\right)$$

$$+\sum_{j=1}^{k}\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}}|f(t_j,x_n(t_j))-f(t_j,x(t_j))|+|\varphi_j(x_n(t_j))-\varphi_j(x(t_j))|\right)$$
$$+\frac{\log(t/t_0)}{\log(T/t_0)}\Bigg[\mathscr{J}_{t_m}^{\nu}|h(\eta,x_n(\eta))-h(\eta,x(\eta))|+\mathscr{J}_{t_m}^{\alpha_m}|f(T,x_n(T))-f(T,x(T))|$$
$$+\sum_{j=1}^{m}\left(\log\frac{T}{t_j}\right)\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1}|f(t_j,x_n(t_j))-f(t_j,x(t_j))|+|\varphi_j^*(x_n(t_j))-\varphi_j^*(x(t_j))|\right)$$
$$+\sum_{j=1}^{m}\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}}|f(t_j,x_n(t_j))-f(t_j,x(t_j))|+|\varphi_j(x_n(t_j))-\varphi_j(x(t_j))|\right)\Bigg]$$
$$+\frac{\log(T/t)}{\log(T/t_0)}\mathscr{J}_{t_0}^{\mu}|g(\xi,x_n(\xi))-g(\xi,x(\xi))|.$$

As $n\to\infty$, by continuity of functions in hypotheses (8.5.1) and (8.5.2), the right hand side of the above inequality converges to zero. Hence, we deduce that the operator $\mathscr{K}$ is continuous.

Given $r>0$, we define $N=\{x\in E:\ \|x\|\le r\}$, $\|f\|:=\max_{\|x\|\le r}|f(t,x(t))|$. Then we have

$$\begin{aligned}
&|\mathscr{K}x(t)|\\
&\le \mathscr{J}_{t_k}^{\alpha_k}|f(t,x(t))|\\
&\quad+\sum_{j=1}^{k}\left(\log\frac{t}{t_j}\right)\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1}|f(t_j,x(t_j))|+|\varphi_j^*(x(t_j))|\right)\\
&\quad+\sum_{j=1}^{k}\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}}|f(t_j,x(t_j))|+|\varphi_j(x(t_j))|\right)\\
&\quad+\frac{\log(t/t_0)}{\log(T/t_0)}\Bigg[\mathscr{J}_{t_m}^{\nu}|h(\eta,x(\eta))|+\mathscr{J}_{t_m}^{\alpha_m}|f(T,x(T))|\\
&\quad+\sum_{j=1}^{m}\left(\log\frac{T}{t_j}\right)\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1}|f(t_j,x(t_j))|+|\varphi_j^*(x(t_j))|\right)\\
&\quad+\sum_{j=1}^{m}\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}}|f(t_j,x(t_j))|+|\varphi_j(x(t_j))|\right)\Bigg]+\frac{\log(T/t)}{\log(T/t_0)}\mathscr{J}_{t_0}^{\mu}|g(\xi,x(\xi))|\\
&\le\Bigg[\frac{(\log(T/t_k))^{\alpha_k}}{\Gamma(\alpha_k+1)}+\sum_{j=1}^{k}\left(\log\frac{T}{t_j}\right)\left(\frac{\left(\log(t_j/t_{j-1})\right)^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1})}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{k}\left(\frac{(\log(t_j/t_{j-1}))^{\alpha_{j-1}}}{\Gamma(\alpha_{j-1}+1)}\right)+\frac{(\log(T/t_m))^{\alpha_m}}{\Gamma(\alpha_m+1)}\\
&+\sum_{j=1}^{m}\left(\log\frac{T}{t_j}\right)\left(\frac{(\log(t_j/t_{j-1}))^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1})}\right)+\sum_{j=1}^{m}\left(\frac{(\log(t_j/t_{j-1}))^{\alpha_{j-1}}}{\Gamma(\alpha_{j-1}+1)}\right)\Bigg]\|f\|\\
&+\frac{(\log(\eta/t_m))^{\nu}}{\Gamma(\nu+1)}\|h\|+\frac{\log(\xi/t_0)^{\mu}}{\Gamma(\mu+1)}\|g\|+2\sum_{j=1}^{k}\left(\log\frac{T}{t_j}\right)\|\varphi_j^*\|+2\sum_{j=1}^{k}\|\varphi_j\|\\
\leq{}& \Lambda_1\|f\|+\Lambda_2 r,
\end{aligned}
$$

which yields $\|\mathscr{K}x\| \leq \Lambda_1\|f\| + \Lambda_2 r$. Therefore, $\mathscr{K}(N)$ is uniformly bounded. Next, we claim that $\mathscr{K}(N)$ is equicontinuous. Let $\tau_1, \tau_2 \in J$, with $\tau_2 < \tau_1$. Then, we obtain

$$
\begin{aligned}
&|\mathscr{K}x(\tau_1)-\mathscr{K}(\tau_2)|\\
\leq{}& \mathscr{J}_{t_k}^{\alpha_k}|f(\tau_1,x(\tau_1))-f(\tau_2,x(\tau_2))|\\
&+\sum_{j=1}^{k}\left|\log\frac{\tau_1}{t_j}-\log\frac{\tau_2}{t_j}\right|\left|\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1}f(t_j,(x(t_j))+\varphi_j^*(x(t_j))\right|\\
&+\left|\frac{\log(\tau_1/t_0)}{\log(T/t_0)}-\frac{\log(\tau_2/t_0)}{\log(T/t_0)}\right|\Bigg[\left|\mathscr{J}_{t_m}^{\nu}h(\eta,x(\eta))+\mathscr{J}_{t_m}^{\alpha_m}f(T,x(T))\right.\\
&+\sum_{j=1}^{m}\left(\log\frac{T}{t_j}\right)\left|\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1}f(t_j,x(t_j))+\varphi_j^*(x(t_j))\right|\\
&\left.+\sum_{j=1}^{m}\left|\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}}f(t_j,x(t_j))+\varphi_j(x(t_j))\right|\right|\Bigg]\\
&+\left|\frac{\log(T/\tau_1)}{\log(T/t_0)}-\frac{\log(T/\tau_2)}{\log(T/t_0)}\right|\left|\mathscr{J}_{t_0}^{\mu}g(\xi,x(\xi))\right|.
\end{aligned}
$$

It is clear that $|\mathscr{K}x(\tau_1) - \mathscr{K}x(\tau_2)| \to 0$ as $\tau_1 \to \tau_2$. Consequently $\mathscr{K}(N)$ is relatively compact in E.

Next, we consider the problem (8.8) as a linear problem by setting $f(t, x(t)) = \lambda(t)x(t)$. Using Lemma 8.2, we define the operator $\mathscr{L}$ by

$$
\mathscr{L}x(t) = \mathscr{J}_{t_k}^{\alpha_k}\lambda(t)x(t)+\sum_{j=1}^{k}\left(\log\frac{t}{t_j}\right)\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1}\lambda(t_j)x(t_j)+\varphi_j^*(x(t_j))\right)
$$

$$+\sum_{j=1}^{k}\left(\mathscr{I}_{t_{j-1}}^{\alpha_{j-1}}\lambda(t_j)x(t_j)+\varphi_j(x(t_j))\right)+\frac{\log(t/t_0)}{\log(T/t_0)}\Bigg[\mathscr{I}_{t_m}^{\nu}h(\eta,x(\eta))$$

$$-\mathscr{I}_{t_m}^{\alpha_m}\lambda(T)x(T)-\sum_{j=1}^{m}\left(\log\frac{T}{t_j}\right)\left(\mathscr{I}_{t_{j-1}}^{\alpha_{j-1}-1}\lambda(t_j)x(t_j)+\varphi_j^*(x(t_j))\right)$$

$$-\sum_{j=1}^{m}\left(\mathscr{I}_{t_{j-1}}^{\alpha_{j-1}}\lambda(t_j)x(t_j)+\varphi_j(x(t_j))\right)\Bigg]+\frac{\log(T/t)}{\log(T/t_0)}\mathscr{I}_{t_0}^{\mu}g(\xi,x(\xi)).$$

We claim that 1 is not an eigenvalue of the operator $\mathscr{L}$. If 1 is an eigenvalue of $\mathscr{L}$, then we get

$$\begin{aligned}
\|x\| &= \sup_{t\in J}|\mathscr{L}x(t)| \\
&\le \sup_{t\in J}\Bigg\{\mathscr{I}_{t_k}^{\alpha_k}|\lambda(t)x(t)|+\sum_{j=1}^{k}\left(\log\frac{t}{t_j}\right)\left(\mathscr{I}_{t_{j-1}}^{\alpha_{j-1}-1}|\lambda(t_j)x(t_j)|+|\varphi_j^*(x(t_j))|\right) \\
&\quad+\sum_{j=1}^{k}\left(\mathscr{I}_{t_{j-1}}^{\alpha_{j-1}}|\lambda(t_j)x(t_j)|+|\varphi_j(x(t_j))|\right)+\frac{\log(t/t_0)}{\log(T/t_0)}\Bigg[\mathscr{I}_{t_m}^{\nu}|h(\eta,x(\eta))| \\
&\quad+\mathscr{I}_{t_m}^{\alpha_m}|\lambda(T)x(T)|-\sum_{j=1}^{m}\left(\log\frac{T}{t_j}\right)\left(\mathscr{I}_{t_{j-1}}^{\alpha_{j-1}-1}|\lambda(t_j)x(t_j)|+|\varphi_j^*(x(t_j))|\right) \\
&\quad+\sum_{j=1}^{m}\left(\mathscr{I}_{t_{j-1}}^{\alpha_{j-1}}|\lambda(t_j)x(t_j)|+|\varphi_j(x(t_j))|\right)\Bigg]+\frac{\log(T/t)}{\log(T/t_0)}\mathscr{I}_{t_0}^{\mu}|g(\xi,x(\xi))|\Bigg\} \\
&\le \Bigg[\frac{(\log(T/t_k))^{\alpha_k}}{\Gamma(\alpha_k+1)}+\sum_{j=1}^{m}\left(\log\frac{T}{t_j}\right)\left(\frac{\left(\log(t_j/t_{j-1})\right)^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1})}\right) \\
&\quad+\sum_{j=1}^{m}\left(\frac{(\log(t_j/t_{j-1}))^{\alpha_{j-1}}}{\Gamma(\alpha_{j-1}+1)}\right)+\frac{(\log(T/t_m))^{\alpha_m}}{\Gamma(\alpha_m+1)} \\
&\quad+\sum_{j=1}^{m}\left(\log\frac{T}{t_j}\right)\left(\frac{(\log(t_j/t_{j-1}))^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1})}\right) \\
&\quad+\sum_{j=1}^{m}\left(\frac{(\log(t_j/t_{j-1}))^{\alpha_{j-1}}}{\Gamma(\alpha_{j-1}+1)}\right)\Bigg]\lambda_{\max}\|x\|+\frac{(\log(\eta/t_m))^{\nu}}{\Gamma(\nu+1)}A\|x\| \\
&\quad+\frac{\log(\xi/t_0)^{\mu}}{\Gamma(\mu+1)}B\|x\|+2\sum_{j=1}^{k}\left(\log\frac{T}{t_j}\right)C\|x\|+2\sum_{j=1}^{k}D\|x\| \\
&= (\lambda_{\max}\Lambda_1+\Lambda_2)\|x\| < \|x\|,
\end{aligned}$$

which is a contradiction. Therefore, 1 is not an eigenvalue of the operator $\mathscr{L}$. Next, we will show that $\|\mathscr{K}x - \mathscr{L}x\|/\|x\|$ vanish as $\|x\| \to \infty$. For $t \in J$, we have

$$
\begin{aligned}
|\mathscr{K}x(t) - \mathscr{L}x(t)| \leq\; & \mathscr{J}_{t_k}^{\alpha_k}|f(t,x(t)) - \lambda(t)x(t)| \\
&+ \sum_{j=1}^{k}\left(\log\frac{t}{t_j}\right)\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1}|f(t_j,(x(t_j)) - \lambda(t_j)x(t_j)|\right) \\
&+ \sum_{j=1}^{k}\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}}|f(t_j,(x(t_j)) - \lambda(t_j)x(t_j)|\right) \\
&+ \frac{\log(t/t_0)}{\log(T/t_0)}\Bigg[\mathscr{J}_{t_m}^{\alpha_m}|f(T,(x(T)) - \lambda(T)x(T)| \\
&+ \sum_{j=1}^{m}\left(\log\frac{T}{t_j}\right)\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1}|f(t_j,(x(t_j)) - \lambda(t_j)x(t_j)|\right) \\
&+ \sum_{j=1}^{m}\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}}|f(t_j,(x(t_j)) - \lambda(t_j)x(t_j)|\right)\Bigg] \\
\leq\; & \mathscr{J}_{t_k}^{\alpha_k}\left(\left|\frac{f(t,x(t))}{x(t)} - \lambda(t)\right||x(t)|\right) \\
&+ \sum_{j=1}^{k}\left(\log\frac{t}{t_j}\right)\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1}\left(\left|\frac{f(t_j,(x(t_j))}{x(t_j)} - \lambda(t_j)\right||x(t_j)|\right)\right) \\
&+ \sum_{j=1}^{k}\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}}\left(\left|\frac{f(t_j,(x(t_j))}{x(t_j)} - \lambda(t_j)\right||x(t_j)|\right)\right) \\
&+ \frac{\log(t/t_0)}{\log(T/t_0)}\Bigg[\mathscr{J}_{t_m}^{\alpha_m}\left(\left|\frac{f(T,(x(T))}{x(T)} - \lambda(T)\right||x(T)|\right) \\
&+ \sum_{j=1}^{m}\left(\log\frac{T}{t_j}\right)\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1}\left(\left|\frac{f(t_j,(x(t_j))}{x(t_j)} - \lambda(t_j)\right||x(t_j)|\right)\right) \\
&+ \sum_{j=1}^{m}\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}}\left(\left|\frac{f(t_j,(x(t_j))}{x(t_j)} - \lambda(t_j)\right||x(t_j)|\right)\right)\Bigg].
\end{aligned}
$$

This means that

$$
\frac{\|\mathscr{K}x - \mathscr{L}x\|}{\|x\|} \leq \mathscr{J}_{t_k}^{\alpha_k}\left|\frac{f(t,x(t))}{x(t)} - \lambda(t)\right|
$$

$$+\sum_{j=1}^{k}\left(\log\frac{t}{t_j}\right)\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1}\left|\frac{f(t_j,(x(t_j))}{x(t_j)}-\lambda(t_j)\right|\right)$$
$$+\sum_{j=1}^{k}\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}}\left|\frac{f(t_j,(x(t_j))}{x(t_j)}-\lambda(t_j)\right|\right)$$
$$+\frac{\log(t/t_0)}{\log(T/t_0)}\left[\mathscr{J}_{t_m}^{\alpha_m}\left|\frac{f(T,(x(T))}{x(T)}-\lambda(T)\right|\right.$$
$$+\sum_{j=1}^{m}\left(\log\frac{T}{t_j}\right)\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1}\left|\frac{f(t_j,(x(t_j))}{x(t_j)}-\lambda(t_j)\right|\right)$$
$$\left.+\sum_{j=1}^{m}\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}}\left|\frac{f(t_j,(x(t_j))}{x(t_j)}-\lambda(t_j)\right|\right)\right].$$

Letting $\|x\|\to\infty$ implies $\left|\frac{f(\cdot,x)}{x}-\lambda\right|\to 0$. Thus, we obtain

$$\lim_{\|x\|\to\infty}\frac{\|\mathscr{K}x-\mathscr{L}x\|}{\|x\|}=0.$$

Consequently, by Theorem 1.10, the problem (8.8) has at least one nontrivial solution on J. □

Example 8.5 Consider the following impulsive Caputo-Hadamard fractional differential equations

$$\begin{cases} {}^C\mathscr{D}^{\alpha_k}x(t) & =\dfrac{tx(t)}{7(1+t)^3}\left(\dfrac{|x(t)|+2}{|x(t)|+1}\right)^{|x(t)|}+\dfrac{1}{2}, \quad t\in[1,e]\setminus\{t_k\}, \\ \Delta x(t_k) & =4x^2(t_k)\log\left(\dfrac{\sin t_k}{9\sqrt{|x(t_k)|+1}}\right), \\ \Delta\delta x(t_k) & =\dfrac{k|x(t_k)|}{5\sqrt{2}(k+1)}\left(\dfrac{\cos(x(t_k))}{5(|x(t_k)|+1)}-\dfrac{\sin(x(t_k))}{2\sqrt{5}(|x(t_k)|+1)}\right), \end{cases} \tag{8.15}$$

subject to nonlinear fractional integral conditions

$$x(1)=\mathscr{J}_1^{7/3}g\left(1+e^{-2},x\left(1+e^{-2}\right)\right) \quad \text{with } g(t,x)=\frac{5t^2}{6}\left(\frac{x^2-1}{2|x|+1}\right),$$

and

$$x(e)=\mathscr{J}_{1+\frac{9(e-1)}{10}}^{5/2}h\left(\frac{7(e+1)}{10},x\left(\frac{7(e+1)}{10}\right)\right) \quad \text{with } h(t,x)=\frac{11\sin(2x+e^{-2t})-3}{t^2+2t+4},$$

where $\alpha_k = 3\sqrt{(19-k)(21+k)}/70 + 4\sqrt{(25-k)(27+k)}/91$, $t_k = 1 + ((e-1)k/10)$ for $k = 1, 2, \ldots, 9$.

Here $m = 9$, $t_0 = 1$, $T = e$, $\mu = 7/3$, $\xi = 1 + e^{-2}$, $\nu = 5/2$, $\eta = 7(e+1)/10$, the functions $f(t, x(t))$ is defined by

$$f(t,x) = \frac{tx}{7(1+t)^3}\left(\frac{|x|+2}{|x|+1}\right)^{|x|} + \frac{1}{2}, \quad t \in [1,e] \setminus \{t_k\},$$

and the impulse functions $\varphi_k(x)$, $\varphi_k^*(x)$ at impulse moments t_k for $k = 1, 2, \ldots, 9$, are given by

$$\varphi_k(x) = 4x^2 \log\left(\frac{\sin(1+((e-1)k/10))}{9\sqrt{|x|+1}}\right),$$

$$\varphi_k^*(x) = \frac{k|x|}{5\sqrt{2}(k+1)}\left(\frac{\cos x}{5(|x|+1)} - \frac{\sin x}{2\sqrt{5}(|x|+1)}\right).$$

It is clear that the function $f(t,x)$ is continuous and $f(t,0) = 1/2$. Dividing $f(t,x)$ by x, we have

$$\frac{f(t,x)}{x} = \frac{t}{7(1+t)^3}\left(1 + \frac{1}{|x|+1}\right)^{|x|} + \frac{1}{2x}.$$

Hence

$$\lim_{\|x\|\to\infty} \frac{f(t,x)}{x} = \frac{et}{7(1+t)^3}.$$

Setting $\lambda(t) = \dfrac{et}{7(1+t)^3}$, we get $\lambda_{\max} = 0.057530$. Furthermore, we have

$$|g(t,x)| \le \frac{15}{4}|x|,\ |h(t,x)| \le \frac{22}{7}|x|,\ |\varphi_j^*(x)| \le \frac{3}{50}|x|,\ |\varphi_j(x)| \le \frac{2}{81}|x|,\ j=1,2,\ldots,9.$$

Letting $A = 15/4$, $B = 22/7$, $C = 3/50$ and $D = 2/81$, we obtain $\Lambda_1 = 1.584506$ and $\Lambda_2 = 0.896396$. Since $(1-\Lambda_2)/\Lambda_1 = 0.065385 > \lambda_{\max}$, therefore, by theorem 8.5, the problem (8.15) has at least one solution on $[1, e]$.

8.3.2 Existence Result via Sadovskii's Fixed Point Theorem

Here, we establish an existence result for the problem (8.8) by using Sadovskii's fixed point theorem (Theorem 1.13).

To apply Theorem 1.13, we decompose the operator $\mathscr{K}$ (defined by (8.13)) into the sum of two operators as

$$\mathscr{K}x(t) = \mathscr{K}_1 x(t) + \mathscr{K}_2 x(t), \quad t \in J,$$

where

$$\begin{aligned}
\mathscr{K}_1 x(t) = &\ \mathscr{J}_{t_k}^{\alpha_k} f(t, x(t)) + \sum_{j=1}^{k} \left(\log \frac{t}{t_j}\right) \left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1} f(t_j, x(t_j)) + \varphi_j^*(x(t_j))\right) \\
&+ \sum_{j=1}^{k} \left(\mathscr{J}_{t_{j-1}}^{\alpha_j-1} f(t_j, x(t_j)) + \varphi_j(x(t_j))\right) - \frac{\log(t/t_0)}{\log(T/t_0)} \Big[\mathscr{J}_{t_m}^{\alpha_m} f(T, x(T)) \\
&+ \sum_{j=1}^{m} \left(\log \frac{T}{t_j}\right) \left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1} f(t_j, x(t_j)) + \varphi_j^*(x(t_j))\right) \\
&+ \sum_{j=1}^{m} \left(\mathscr{J}_{t_{j-1}}^{\alpha_j-1} f(t_j, x(t_j)) + \varphi_j(x(t_j))\right) \Big],
\end{aligned} \tag{8.16}$$

and

$$\mathscr{K}_2 x(t) = \frac{\log(t/t_0)}{\log(T/t_0)} \mathscr{J}_{t_m}^{\nu} h(\eta, x(\eta)) + \frac{\log(T/t)}{\log(T/t_0)} \mathscr{J}_{t_0}^{\mu} g(\xi, x(\xi)). \tag{8.17}$$

Theorem 8.6 *Let $f, g, h : J \times \mathbb{R} \to \mathbb{R}$ and $\varphi, \varphi^* : \mathbb{R} \to \mathbb{R}$ be continuous functions. Further, it is assumed that:*

(8.6.1) there exist positive constants L_1, L_2, L_3 such that

$$\begin{aligned}
|f(t,x) - f(t,y)| &\le L_1 |x - y|, && \forall (t,x), (t,y) \in J \times \mathbb{R}, \\
|\varphi_j(x) - \varphi_j(y)| &\le L_2 |x - y|, && \forall x, y \in \mathbb{R}, \quad \forall j = 1, 2, \ldots, m, \\
|\varphi_j^*(x) - \varphi_j^*(y)| &\le L_3 |x - y|, && \forall x, y \in \mathbb{R}, \quad \forall j = 1, 2, \ldots, m;
\end{aligned}$$

(8.6.2) there are functions $k_1, k_2 \in C(J, \mathbb{R})$, and nondecreasing functions $\psi_1, \psi_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|g(t,x)| \le k_1(t)\psi_1(\|x\|), \ |h(t,x)| \le k_2(t)\psi_2(\|x\|), \qquad \forall (t,x) \in J \times \mathbb{R}.$$

Then the problem (8.8) has at least one solution on J, provided that $\Omega_1 < 1$, where

$$\begin{aligned}
\Omega_1 = L_1 \Bigg[&\sup_{j=1,2,\ldots,m} \left\{ \frac{(\log(T/t_j))^{\alpha_j}}{\Gamma(\alpha_j + 1)} \right\} + \frac{(\log(T/t_m))^{\alpha_m}}{\Gamma(\alpha_m + 1)} \\
&+ 2\sum_{j=1}^{m} \left(\frac{(\log(t_j/t_{j-1}))^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1}+1)} \cdot \log \frac{(T/t_j)^{\alpha_{j-1}}}{t_{j-1}/t_j} \right) \Bigg] + 2L_2 \log\left(\frac{T^m}{\prod_{j=1}^m t_j}\right) + 2mL_3.
\end{aligned}$$

Proof Define the ball $B_\rho = \{x \in E;\ \|x\| \le \rho\}$ and set $M = \sup_{t\in J}|f(t,0)|$, $N = \max|\varphi_j^*(0)|$, $P = \max|\varphi_j(0)|$ for each $j = 1, 2, \ldots, m$,

$$\Omega_2 = M\Bigg[\sup_{j=1,2,\ldots,m}\left\{\frac{(\log(T/t_j))^{\alpha_j}}{\Gamma(\alpha_j+1)}\right\} + \frac{(\log(T/t_m))^{\alpha_m}}{\Gamma(\alpha_m+1)}$$
$$+2\sum_{j=1}^{m}\left(\frac{(\log(t_j/t_{j-1}))^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1}+1)}\cdot\log\frac{(T/t_j)^{\alpha_{j-1}}}{t_j/t_{j-1}}\right)\Bigg]$$
$$+2N\log\left(\frac{T^m}{\prod_{j=1}^{m}t_j}\right) + 2mP,$$

and

$$\Omega_3 = \frac{\|k_1\|\psi(\rho)}{\Gamma(\nu+1)}\left(\log\frac{\eta}{t_m}\right)^{\nu} + \frac{\|k_2\|\psi(\rho)}{\Gamma(\mu+1)}\left(\log\frac{\xi}{t_0}\right)^{\mu}.$$

Let $x \in B_\rho$ with $\rho > \dfrac{\Omega_2+\Omega_3}{1-\Omega_1}$. Then, we have

$$|\mathcal{K}_1x(t)|$$
$$\le \mathcal{J}_{t_k}^{\alpha_k}|f(t,x(t))| + \sum_{j=1}^{k}\left(\log\frac{t}{t_j}\right)\left(\mathcal{J}_{t_{j-1}}^{\alpha_{j-1}-1}|f(t_j,x(t_j))| + |\varphi_j^*(x(t_j))|\right)$$
$$+\sum_{j=1}^{k}\left(\mathcal{J}_{t_{j-1}}^{\alpha_{j-1}}|f(t_j,x(t_j))| + |\varphi_j(x(t_j))|\right) + \frac{\log(t/t_0)}{\log(T/t_0)}\Bigg[\mathcal{J}_{t_m}^{\alpha_m}|f(T,x(T))|$$
$$+\sum_{j=1}^{m}\left(\log\frac{T}{t_j}\right)\left(\mathcal{J}_{t_{j-1}}^{\alpha_{j-1}-1}|f(t_j,x(t_j))| + |\varphi_j^*(x(t_j))|\right)$$
$$+\sum_{j=1}^{m}\left(\mathcal{J}_{t_{j-1}}^{\alpha_{j-1}}|f(t_j,x(t_j))| + |\varphi_j(x(t_j))|\right)\Bigg]$$
$$\le \mathcal{J}_{t_k}^{\alpha_k}(|f(t,x(t))-f(t,0)| + |f(t,0)|)$$
$$+\sum_{j=1}^{k}\left(\log\frac{t}{t_j}\right)\Big(\mathcal{J}_{t_{j-1}}^{\alpha_{j-1}-1}(|f(t_j,x(t_j))-f(t_j,0)| + |f(t_j,0)|)$$
$$+|\varphi_j^*(x(t_j))-\varphi_j^*(0)| + |\varphi_j^*(0)|\Big)$$

$$
\begin{aligned}
&+\sum_{j=1}^{k}\Big(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}}(|f(t_j,x(t_j))-f(t_j,0)|+|f(t_j,0)|)\\
&+|\varphi_j(x(t_j))-\varphi_j(0)|+|\varphi_j(0)|\Big)\\
&+\frac{\log(t/t_0)}{\log(T/t_0)}\Big[\mathscr{J}_{t_m}^{\alpha_m}(|f(T,x(T))-f(T,0)|+|f(T,0)|)\\
&+\sum_{j=1}^{m}\left(\log\frac{T}{t_j}\right)\Big(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1}(|f(t_j,x(t_j))-f(t_j,0)|+|f(t_j,0)|)\\
&+|\varphi_j^*(x(t_j))-\varphi_j^*(0)|+|\varphi_j^*(0)|\Big)\\
&+\sum_{j=1}^{m}\Big(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}}(|f(t_j,x(t_j))-f(t_j,0)|+|f(t_j,0)|)\\
&+|\varphi_j(x(t_j))-\varphi_j(0)|+|\varphi_j(0)|\Big)\Big]\\
\le&\left[\frac{(\log(T/t_k))^{\alpha_k}}{\Gamma(\alpha_k+1)}+\frac{(\log(T/t_m))^{\alpha_m}}{\Gamma(\alpha_m+1)}\right.\\
&+2\sum_{j=1}^{m}\left(\log\frac{T}{t_j}\right)\left(\frac{\big(\log(t_j/t_{j-1})\big)^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1})}(L_1\rho+M)+(L_2\rho+N)\right)\\
&\left.+2\sum_{j=1}^{m}\left(\frac{(\log(t_j/t_{j-1}))^{\alpha_{j-1}}}{\Gamma(\alpha_{j-1}+1)}(L_1\rho+M)+(L_3\rho+P)\right)\right]\\
=&\ \Omega_1\rho+\Omega_2,
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathscr{K}_2x(t)|&\le\frac{\log(T/t_0)}{\log(T/t_0)}\mathscr{J}_{t_m}^{\nu}|h(\eta,x(\eta))|+\frac{\log(T/t)}{\log(T/t_0)}\mathscr{J}_{t_0}^{\mu}|g(\xi,x(\xi))|\\
&\le\frac{\|k_1\|\psi(\rho)}{\Gamma(\nu+1)}\left(\log\frac{\eta}{t_m}\right)^{\nu}+\frac{\|k_2\|\psi(\rho)}{\Gamma(\mu+1)}\left(\log\frac{\xi}{t_0}\right)^{\mu}=\Omega_3.
\end{aligned}
$$

From the preceding inequalities, it follows that

$$|\mathscr{K}x(t)|=|\mathscr{K}_1x(t)+\mathscr{K}_2x(t)|\le|\mathscr{K}_1x(t)|+|\mathscr{K}_2x(t)|\le\Omega_1\rho+\Omega_2+\Omega_3,$$

which leads to $\mathscr{K}(B_\rho)\subset B_\rho$.

Now, we will show that $\mathscr{K}_1$ is contractive and $\mathscr{K}_2$ is compact. Let $\{x_n\}$ be a sequence in B_ρ. For $t \in J$, we have

$$
\begin{aligned}
|\mathscr{K}_1 x_n(t) - \mathscr{K}_1 x(t)| \leq\ & \mathscr{J}_{t_k}^{\alpha_k}(|f(t,x_n(t)) - f(t,x(t))|) \\
& + \sum_{j=1}^{k}\left(\log\frac{t}{t_j}\right)\Big(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1}(|f(t_j,x_n(t_j)) - f(t_j,x(t_j))|) \\
& + |\varphi_j^*(x_n(t_j)) - \varphi_j^*(x(t_j))|\Big) \\
& + \sum_{j=1}^{k}\Big(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}}(|f(t_j,x_n(t_j)) - f(t_j,x(t_j))|) \\
& + |\varphi_j(x_n(t_j)) - \varphi_j(x(t_j))|\Big) \\
& + \frac{\log(t/t_0)}{\log(T/t_0)}\Bigg[\mathscr{J}_{t_m}^{\alpha_m}(|f(T,x_n(T)) - f(T,x(T))|) \\
& + \sum_{j=1}^{m}\left(\log\frac{T}{t_j}\right)\Big(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1}(|f(t_j,x_n(t_j)) - f(t_j,x(t_j))|) \\
& + |\varphi_j^*(x_n(t_j)) - \varphi_j^*(x(t_j))|\Big) \\
& + \sum_{j=1}^{m}\Big(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}}(|f(s,x_n(s)) - f(s,x(s))|)(t_j) \\
& + |\varphi_j(x_n(t_j)) - \varphi_j(x(t_j))|\Big)\Bigg] \\
\leq\ & \Omega_1\|x_n - x\|.
\end{aligned}
$$

As $x_n \to x$, we get $\|\mathscr{K}_1 x_n - \mathscr{K}_1 x\| \to 0$. Therefore, $\mathscr{K}_1$ is continuous. Moreover $\mathscr{K}_1$ is also contractive, since for $x, y \in B_\rho$, we get

$$\|\mathscr{K}_1 x - \mathscr{K}_1 y\| \leq \Omega_1\|x - y\|,$$

with $\Omega_1 < 1$.

Next, we claim that $\mathscr{K}_2$ is compact. By the hypothesis (8.6.2), we deduce that $\mathscr{K}_2$ is uniformly bounded. To show that $\mathscr{K}_2$ is equicontinuous, let $\tau_1, \tau_2 \in J$. For $x \in B_\rho$, we have

$$|\mathscr{K}_2 x(\tau_2) - \mathscr{K}_2 x(\tau_1)| = \left|\frac{\log(\tau_2/\tau_1)}{\log(T/t_0)}\mathscr{J}_{t_m}^{\nu} h(\eta, x(\eta)) + \frac{\log(\tau_1/\tau_2)}{\log(T/t_0)}\mathscr{J}_{t_0}^{\mu} g(\xi, x(\xi))\right|,$$

which tends to zero as $\tau_1 \to \tau_2$. Thus, by Arzelá-Ascoli Theorem, it follows that $\mathscr{K}_2$ is compact. Hence the operator $\mathscr{K}$ satisfies the hypotheses of Theorem 1.13. Therefore $\mathscr{K}$ is a condensing operator on B_ρ. Thus, by Theorem 1.13, the problem (8.8) has at least one solution on J. □

Example 8.6 Consider the impulsive Caputo-Hadamard fractional differential equations:

$$\begin{cases} {}^C\mathscr{D}^{\alpha_k}x(t) &= \dfrac{\cos t}{80-e^t}\left(\dfrac{x^2(t)+|x(t)|\log e^2 t}{|x(t)|+\log t}\right), \quad t\in\left[e^{1/4},e\right]\setminus\{t_k\}, \\ \Delta x(t_k) &= \dfrac{\cos^2(kx(t_k)/9)}{9k}, \quad k=1,2,\ldots,8, \\ \Delta\delta x(t_k) &= \dfrac{(-1)^k}{30}\dfrac{x^2(t_k)+10|x(t_k)|}{|x(t)|+9}, \quad k=1,2,\ldots,8, \end{cases} \tag{8.18}$$

supplemented with the nonlinear fractional integral conditions:

$$x(e^{1/4}) = \mathscr{J}^{9/5}_{e^{1/4}}g\left(e^{3/10},x\left(e^{3/10}\right)\right) \quad \text{with } g(t,x)=\frac{t|x|+e^{\cos t}}{t^4}-|x|t\sin^2 t,$$

and

$$x(e)=\mathscr{J}^{10/3}_{e^{11/12}}h\left(e^{14/15},x\left(e^{14/15}\right)\right) \quad \text{with } h(t,x)=\frac{e^t(|x|+1)}{2}+\frac{\cos t}{|x|+1}+\sqrt{2}\sin t,$$

where $\alpha_k = (10k+16)/(6k+9)$ and $t_k = e^{(3+k)/12}$, for $k = 1,2,\ldots,8$.

Here $m = 8$, $t_0 = e^{1/4}$, $T = e$, $\mu = 9/5$, $\xi = e^{3/10}$, $\nu = 10/3$, $\eta = e^{14/15}$,

$$f(t,x) = \frac{\cos t}{80-e^t}\left(\frac{x^2+|x|\log e^2 t}{|x|+\log t}\right),\ \varphi_k(x) = \frac{\cos^2(kx/9)}{9k},$$
$$\varphi_k^*(x) = \frac{(-1)^k}{30}\frac{x^2+10|x|}{|x|+9},$$

$k = 1,\ldots,8$. Clearly $|f(t,x)-f(t,y)| \le \frac{3}{20}|x-y|$, $|\varphi_j(x)-\varphi_j(y)| \le \frac{2}{81}|x-y|$, $|\varphi_j^*(x)-\varphi_j^*(y)| \le \frac{1}{27}|x-y|$, and function g, h satisfy the inequalities

$$|g(t,x)| \le e(\|x\|+1) \text{ and } |h(t,x)| \le \frac{\sqrt{e^e+4}}{2}\left(\|x\|+\frac{1}{\|x\|}\right).$$

Choosing $k_1 = e$, $k_2 = (\sqrt{e^e+4})/2$ and setting $\psi_1(x) = e(x+1)$ and $\psi_2(t) = x+(1/x)$, we have that ψ_1 and ψ_2 are nondecreasing positive functions on $\mathbb{R}^+$.

Therefore, the hypothesis (8.6.2) is satisfied. With $L_1 = 3/20$, $L_2 = 2/81$ and $L_3 = 1/27$, we find that $\Omega_1 \approx 0.982119 < 1$. Thus, all the conditions of Theorem 1.13 hold and hence the problem (8.18) has at least one solution on $[e^{1/4}, e]$.

8.3.3 Existence Result via O'Regan's Fixed Point Theorem

This section is devoted to the third existence result for the problem (8.8), which is based on a fixed point theorem due to O'Regan (Theorem 1.6).

For the sake of convenience, we introduce the notations:

$$\Psi_1 = \|b\|\Lambda_1,\ \Psi_2 = 2d_1 \log\left(\frac{T^m}{\prod_{j=1}^m t_j}\right) + 2md_2,\ \Psi_3 = \frac{(\log(\eta/t_m))^\nu}{\Gamma(\nu+1)} + \frac{(\log(\xi/t_0))^\mu}{\Gamma(\mu+1)}.$$

Theorem 8.7 *Let $f, g, h : J \times \mathbb{R} \to \mathbb{R}$ and $\varphi_k, \varphi_k^* : \mathbb{R} \to \mathbb{R}$ for $k = 1, 2, \ldots, m$ be continuous functions. Assume that:*

(8.7.1) there exists a nonnegative function $b \in C(J, [0, \infty))$ and a nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that

$$|f(t, x)| \le b(t)\psi(\|x\|), \quad \forall (t, x) \in J \times \mathbb{R};$$

(8.7.2) there exist positive constants d_1 and d_2 such that

$$|\varphi_j^*(x(t))| < d_1 \text{ and } |\varphi_j(x(t))| < d_2 \text{ for } j = 1, 2, \ldots, m;$$

(8.7.3) there exist positive constants c_1, c_2 and continuous functions $\phi_1, \phi_2 : [0, \infty) \to [0, \infty)$ such that

$$\phi_1(|x|) \le c_1|x| \text{ and } |g(t, x) - g(t, y)| \le \phi_1(\|x - y\|),$$
$$\phi_2(|x|) \le c_2|x| \text{ and } |h(t, x) - h(t, y)| \le \phi_2(\|x - y\|),$$

for all $t \in J$ and $x, y \in \mathbb{R}$;

(8.7.4) $\sup_{r\in(0,\infty)} \left\{\frac{r}{\Psi_1\psi(r) + \Psi_2 + l\Psi_3}\right\} > \frac{1}{1 - \kappa\Psi_3}$, where $\kappa = \max\{c_1, c_2\}$, $l = \sup_{t\in J}\{|g(t, 0)|, |h(t, 0)|\}$ and $\kappa\Psi_3 < 1$.

Then the boundary value problem (8.8) has at least one solution on J.

Proof Consider the operator $\mathscr{K} : E \to E$ defined by (8.13) in the form:

$$\mathscr{K}x(t) = \mathscr{K}_1 x(t) + \mathscr{K}_2 x(t), \quad t \in J,$$

where $\mathscr{K}_1$ and $\mathscr{K}_2$ are respectively defined by (8.16) and (8.17). From (8.7.4), there exists a number $\omega > 0$ such that

$$\frac{\omega}{\Psi_1 \psi(\omega) + \Psi_2 + l\Psi_3} > \frac{1}{1 - k\Psi_3}.$$

Let $B_\omega = \{x \in E : \|x\| < \omega\}$. We need to show that $\mathscr{K}_1$ is continuous and completely continuous. First, we show that $\mathscr{K}_1(\bar{B}_\omega)$ is bounded. For any $x \in \bar{B}_\omega$, we have

$$\begin{aligned}
|\mathscr{K}_1 x(t)| \leq & \Bigg[\frac{(\log(T/t_k))^{\alpha_k}}{\Gamma(\alpha_k+1)} \|f\| + \frac{(\log(T/t_m))^{\alpha_m}}{\Gamma(\alpha_m+1)} \|f\| \\
& + 2\sum_{j=1}^{m} \left(\log \frac{T}{t_j} \right) \left(\frac{\left(\log(t_j/t_{j-1})\right)^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1})} \|f\| + \|\varphi_j^*\| \right) \\
& + 2\sum_{j=1}^{m} \left(\frac{(\log(t_j/t_{j-1}))^{\alpha_{j-1}}}{\Gamma(\alpha_{j-1}+1)} \|f\| + \|\varphi_j\| \right) \Bigg] \\
\leq & \|f\| \Bigg[\sup_{j=1,2,\ldots,m} \left\{ \frac{(\log(T/t_j))^{\alpha_j}}{\Gamma(\alpha_j+1)} \right\} + \frac{(\log(T/t_m))^{\alpha_m}}{\Gamma(\alpha_m+1)} \\
& + 2\sum_{j=1}^{m} \left(\frac{(\log(t_j/t_{j-1}))^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1}+1)} \cdot \log \frac{(T/t_j)}{t_{j-1}/t_j}^{\alpha_{j-1}} \right) \Bigg] \\
& + 2m\|\varphi_j\| + 2\left(m \log T - \log \prod_{j=1}^{m} t_j \right) \|\varphi_j^*\|.
\end{aligned}$$

Thus $\mathscr{K}_1$ is uniformly bounded. Let $\tau_1, \tau_2 \in J$ such that $\tau_1 < \tau_2$. Then

$$\begin{aligned}
& |\mathscr{K}_1 x(\tau_2) - \mathscr{K}_1 x(\tau_1)| \\
= & |\mathscr{J}_{t_k}^{\alpha_k} f(\tau_2, x(\tau_1)) - \mathscr{J}_{t_k}^{\alpha_k} f(\tau_1, x(\tau_1))| \\
& + \Bigg| \sum_{j=1}^{k} \left(\log \frac{\tau_2}{t_j} \right) \left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1} f(t_j, x(t_j)) + \varphi_j^*(x(t_j)) \right) \\
& - \sum_{j=1}^{k} \left(\log \frac{\tau_1}{t_j} \right) \left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1} f(t_j, x(t_j)) + \varphi_j^*(x(t_j)) \right) \Bigg| \\
& + \Bigg| \frac{\log(\tau_1/t_0)}{\log(T/t_0)} \Bigg[\mathscr{J}_{t_m}^{\alpha_m} f(T, x(T)) + \sum_{j=1}^{m} \left(\log \frac{T}{t_j} \right) \left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1} f(t_j, x(t_j)) + \varphi_j^*(x(t_j)) \right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}}f(t_j,x(t_j))+\varphi_j(x(t_j))\right)\Bigg]\\
&-\frac{\log(\tau_2/t_0)}{\log(T/t_0)}\Bigg[\mathscr{J}_{t_m}^{\alpha_m}f(T,x(T))+\sum_{j=1}^{m}\left(\log\frac{T}{t_j}\right)\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1}f(t_j,x(t_j))+\varphi_j^*(x(t_j))\right)\\
&+\sum_{j=1}^{m}\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}}f(t_j,x(t_j))+\varphi_j(x(t_j))\right)\Bigg]\Bigg|\\
\le\ &\mathscr{J}_{t_k}^{\alpha_k}|f(\tau_2,x(\tau_2))-f(\tau_1,x(\tau_1))|+\sum_{j=1}^{m}\left|\log\frac{\tau_2}{\tau_1}\right|\left|\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1}f(t_j,x(t_j))+\varphi_j^*(x(t_j))\right|\\
&+\left|\frac{\log(\tau_1/\tau_2)}{\log(T/t_0)}\right|\Bigg|\mathscr{J}_{t_m}^{\alpha_m}f(T,x(T))+\sum_{j=1}^{m}\left(\log\frac{T}{t_j}\right)\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}-1}f(t_j,x(t_j))+\varphi_j^*(x(t_j))\right)\\
&+\sum_{j=1}^{m}\left(\mathscr{J}_{t_{j-1}}^{\alpha_{j-1}}f(t_j,x(t_j))+\varphi_j(x(t_j))\right)\Bigg|,
\end{aligned}
$$

which tends to zero as $\tau_2 \to \tau_1$. Hence $\mathscr{K}_1$ is equicontinuous. Hence, it follows by Arzelá-Ascoli Theorem that $\mathscr{K}_1(\bar{B}_\omega)$ is compact and hence completely continuous. Moreover, as in Theorem 8.5, we can show that the operator $\mathscr{K}_1$ is continuous.

Next, we show that $\mathscr{K}_2$ is a nonlinear contraction. For $x, y \in B_\omega$, we have

$$
\begin{aligned}
|\mathscr{K}_2x(t)-\mathscr{K}_2y(t)| &\le \frac{(\log(\eta/t_m))^\nu}{\Gamma(\nu+1)}\phi_1(\|x-y\|)+\frac{(\log(\xi/t_0))^\mu}{\Gamma(\mu+1)}\phi_2(\|x-y\|)\\
&\le \left(c_1\frac{(\log(\eta/t_m))^\nu}{\Gamma(\nu+1)}+c_2\frac{(\log(\xi/t_0))^\mu}{\Gamma(\mu+1)}\right)\|x-y\|\\
&\le \kappa\Psi_3\|x-y\|.
\end{aligned}
$$

Setting $\phi(x) = \kappa\Psi_3 x$, note that $\phi(0) = 0$ and $\phi(x) = \kappa\Psi_3 x < x$ for $x > 0$. Thus

$$
\|\mathscr{K}_2x-\mathscr{K}_2y\| \le \phi(\|x-y\|),
$$

which implies that $\mathscr{K}_2$ is a nonlinear contraction.

For any $x \in B_\omega$, we have

$$
|g(t,x)| \le |g(t,x)-g(t,0)|+|g(t,0)| \le \phi_1(\|x\|)+|g(t,0)| \le c_1\omega+\sup_{t\in J}|g(t,0)|,
$$

and

$$
|h(t,x)| \le c_2\omega+\sup_{t\in J}|h(t,0)|.
$$

Thus, letting $k = \max\{c_1, c_2\}$ and $l = \sup_{t\in J}\{|g(t,0)|, |h(t,0)|\}$, we obtain

$$\|\mathscr{K}_2 x\| \le \left[\frac{(\log(\eta/t_m))^{\nu}}{\Gamma(\nu+1)} + \frac{(\log(\xi/t_0))^{\mu}}{\Gamma(\mu+1)}\right](k\omega + l).$$

Therefore, both $\mathscr{K}_1(\bar{B}_\omega)$ and $\mathscr{K}_2(\bar{B}_\omega)$ are bounded, which implies the boundedness of $\mathscr{K}(\bar{B}_\omega)$.

Finally, we show that the case (C2) in Theorem 1.6 does not occur. To this end, let us suppose that the condition (C2) holds. This implies that there exists $\lambda \in (0,1)$ and $x \in \partial B_\omega$ such that $x = \lambda \mathscr{K} x$. So, we have $\|x\| = \omega$ and

$$\begin{aligned}
|x(t)| &= \lambda|\mathscr{K}_1 x(t) + \mathscr{K}_2 x(t)| \\
&\le |\mathscr{K}_1 x(t)| + |\mathscr{K}_2 x(t)| \\
&\le \|f\|\Bigg[\sup_{j=1,2,\ldots,m}\left\{\frac{(\log(T/t_j))^{\alpha_j}}{\Gamma(\alpha_j+1)}\right\} + \frac{(\log(T/t_m))^{\alpha_m}}{\Gamma(\alpha_m+1)} \\
&\quad + 2\sum_{j=1}^{m}\left(\frac{(\log(t_j/t_{j-1}))^{\alpha_{j-1}-1}}{\Gamma(\alpha_{j-1}+1)} \cdot \log\frac{(T/t_j)^{\alpha_{j-1}}}{t_{j-1}/t_j}\right)\Bigg] \\
&\quad + 2m\|\varphi_j\| + 2\Big(m\log T - \log\prod_{j=1}^{m} t_j\Big)\|\varphi_j^*\|. \\
&\quad + \left[\frac{(\log(\eta/t_m))^{\nu}}{\Gamma(\nu+1)} + \frac{(\log(\xi/t_0))^{\mu}}{\Gamma(\mu+1)}\right](k\omega + l) \\
&\le \Psi_1\psi(\omega) + \Psi_2 + (k\omega + l)\Psi_3,
\end{aligned}$$

which, on taking the supremum for all $t \in J$, gives

$$\|x\| \le \Psi_1\psi(\omega) + \Psi_2 + (k\omega + l)\Psi_3.$$

In consequence, we get

$$\frac{\omega}{\Psi_1\psi(\omega) + \Psi_2 + l\Psi_3} \le \frac{1}{1 - k\Psi_3}$$

which contradicts (8.7.4). Thus the operators $\mathscr{K}_1$ and $\mathscr{K}_2$ satisfy all the conditions of Theorem 1.6. Hence, the operator $\mathscr{K}$ has at least one fixed point on $\bar{B}_\omega$, which is a solution of the problem (8.8). This completes the proof. □

Example 8.7 Consider the following problem of impulsive Caputo-Hadamard fractional differential equations

$$\begin{cases} {}^{C}\mathscr{D}^{\alpha_k}x(t) &= \dfrac{1}{27}\left(\dfrac{t^3+t^2\cos(x(t))}{5+4t+|x(t)|}\right)\left(\dfrac{7|x(t)|+9}{21+e^t}\right),\ t\in\left[\dfrac{3}{2},5\right]\setminus\{t_k\} \\ \Delta x(t_k) &= \dfrac{e^{x(t_k)}-15k^2}{15e^{x(t_k)}}, \quad k=1,2,\ldots,7, \\ \Delta\delta x(t_k) &= \dfrac{2}{5}\sin\left(\dfrac{x(t_k)}{9x(t_k)+5}\right)\cos\left(\dfrac{k\pi}{2}\right), \quad k=1,2,\ldots,7, \end{cases} \tag{8.19}$$

and the nonlinear fractional integral conditions

$$x\left(\frac{3}{2}\right)=\mathscr{J}^{5/3}_{3/2}g\left(\frac{7}{4},x\left(\frac{7}{4}\right)\right), \quad \text{with } g(t,x)=\frac{(9-e^{\sin t})\arctan(|x|/3)}{4t^2+2t-1}+\frac{3}{7},$$

$$\begin{aligned} x(5) &= \mathscr{J}^{3/2}_{573/128}h\left(\frac{9}{2},x\left(\frac{9}{2}\right)\right), \ \text{with } h(t,x) \\ &= \left(\frac{10\sin^2(\pi t/15)}{(2t+3)^2}\right)\left(\frac{|x|^2+12|x|}{|x|+11}\right)-\frac{1}{2t}, \end{aligned}$$

where $\alpha_k=(4-e^{-k/2})(2)$ and $t_k=2^{-1}+2^2+2^{-k}-2^{2-k}$, for $k=1,2,\ldots,7$. Here $m=7$, $t_0=3/2$, $T=5$, $\mu=5/3$, $\xi=7/4$, $\nu=3/2$, $\eta=9/2$,

$$f(t,x)=\frac{1}{27}\left(\frac{t^3+t^2\cos x}{5+4t+|x|}\right)\left(\frac{7|x|+9}{21+e^t}\right),$$

and the impulsive functions $\varphi_k(x)$, $\varphi_k^*(x)$ at impulse moments t_k for $k=1,2,\ldots,9$, are defined by

$$\varphi_k(x)=\frac{e^x-15k^2}{15e^x}, \quad \varphi_k^*(x)=\frac{2}{5}\sin\left(\frac{x}{9x+5}\right)\cos\left(\frac{k\pi}{2}\right).$$

For $(t,x)\in[3/2,5]\times\mathbb{R}$, we have

$$|f(t,x)|\le\frac{t^2(t+1)}{27(5+4t)}\left(\frac{\|x\|}{3}+\frac{3}{7}\right).$$

Let $b\in C([3/2,5],\mathbb{R})$ and $\psi:[0,\infty)\to(0,\infty)$ be defined by

$$b(t)=\frac{t^2(t+1)}{27(5+4t)} \quad \text{and} \quad \psi(x)=\frac{x}{3}+\frac{3}{7}.$$

Obviously b is a nonnegative function and ψ is a nondecreasing function. The impulsive functions φ_k^* and φ_k are bounded by constants $d_1 = 2/45$ and $d_2 = 1/15$ respectively. Selecting $\phi_1, \phi_2 : [0, \infty) \to [0, \infty)$ as

$$\phi_1(x) = \frac{9}{11} \arctan\left(\frac{x}{3}\right), \qquad \phi_2(x) = \frac{10}{33}x,$$

and noting that $\phi_1(x) \le (3/11)x$, we obtain

$$|g(t,x) - g(t,y)| \le \frac{9}{11} \arctan\left(\frac{\|x-y\|}{3}\right) = \phi_1(\|x-y\|),$$

$$|h(t,x) - h(t,y)| \le \frac{10}{36}\left(\|x-y\| + \frac{11\|x-y\|}{121 + |x| + |y|}\right) \le \phi_2(\|x-y\|).$$

Thus, we deduce that $\|b\| = 2/9$, $d_1 = 2/45$, $d_2 = 1/15$, $c_1 = 3/11$ and $c_2 = 10/33$. Furthermore, $\Psi_1 = 0.944897$, $\Psi_2 = 0.029739$ and $\Psi_3 = 1.065922$, $\kappa = 10/33$, and $l = 1/2$. With the given data, the hypothesis (8.7.4) is satisfied for $r > 9.141700$. Therefore, the problem (8.19) has at least one solution on $[3/2, 5]$ by the conclusion of Theorem 8.7.

8.4 Notes and Remarks

In this chapter, we studied boundary value problems for first and second order impulsive multi-order Hadamard fractional differential equations supplemented with nonlinear fractional integral conditions by using classical fixed point theorems. The results in this chapter are based on the papers [176] and [177].

Chapter 9
Initial and Boundary Value Problems for Hybrid Hadamard Fractional Differential Equations and Inclusions

9.1 Introduction

This chapter is devoted to the study of initial and boundary value problems of hybrid fractional differential equations and inclusions involving Hadamard derivative and integral. Several existence results for local and nonlocal cases of the given problems are obtained.

By hybrid differential equation, we mean that the terms in the equation are perturbed either linearly or quadratically or through the combination of first and second types. Perturbation taking place in form of the sum or difference of terms in an equation is called linear. On the other hand, if the equation is perturbed through the product or quotient of the terms in it, then it is called quadratic perturbation. So the study of hybrid differential equation is more general and covers several dynamic systems as particular cases.

9.2 Initial Value Problems for Hybrid Hadamard Fractional Differential Equations

In this section, we study the existence of solutions for an initial value problem of hybrid fractional differential equations of Hadamard type given by

$$\begin{cases} {}_HD^{\alpha}\left(\dfrac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \quad 1 \le t \le T, \quad 0 < \alpha \le 1, \\ {}_HJ^{1-\alpha}x(t)|_{t=1} = \eta, \end{cases} \tag{9.1}$$

B. Ahmad et al., *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, DOI 10.1007/978-3-319-52141-1_9

where ${}_HD^{\alpha}$ is the Hadamard fractional derivative, $f \in C([1,T] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g : C([1,T] \times \mathbb{R}, \mathbb{R})$, ${}_HJ^{(\cdot)}$ is the Hadamard fractional integral and $\eta \in \mathbb{R}$.

From Theorem 2.1, we have:

Lemma 9.1 *Given $y \in C([1,T], \mathbb{R})$, the solution of initial value problem*

$$\begin{cases} {}_HD^{\alpha}\left(\dfrac{x(t)}{f(t,x(t))}\right) = y(t), \quad 1 \le t \le T,\ 0 < \alpha \le 1, \\ \qquad\qquad {}_HJ^{1-\alpha}x(t)|_{t=1} = \eta, \end{cases} \tag{9.2}$$

is given by

$$x(t) = f(t,x(t))\left(\frac{\eta}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds\right), \quad t \in [1,T].$$

Let $X = C([1,T], \mathbb{R})$ denote the Banach space of all continuous real valued functions defined on $[1,T]$ with the norm $\|x\| = \sup\{|x(t)| : t \in [1,T]\}$. For $t \in [1,T]$, we define $x_r(t) = (\log t)^r x(t), r \ge 0$. Let $C_r([1,T], \mathbb{R})$ be the space of all continuous functions x such that $x_r \in C([1,T], \mathbb{R})$ which is indeed a Banach space endowed with the norm $\|x\|_C = \sup\{(\log t)^r |x(t)| : t \in [1,T]\}$.

Let $0 \le \gamma < 1$ and $C_{\gamma,\log}[1,T]$ denote the weighted space of continuous functions defined by

$$C_{\gamma,\log}[1,T] = \left\{g(t) : (\log t)^{\gamma}g(t) \in C[1,T],\ \|y\|_{C_{\gamma,\log}} = \|(\log t)^{\gamma}g(t)\|_C\right\}.$$

In the following, we denote $\|y\|_{C_{\gamma,\log}}$ by $\|y\|_C$.

Theorem 9.1 *Assume that:*

(9.1.1) the function $f : [1,T]\times\mathbb{R} \to \mathbb{R}\setminus\{0\}$ is bounded (i.e. $|f(t,x)| \le K,\ \forall (t,x) \in [1,T] \times \mathbb{R}$), continuous and there exists a bounded function ϕ, with bound $\|\phi\|$, such that $\phi(t) > 0$, a.e. $t \in [1,T]$ and

$$|f(t,x) - f(t,y)| \le \phi(t)|x(t) - y(t)|, \quad a.e.\ \ t \in [1,T] \ \ \text{and for all}\ \ x, y \in \mathbb{R};$$

(9.1.2) there exist a function $p \in C([1,T], \mathbb{R}^+)$ and a continuous nondecreasing function $\Omega : [0,\infty) \to (0,\infty)$ such that

$$|g(t,x(t))| \le p(t)\Omega(\|x\|_C), \quad (t,x) \in [1,T] \times \mathbb{R};$$

(9.1.3) there exists a number $r > 0$ such that

$$r \ge K\left[\frac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha}\frac{1}{\Gamma(\alpha+1)}\|p\|\Omega(r)\right], \tag{9.3}$$

and

$$\|\phi\| \left[\frac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha} \frac{1}{\Gamma(\alpha+1)} \|p\| \Omega(r) \right] < 1.$$

Then the initial value problem (9.1) has at least one solution on $[1, T]$.

Proof Define a subset S of X as

$$S = \{x \in X : \|x\|_C \leq r\},$$

where r satisfies the inequality (9.3).

Clearly S is closed, convex and bounded subset of the Banach space X. By Lemma 9.1, the initial value problem (9.1) is equivalent to the integral equation

$$\begin{aligned} x(t) &= f(t, x(t)) \left(\frac{\eta}{\Gamma(\alpha)} (\log t)^{\alpha-1} \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \frac{g(s, x(s))}{s} ds \right), \quad t \in [1, T]. \end{aligned} \tag{9.4}$$

Define two operators $\mathscr{A} : X \to X$ by

$$\mathscr{A} x(t) = f(t, x(t)), \ \ t \in [1, T], \tag{9.5}$$

and $\mathscr{B} : S \to X$ by

$$\mathscr{B} x(t) = \frac{\eta}{\Gamma(\alpha)} (\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \frac{g(s, x(s))}{s} ds, \quad t \in [1, T]. \tag{9.6}$$

Then $x = \mathscr{A} x \mathscr{B} x$. We shall show that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 1.7. For the sake of clarity, we split the proof into a sequence of steps.

Step 1. *We first show that* $\mathscr{A}$ *is a Lipschitz on* X, *i.e., (a) of Theorem 1.7 holds.*

Let $x, y \in X$. Then, by (9.1.1), we have

$$\begin{aligned} |(\log t)^{1-\alpha} \mathscr{A} x(t) - (\log t)^{1-\alpha} \mathscr{A} y(t)| &= (\log t)^{1-\alpha} |f(t, x(t)) - f(t, y(t))| \\ &\leq \phi(t) (\log t)^{1-\alpha} |x(t) - y(t)| \\ &\leq \|\phi\| \|x - y\|_C, \end{aligned}$$

for all $t \in [1, T]$. Taking the supremum over the interval $[1, T]$, we obtain

$$\|\mathscr{A} x - \mathscr{A} y\|_C \leq \|\phi\| \|x - y\|_C,$$

for all $x, y \in X$. So $\mathscr{A}$ is a Lipschitz on X with Lipschitz constant $\|\phi\|$.

Step 2. *The operator $\mathscr{B}$ is completely continuous on S, i.e., (b) of Theorem 1.7 holds.*

First, we show that $\mathscr{B}$ is continuous on S. Let $\{x_n\}$ be a sequence in S converging to a point $x \in S$. Then, by Lebesque dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{n\to\infty} (\log t)^{1-\alpha} \mathscr{B}x_n(t) \\
&= \lim_{n\to\infty} \left(\frac{\eta}{\Gamma(\alpha)} + (\log t)^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s, x_n(s))}{s} ds \right) \\
&= \frac{\eta}{\Gamma(\alpha)} + (\log t)^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{\lim_{n\to\infty} g(s, x_n(s))}{s} ds \\
&= \frac{\eta}{\Gamma(\alpha)} + (\log t)^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s, x(s))}{s} ds \\
&= (\log t)^{1-\alpha} \mathscr{B}x(t),
\end{aligned}$$

for all $t \in [1, T]$. This shows that $\mathscr{B}$ is continuous os S. It is enough to show that $\mathscr{B}(S)$ is a uniformly bounded and equicontinuous set in X. First, we note that

$$\begin{aligned}
(\log t)^{1-\alpha} |\mathscr{B}x(t)| &= \left| \frac{\eta}{\Gamma(\alpha)} + (\log t)^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s, x(s))}{s} ds \right| \\
&\leq \frac{|\eta|}{\Gamma(\alpha)} + \|p\| \Omega(r) (\log T)^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{1}{s} ds \\
&= \frac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha} \frac{1}{\Gamma(\alpha+1)} \|p\| \Omega(r),
\end{aligned}$$

for all $t \in [1, T]$. Taking supremum over the interval $[1, T]$, the above inequality becomes

$$\|\mathscr{B}x\|_C \leq \frac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha} \frac{1}{\Gamma(\alpha+1)} \|p\| \Omega(r),$$

for all $x \in S$. This shows that $\mathscr{B}$ is uniformly bounded on S.

Next, we show that $\mathscr{B}$ is an equicontinuous set in X. Let $\tau_1, \tau_2 \in [1, T]$ with $\tau_1 < \tau_2$ and $x \in S$. Then, we have

$$\begin{aligned}
&|(\log \tau_2)^{1-\alpha} (\mathscr{B}x)(\tau_2) - (\log \tau_1)^{1-\alpha} (\mathscr{B}x)(\tau_1)| \\
&\leq \frac{\|p\| \Omega(r)}{\Gamma(\alpha)} \left| \int_1^{\tau_2} (\log \tau_2)^{1-\alpha} \left(\log \frac{\tau_2}{s}\right)^{\alpha-1} \frac{1}{s} ds - \int_1^{\tau_1} (\log \tau_1)^{1-\alpha} \left(\log \frac{\tau_1}{s}\right)^{\alpha-1} \frac{1}{s} ds \right|
\end{aligned}$$

$$\leq \frac{\|p\|\Omega(r)}{\Gamma(\alpha)}\left|\int_1^{\tau_1}\left[(\log\tau_2)^{1-\alpha}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}-(\log\tau_1)^{1-\alpha}\left(\log\frac{\tau_1}{s}\right)^{\alpha-1}\right]\frac{1}{s}ds\right|$$
$$+\frac{\|p\|\Omega(r)}{\Gamma(\alpha)}\left|\int_{\tau_1}^{\tau_2}(\log\tau_2)^{1-\alpha}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{1}{s}ds\right|.$$

Obviously the right hand side of the above inequality tends to zero, independently of $x \in S$ as $\tau_2 - \tau_1 \to 0$. Therefore, it follows from the Arzelá-Ascoli Theorem that $\mathscr{B}$ is a completely continuous operator on S.

Step 3. *Next, we show that hypothesis (c) of Theorem 1.7 is satisfied.* Let $x \in X$ and $y \in S$ be arbitrary elements such that $x = \mathscr{A}x\mathscr{B}y$. Then, we have

$$\begin{aligned}
&(\log t)^{1-\alpha}|x(t)| \\
&= (\log t)^{1-\alpha}|\mathscr{A}x(t)||\mathscr{B}y(t)| \\
&= |f(t,x(t))|\left|\left(\frac{\eta}{\Gamma(\alpha)}+(\log t)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s,y(s))}{s}ds\right)\right| \\
&\leq K\left|\left(\frac{\eta}{\Gamma(\alpha)}+(\log t)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s,y(s))}{s}ds\right)\right| \\
&\leq K\left[\frac{|\eta|}{\Gamma(\alpha)}+(\log T)^{1-\alpha}\|p\|\Omega(r)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{s}ds\right] \\
&= K\left[\frac{|\eta|}{\Gamma(\alpha)}+(\log T)^{1-\alpha}\frac{1}{\Gamma(\alpha+1)}\|p\|\Omega(r)\right].
\end{aligned}$$

Taking supremum for $t \in [1, T]$, we obtain

$$\|x\|_C \leq K\left[\frac{|\eta|}{\Gamma(\alpha)}+(\log T)^{1-\alpha}\frac{1}{\Gamma(\alpha+1)}\|p\|\Omega(r)\right] \leq r,$$

that is, $x \in S$.

Step 4. *Now, we show that $Mk < 1$, that is, (d) of Theorem 1.7 holds.*

This is obvious by (9.1.4), since, we have $M = \|B(S)\| = \sup\{\|\mathscr{B}x\| : x \in S\} \leq \dfrac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha}\dfrac{1}{\Gamma(\alpha+1)}\|p\|\Omega(r)$ and $k = \|\phi\|$.

Thus all the conditions of Theorem 1.7 are satisfied and hence the operator equation $x = \mathscr{A}x\mathscr{B}x$ has a solution in S. In consequence, the problem (9.1) has a solution on $[1, T]$. This completes the proof. □

Example 9.1 Consider the hybrid initial value problem

$$\begin{cases} {}_HD^{1/2}\left(\dfrac{x(t)}{f(t,x)}\right) = g(t,x), \ \ 1 < t < e, \\ {}_HJ^{1/2}x(t)|_{t=1} = 1, \end{cases} \tag{9.7}$$

where

$$f(t,x) = \frac{1}{5\sqrt{\pi}}(|\sin t \tan^{-1} x| + \pi/2), \ g(t,x) = \frac{1}{10}\left(\frac{1}{6}|x| + \frac{1}{8}\cos x + \frac{|x|}{4(1+|x|)} + \frac{1}{16}\right).$$

Obviously $|f(t,x)| \leq \dfrac{\sqrt{\pi}}{5} = K$, $\|\phi\| = \dfrac{1}{5\sqrt{\pi}}$, $|g(t,x)| \leq \dfrac{1}{10}\left(\dfrac{1}{6}|x| + \dfrac{7}{16}\right)$. We choose $\|p\| = \dfrac{1}{10}$, $\Omega(r) = \dfrac{1}{6}r + \dfrac{7}{16}$. By the condition (9.1.3), it is found that $\dfrac{261}{1192} \leq r < \dfrac{3}{8}(400\pi - 87)$. Clearly all the conditions of Theorem 9.1 are satisfied. Hence, by the conclusion of Theorem 9.1, it follows that the problem (9.7) has a solution on $[1, e]$.

9.3 Fractional Hybrid Differential Inclusions of Hadamard Type

In this section, we investigate the existence of solutions for the following inclusion problem

$$\begin{cases} {}_HD^{\alpha}\left(\dfrac{x(t)}{f(t,x(t))}\right) \in F(t,x(t)), \ \ 1 \leq t \leq T, \ \ 0 < \alpha \leq 1, \\ {}_HJ^{1-\alpha}x(t)|_{t=1} = \eta, \end{cases} \tag{9.8}$$

where $F : [1,T] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map, $\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

Theorem 9.2 *Assume that (9.1.1) holds. In addition, we suppose that:*

(9.2.1) $F : [1,T] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ *is* L^1*-Carathéodory and has nonempty compact and convex values;*

(9.2.2) $2\|\phi\|\left(\dfrac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1}\frac{g(s)}{s}ds\right) < 1.$

Then the boundary value problem (9.8) has at least one solution on $[1, T]$.

Proof Transform the problem (9.8) into a fixed point problem. Consider the operator $\mathscr{N}: X \to \mathscr{P}(X)$ defined by

$$\mathscr{N}(x) = \Bigg\{ h \in X : \ h(t) = f(t, x(t)) \Bigg(\frac{\eta}{\Gamma(\alpha)} (\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \Bigg), \ v \in S_{F,x} \Bigg\}.$$

Now, we define two operators $\mathscr{A}_1 : X \to X$ by

$$\mathscr{A}_1 x(t) = f(t, x(t)), \ t \in [1, T], \tag{9.9}$$

and $\mathscr{B}_1 : X \to \mathscr{P}(X)$ by

$$\mathscr{B}_1(x) = \Bigg\{ h \in C([1,T], \mathbb{R}) : h(t) = \frac{\eta}{\Gamma(\alpha)} (\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds, \ v \in S_{F,x} \Bigg\}. \tag{9.10}$$

Observe that $\mathscr{N}(x) = \mathscr{A}_1 x \mathscr{B}_1 x$. We shall show that the operators $\mathscr{A}_1$ and $\mathscr{B}_1$ satisfy all the conditions of Theorem 1.8. For the sake of convenience, we split the proof into several steps.

Step 1. *$\mathscr{A}_1$ is a Lipschitz on X, i.e., (a) of Theorem 1.8 holds.*

This was proved in Step 1 of Theorem 9.1.

Step 2. *The multivalued operator $\mathscr{B}_1$ is compact and upper semicontinuous on X, i.e., (b) of Theorem 1.8 holds.*

First, we show that $\mathscr{B}_1$ has convex values. Let $u_1, u_2 \in \mathscr{B}_1 x$. Then there are $v_1, v_2 \in S_{F,x}$ such that

$$u_i(t) = \frac{\eta}{\Gamma(\alpha)} (\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_i(s)}{s} ds,$$

$i = 1, 2, \ t \in [1, T]$. For any $\theta \in [0, 1]$, we have

$$\theta u_1(t) + (1-\theta) u_2(t) = \frac{\eta}{\Gamma(\alpha)} (\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{[\theta v_1(s) + (1-\theta) v_2(s)]}{s} ds,$$

with $\theta v_1(t) + (1-\theta) v_2(t) \in F(t, x(t))$ for all $t \in [1, T]$. Hence $\theta u_1(t) + (1-\theta) u_2(t) \in \mathscr{B}_1 x$ and consequently $\mathscr{B}_1 x$ is convex for each $x \in X$. As a result $\mathscr{B}_1$ defines a multivalued operator $\mathscr{B}_1 : X \to \mathscr{P}_{cv}(X)$.

Next, we show that $\mathscr{B}_1$ maps bounded sets into bounded sets in X. To see this, let Q be a bounded set in X. Then there exists a real number $r > 0$ such that $\|x\|_C \leq r, \forall x \in Q$.

Now, for each $h \in \mathscr{B}_1 x$, there exists a $v \in S_{F,x}$ such that

$$h(t) = \frac{\eta}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s}ds.$$

Then for each $t \in [1, T]$, using (9.2.2), we have

$$\begin{aligned}(\log t)^{1-\alpha}|h(t)| &= \left|\frac{\eta}{\Gamma(\alpha)} + (\log t)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s}ds\right| \\ &\leq \frac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s)}{s}ds \\ &\leq \frac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \frac{g(s)}{s}ds.\end{aligned}$$

This further implies that

$$\|h\|_C \leq \frac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \frac{g(s)}{s}ds,$$

and so $\mathscr{B}_1(X)$ is uniformly bounded.

Next, we show that $\mathscr{B}_1$ maps bounded sets into equicontinuous sets. Let Q be, as above, a bounded set and $h \in \mathscr{B}_1 x$ for some $x \in Q$. Then there exists a $v \in S_{F,x}$ such that

$$h(t) = \frac{\eta}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s}ds, \ t \in [1, T].$$

Then, for any $\tau_1, \tau_2 \in [1, T]$ with $\tau_1 < \tau_2$, we have

$$\begin{aligned}&|(\log \tau_2)^{1-\alpha}(\mathscr{B}_1 x)(\tau_2) - (\log \tau_1)^{1-\alpha}(\mathscr{B}_1 x)(\tau_1)| \\ &\leq \left|\int_1^{\tau_2}(\log \tau_2)^{1-\alpha}\left(\log \frac{\tau_2}{s}\right)^{\alpha-1} \frac{g(s)}{s}ds - \int_1^{\tau_1}(\log \tau_1)^{1-\alpha}\left(\log \frac{\tau_1}{s}\right)^{\alpha-1} \frac{g(s)}{s}ds\right| \\ &\leq \left|\int_1^{\tau_1}\left[(\log \tau_2)^{1-\alpha}\left(\log \frac{\tau_2}{s}\right)^{\alpha-1} - (\log \tau_1)^{1-\alpha}\left(\log \frac{\tau_1}{s}\right)^{\alpha-1}\right]\frac{g(s)}{s}ds\right| \\ &+\left|\int_{\tau_1}^{\tau_2}(\log \tau_2)^{1-\alpha}\left(\log \frac{\tau_2}{s}\right)^{\alpha-1} \frac{g(s)}{s}ds\right|.\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero, independently of $x \in Q$ as $\tau_2 - \tau_1 \to 0$. Therefore it follows by the Arzelá-Ascoli Theorem that $\mathscr{B}_1 : X \to \mathscr{P}(X)$ is completely continuous.

In our next step, we show that $\mathscr{B}_1$ is upper semicontinuous. By Lemma 1.1, $\mathscr{B}_1$ will be upper semicontinuous if we prove that it has a closed graph, since $\mathscr{B}_1$ is already shown to be completely continuous.

Thus, in our next step, we show that $\mathscr{B}_1$ *has a closed graph*. Let $x_n \to x_*, h_n \in \mathscr{B}_1(x_n)$ and $h_n \to h_*$. Then, we need to show that $h_* \in \mathscr{B}_1$. Associated with $h_n \in \mathscr{B}_1(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in [1, T]$,

$$h_n(t) = \frac{\eta}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_n(s)}{s} ds.$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [1, T]$,

$$h_*(t) = \frac{\eta}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_*(s)}{s} ds.$$

Let us consider the linear operator $\Theta : L^1([1, T], \mathbb{R}) \to X$ given by

$$f \mapsto \Theta(v)(t) = \frac{\eta}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds.$$

Observe that

$$\|h_n(t) - h_*(t)\| = \left\| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{(v_n(s) - v_*(s))}{s} ds \right\| \to 0, \quad \text{as } n \to \infty.$$

Thus, it follows by Lemma 1.2 that $\Theta \circ S_{F,x}$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = \frac{\eta}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_*(s)}{s} ds,$$

for some $v_* \in S_{F,x_*}$.

As a result, we have that the operator $\mathscr{B}_1$ is compact and upper semicontinuous operator on X.

Step 3. *Now, we show that* $2Mk < 1$, *i.e., (c) of Theorem 1.8 holds.*

This is obvious by (9.2.3) since, we have $M = \|B(X)\| = \sup\{|\mathscr{B}_1 x : x \in X\} \leq \frac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} \frac{g(s)}{s} ds$ and $k = \|\phi\|$.

Thus all the conditions of Theorem 1.8 are satisfied and a direct application of this theorem yields that either the conclusion (i) or the conclusion (ii) holds. We show that the conclusion (ii) is not possible.

Let $\bar{\mathscr{E}} = \{u \in X | \lambda u \in \mathscr{A}_1 u \mathscr{B}_1 u,\ \lambda > 1\}$ and $u \in \bar{\mathscr{E}}$ be arbitrary. Then, we have for $\lambda > 1,\ \lambda u(t) \in \mathscr{A}_1 u(t) \mathscr{B}_1 u(t)$. Further, there exists $v \in S_{F,x}$ such that for any $\lambda > 1$, we have

$$u(t) = \lambda^{-1}[f(t,u(t)]\left(\frac{\eta}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds\right),$$

for all $t \in [1,T]$. In consequence, we have

$$\begin{aligned}(\log t)^{1-\alpha}|u(t)| &\le \lambda^{-1}|f(t,u(t)| \times \\ &\quad \times\left(\frac{\eta}{\Gamma(\alpha)} + (\log t)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|v(s)|}{s}ds\right) \\ &\le K\left(\frac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1}\frac{g(s)}{s}ds\right) \\ &\le K\left(\frac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1}\frac{g(s)}{s}ds\right).\end{aligned}$$

Thus

$$\|u\|_C \le K\left(\frac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1}\frac{g(s)}{s}ds\right) := M.$$

Thus the condition (ii) of Theorem 1.8 does not hold. Therefore the operator equation $x = \mathscr{A}_1 x \mathscr{B}_1 x$ and consequently problem (9.8) has a solution on $[1,T]$. This completes the proof. □

Example 9.2 Consider the hybrid initial value inclusion problem

$$\begin{cases} {}_HD^{1/2}\left(\dfrac{x(t)}{f(t,x)}\right) \in F(t,x(t)), \quad 1 < t < e, \\ {}_HJ^{1/2}x(t)|_{t=1} = \dfrac{2}{3}, \end{cases} \tag{9.11}$$

where $f(t,x) = \left|\frac{\log t}{2}\arctan x\right| + \frac{1}{\sqrt{1+t^2}}$ and $F(t,x) = \left[\frac{|x|^5}{15(|x|^5+1)}, \frac{|\sin x|}{7(|\sin x|+1)} + \frac{2}{7}\right]$, and $T = e$. Clearly $\phi(t) = \frac{1}{2}\log t$ with $\|\phi\| = \frac{1}{2}$ (the condition (9.2.1) holds) and $\|F(t,x)\| = \sup\{|y| : y \in F(t,x)\} \le \frac{3}{7} = g(t),\ x \in \mathbb{R}$. With the given values, the condition (9.2.3) is clearly satisfied, that is,

$$2\|\phi\|\left(\frac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1}\frac{g(s)}{s}ds\right) \simeq 0.859717 < 1.$$

In consequence, the conclusion of Theorem 9.2 applies to the problem (9.11).

9.4 Boundary Value Problems for Hybrid Fractional Differential Equations and Inclusions of Hadamard Type

In this section, we study the existence of solutions of a boundary value problem of hybrid fractional differential equations of Hadamard type given by

$$\begin{cases} D^\alpha\left(\dfrac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \quad 1 \le t \le e, \quad 1 < \alpha \le 2, \\ x(1) = 0, \quad x(e) = 0, \end{cases} \tag{9.12}$$

where D^α is the Hadamard fractional derivative, $f \in C([1,e]\times\mathbb{R}, \mathbb{R}\setminus\{0\})$ and $g \in C([1,e]\times\mathbb{R},\mathbb{R})$.

For $1 < \alpha \le 2, \beta > 0$, we also investigate the case when the hybrid part of Hadamard type fractional differential equation contains Hadamard integral for a given nonlinear function. Precisely, we consider the following problem:

$$\begin{cases} D^\alpha\left(\dfrac{x(t)}{f(t,x(t)) + \dfrac{1}{\Gamma(\beta)}\displaystyle\int_1^t\left(\log\frac{t}{s}\right)^{\beta-1}\frac{h(s,x(s))}{s}ds}\right) = g(t,x(t)), \ 1 \le t \le e, \\ x(1) = 0, \quad x(e) = 0, \end{cases} \tag{9.13}$$

where $f,\ h \in C([1,e]\times\mathbb{R},\mathbb{R})$ are such that

$$f(t,x(t)) + \frac{1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{h(s,x(s))}{s}ds \neq 0, \quad \forall (t,x) \in [1,e]\times\mathbb{R}.$$

For some recent work on hybrid fractional differential equations, we refer to [34, 77, 186] and the references cited therein.

Lemma 9.2 *Given $y \in C([1,e],\mathbb{R})$, the boundary value problem*

$$\begin{cases} D^\alpha\left(\dfrac{x(t)}{f(t,x(t))}\right) = y(t), \quad 1 < t < e, \\ x(1) = x(e) = 0, \end{cases} \tag{9.14}$$

is equivalent to the following integral equation

$$x(t) = f(t, x(t))\Bigg(\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds - (\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds\Bigg), \quad t \in [1, e].$$

Proof As argued before, the solution of Hadamard differential equation in (9.14) can be written as

$$x(t) = f(t, x(t))\Bigg(\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds + c_1(\log t)^{\alpha-1} + c_2(\log t)^{\alpha-2}\Bigg), \tag{9.15}$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants. Using the boundary conditions given in (9.14), we find that

$$c_2 = 0, \quad c_1 = -\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds.$$

Substituting the values of c_1, c_2 in (9.15), we get

$$x(t) = f(t, x(t))\Bigg(\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds - (\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds\Bigg), \quad t \in [1, e].$$

The converse follows by direct computation. The proof is completed. □

Theorem 9.3 *Assume that:*

(9.3.1) the function $f : [1, e]\times\mathbb{R} \to \mathbb{R}\setminus\{0\}$ *is continuous and there exists a bounded function* ϕ, *with bound* $\|\phi\|$, *such that* $\phi(t) > 0$, *for* $t \in [1, e]$ *and*

$$|f(t, x) - f(t, y)| \le \phi(t)|x - y|, \quad \textit{for } t \in [1, e] \textit{ and for all } x, y \in \mathbb{R};$$

(9.3.2) there exist a function $p \in C([1, e], \mathbb{R}^+)$ *and a continuous nondecreasing function* $\Omega : [0, \infty) \to (0, \infty)$ *such that*

$$|g(t, x)| \le p(t)\Omega(\|x\|), \quad t \in [1, e], \textit{ and for all } x \in \mathbb{R};$$

(9.3.3) there exists a number $r > 0$ *such that*

$$r \ge \frac{2F_0\|p\|\Omega(r)}{\Gamma(\alpha+1) - 2\|\phi\|\|p\|\Omega(r)}, \tag{9.16}$$

with

$$\frac{2\|\phi\|}{\Gamma(\alpha+1)}\|p\|\Omega(r) < 1,$$

and $F_0 = \sup_{t\in[1,e]} |f(t,0)|$.

Then the boundary value problem (9.12) has at least one solution on $[1, e]$.

Proof Set $\mathscr{E}_1 = C([1,e],\mathbb{R})$ and define a subset S of $\mathscr{E}_1$ as

$$S = \{x \in \mathscr{E}_1 : \|x\| \le r\},$$

where r satisfies the inequality (9.16).

Clearly S is closed, convex and bounded subset of the Banach space $\mathscr{E}_1$. By Lemma 9.2, the boundary value problem (9.12) is equivalent to the integral equation

$$\begin{aligned} x(t) = f(t,x(t))\Bigg(&\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s,x(s))}{s}ds \\ &-(\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{g(s,x(s))}{s}ds\Bigg), \quad t\in[1,e]. \end{aligned} \tag{9.17}$$

Define two operators $\mathscr{A} : \mathscr{E}_1 \to \mathscr{E}_1$ by

$$\mathscr{A}x(t) = f(t,x(t)),\ \ t \in [1,e], \tag{9.18}$$

and $\mathscr{B} : S \to \mathscr{E}_1$ by

$$\begin{aligned} \mathscr{B}x(t) = &\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s,x(s))}{s}ds \\ &-(\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{g(s,x(s))}{s}ds, \quad t\in[1,e]. \end{aligned} \tag{9.19}$$

Then $x = \mathscr{A}x\mathscr{B}x$. We shall show that the operators $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 1.7 in a series of steps.

Step 1. *We first show that* $\mathscr{A}$ *is a Lipschitz on* $\mathscr{E}_1$*, i.e., (a) of Theorem 1.7 holds.*

Let $x, y \in \mathscr{E}_1$. Then, by (9.3.1), we have

$$\begin{aligned} |\mathscr{A}x(t) - \mathscr{A}y(t)| &= |f(t,x(t)) - f(t,y(t))| \\ &\le \phi(t)|x(t) - y(t)| \\ &\le \|\phi\|\|x - y\|, \end{aligned}$$

for all $t \in [1, e]$. Taking the supremum over the interval $[1, e]$, we obtain

$$\|\mathscr{A}x - \mathscr{A}y\| \leq \|\phi\| \|x - y\|,$$

for all $x, y \in \mathscr{E}_1$. So $\mathscr{A}$ is a Lipschitz on $\mathscr{E}_1$ with Lipschitz constant $\|\phi\|$.

Step 2. *The operator $\mathscr{B}$ is completely continuous on S, i.e., (b) of Theorem 1.7 holds.*

First, we show that $\mathscr{B}$ is continuous on S. Let $\{x_n\}$ be a sequence in S converging to a point $x \in S$. Then by Lebesque dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathscr{B}x_n(t) &= \lim_{n\to\infty} \Bigg(\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s, x_n(s))}{s} ds \\
&\quad -(\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{g(s, x_n(s))}{s} ds \Bigg) \\
&= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{\lim_{n\to\infty} g(s, x_n(s))}{s} ds \\
&\quad -(\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{\lim_{n\to\infty} g(s, x_n(s))}{s} ds \\
&= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s, x(s))}{s} ds \\
&\quad -(\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{g(s, x(s))}{s} ds \\
&= \mathscr{B}x(t),
\end{aligned}$$

for all $t \in [1, e]$. This shows that $\mathscr{B}$ is continuous on S. Next we show that $\mathscr{B}(S)$ is a uniformly bounded and equicontinuous set in $\mathscr{E}_1$. First, we note that

$$\begin{aligned}
|\mathscr{B}x(t)| &= \Bigg| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s, x(s))}{s} ds \\
&\quad -(\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{g(s, x(s))}{s} ds \Bigg| \\
&\leq \|p\| \Omega(r) \left[\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{1}{s} ds + \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{1}{s} ds \right] \\
&= \frac{2}{\Gamma(\alpha+1)} \|p\| \Omega(r),
\end{aligned}$$

for all $t \in [1, e]$. Taking supremum over the interval $[1, e]$, the above inequality becomes

$$\|\mathscr{B}x\| \leq \frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r),$$

for all $x \in S$. This shows that $\mathscr{B}$ is uniformly bounded on S.

Next, we show that $\mathscr{B}$ is an equicontinuous set in $\mathscr{E}_1$. Let $\tau_1, \tau_2 \in [1, e]$ with $\tau_1 < \tau_2$ and $x \in S$. Then, we have

$$\begin{aligned}
&|(\mathscr{B}x)(\tau_2) - (\mathscr{B}x)(\tau_1)| \\
&\leq \frac{\|p\|\Omega(r)}{\Gamma(\alpha)}\left|\int_1^{\tau_1}\left(\log\frac{\tau_1}{s}\right)^{\alpha-1}\frac{1}{s}ds - \int_1^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{1}{s}ds\right| \\
&\quad + \frac{\|p\|\Omega(r)|(\log\tau_2)^{\alpha-1} - (\log\tau_1)^{\alpha-1}|}{\Gamma(\alpha)}\int_1^{e}\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}ds \\
&\leq \frac{\|p\|\Omega(r)}{\Gamma(\alpha+1)}\Big[2(\log(\tau_2/\tau_1))^{\alpha} + |(\log\tau_2)^{\alpha} - (\log\tau_1)^{\alpha}|\Big] \\
&\quad + \frac{\|p\|\Omega(r)|(\log\tau_2)^{\alpha-1} - (\log\tau_1)^{\alpha-1}|}{\Gamma(\alpha+1)}.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero, independently of $x \in S$ as $\tau_2 - \tau_1 \to 0$. Therefore, it follows from the Arzelá-Ascoli Theorem that $\mathscr{B}$ is a completely continuous operator on S.

Step 3. *Here we show that hypothesis (c) of Theorem 1.7 is satisfied.* Let $x \in \mathscr{E}_1$ and $y \in S$ be arbitrary elements such that $x = \mathscr{A}x\mathscr{B}y$. Then, we have

$$\begin{aligned}
|x(t)| &= |\mathscr{A}x(t)||\mathscr{B}y(t)| \\
&= |f(t, x(t))|\left|\left(\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s, y(s))}{s}ds\right.\right. \\
&\quad \left.\left. -(\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{g(s, y(s))}{s}ds\right)\right| \\
&\leq [|f(t, x(t)) - f(t, 0)| + |f(t, 0)|] \times \left|\left(\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s, y(s))}{s}ds\right.\right. \\
&\quad \left.\left. -(\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{g(s, y(s))}{s}ds\right)\right| \\
&\leq [\phi(t)|x(t)| + F_0]\|p\|\Omega(r)\left[\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{1}{s}ds\right.
\end{aligned}$$

$$+\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{1}{s}ds\Bigg]$$
$$\leq [\|\phi\||x(t)|+F_0]\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r).$$

Thus

$$|x(t)|\leq\|\phi\||x(t)|\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r)+F_0\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r),$$

which, on taking supremum for $t\in[1,e]$, yields

$$\|x\|\leq\frac{2F_0\|p\|\Omega(r)}{\Gamma(\alpha+1)-2\|\phi\|\|p\|\Omega(r)}\leq r.$$

This shows that $x\in S$.

Step 4. *Now we show that $Mk<1$, that is, (d) of Theorem 1.7 holds.*

This is obvious by (9.3.3) since $M=\|B(S)\|=\sup\{\|\mathscr{B}x\|:x\in S\}\leq\frac{2}{\Gamma(\alpha+1)}\|p\|$ and $k=\|\phi\|$.

Thus all the conditions of Theorem 1.7 are satisfied and hence the operator equation $x=\mathscr{A}x\mathscr{B}x$ has a solution in S. In consequence, the problem (9.12) has a solution on $[1,e]$. This completes the proof. □

Example 9.3 Consider the boundary value problem

$$\begin{cases} D^{3/2}\left(\dfrac{x(t)}{\sin x+2}\right)=\dfrac{1}{4}\cos x(t), & 1<t<e,\\ x(1)=x(e)=0. \end{cases} \tag{9.20}$$

Here $f(t,x)=\sin x+2, g(t,x)=\frac{1}{4}\cos x$. Clearly (9.3.1) and (9.3.2) hold with $\phi(t)=1$ and $p(t)=\frac{1}{4}, \Omega(r)=1$ respectively. Since $\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r)=\frac{2}{3\sqrt{\pi}}<1$, the problem (9.20) has a solution on $[1,e]$ by Theorem 9.3.

Theorem 9.4 *Assume that (9.3.2) and the following conditions hold:*

(9.4.1) *the functions $f,g:[1,e]\times\mathbb{R}\to\mathbb{R}$ are continuous and there exist bounded functions ϕ and ψ with bounds $\|\phi\|$ and $\|\psi\|$ such that $\phi(t)>0,\psi(t)>0$ for $t\in[1,e]$ and $|f(t,x(t))-f(t,y(t))|\leq\phi(t)|x(t)-y(t)|$, $|h(t,x(t))-h(t,y(t))|\leq\psi(t)|x(t)-y(t)|$, for $t\in[1,e]$ and for all $x,\ y\in\mathbb{R}$;*

(9.4.2) there exists a number $r > 0$ *such that*

$$r \geq \frac{2(F_0\Gamma(\beta+1)+H_0)\|p\|\Omega(r)}{[\Gamma(\alpha+1)\Gamma(\beta+1)-2(\|\phi\|\Gamma(\beta+1)+\|\psi\|)\|p\|\Omega(r)]}, \tag{9.21}$$

where $[\Gamma(\alpha+1)\Gamma(\beta+1)-2(\|\phi\|\Gamma(\beta+1)+\|\psi\|)\|p\|\Omega(r)] > 0$, $F_0 = \sup_{t\in[1,e]} |f(t,0)|$ *and* $H_0 = \sup_{t\in[1,e]} |h(t,0)|$.

Then the problem (9.13) has at least one solution on $[1, e]$.

Proof Setting the operator $\mathscr{A} : \mathscr{E}_1 \to \mathscr{E}_1$ as

$$\mathscr{A}x(t) = f(t,x(t)) + \frac{1}{\Gamma(\beta)}\int_1^t \left(\log\frac{t}{s}\right)^{\beta-1} \frac{h(s,x(s))}{s} ds, \ \ t \in [1,e], \tag{9.22}$$

the proof is similar to that of Theorem 9.3. So, we omit it. □

Example 9.4 Consider the problem (9.13) with $\alpha = 3/2$, $f(t,x) = (|\sin x + x| + 1)/\sqrt{t+3}$, $\beta = 3$, $h(t,x) = (|\tan^{-1} x| + \pi)/\sqrt{1+t}$, $g(t,x) = \cos x/(3+t)$, $1 < t < e$. Then $\phi(t) = 2/\sqrt{t+3}$, $\psi(t) = 1/\sqrt{t+1}$, $p(t) = 1/(3+t)$. With $\|\phi\| = 1$, $\|\psi\| = 1/\sqrt{2}$, $\|p\| = 1/4$, $\Omega(r) = 1$ and

$$\Gamma(\alpha+1)\Gamma(\beta+1) - 2(\|\phi\|\Gamma(\beta+1)+\|\psi\|)\|p\|\Omega(r) \simeq 4.622489,$$

all the conditions of Theorem 9.4 are satisfied. Hence the problem (9.13) with the given data has at least one solution on $[1, e]$.

9.5 Boundary Value Problems for Fractional Hybrid Differential Inclusions of Hadamard Type with Dirichlet Boundary Conditions

In this section, we study a Dirichlet boundary value problem of nonlinear fractional hybrid differential inclusions given by

$$\begin{cases} D^{\alpha}\left(\dfrac{x(t)}{f(t,x(t))}\right) \in F(t,x(t)), \ \ 1 < t < e, \ \ 1 < \alpha \leq 2, \\ \qquad\qquad x(1) = x(e) = 0, \end{cases} \tag{9.23}$$

where D^{α} is the Hadamard fractional derivative, $f \in C([1,e]\times\mathbb{R}, \mathbb{R}\setminus\{0\})$, $F : [1,e]\times\mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map, $\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

Definition 9.1 A function $x \in \mathscr{C}^2([1,e],\mathbb{R})$ is called a solution of the problem (9.23) if there exists a function $v \in L^1([1,e],\mathbb{R})$ with $v(t) \in F(t,x(t))$ a.e. on $[1,e]$ such that $D^\alpha\left(\dfrac{x(t)}{f(t,x(t))}\right) = v(t)$ a.e. on $[1,e]$ and $x(1) = x(e) = 0$.

Theorem 9.5 *Assume that (9.3.1) holds. In addition, we suppose that:*

(9.5.1) $F : [1,e]\times\mathbb{R} \to \mathscr{P}(\mathbb{R})$ *is* L^1*-Carathéodory and has nonempty compact and convex values;*

(9.5.2) there exists a function $p \in C([1,e],\mathbb{R}^+)$ *such that*

$$\|F(t,x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t) \ \textit{for each} \ (t,x) \in [1,e]\times\mathbb{R};$$

(9.5.3) $\dfrac{2\|\phi\|}{\Gamma(\alpha+1)}\|p\| < \dfrac{1}{2}$.

Then the problem (9.23) has at least one solution on $[1,e]$.

Proof We transform the problem (9.23) into a fixed point problem. Consider the operator $\mathscr{N} : \mathscr{E}_1 \to \mathscr{P}(\mathscr{E}_1)$ defined by

$$\mathscr{N}x(t) = \left\{ h \in \mathscr{E}_1 : \ h(t) = f(t,x(t))\left(\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds \right.\right.$$
$$\left.\left. -(\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds\right), \ v \in S_{F,x}\right\}.$$

Next, we introduce two operators $\mathscr{A} : \mathscr{E}_1 \to \mathscr{E}_1$ by

$$\mathscr{A}x(t) = f(t,x(t)), \ t \in [1,e], \tag{9.24}$$

and $\mathscr{B} : \mathscr{E}_1 \to \mathscr{P}(\mathscr{E}_1)$ by

$$\mathscr{B}x(t) = \left\{ h \in \mathscr{E}_1 : h(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds \right.$$
$$\left. -(\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds, \ v \in S_{F,x}\right\}. \tag{9.25}$$

Observe that $\mathscr{N}(x) = \mathscr{A}x\mathscr{B}x$. For the sake of clarity, we split the proof into several steps.

Step 1. $\mathscr{A}$ *is a Lipschitz on* $\mathscr{E}_1$*, i.e., (a) of Theorem 1.8 holds.*

This was proved in Step 1 of Theorem 9.3.

Step 2. *The multivalued operator* $\mathscr{B}$ *is compact and upper semi- continuous on* $\mathscr{E}_1$*, i.e., (b) of Theorem 1.8 holds.*

First, we show that $\mathscr{B}$ has convex values. Let $u_1, u_2 \in \mathscr{B}x$. Then there are $v_1, v_2 \in S_{F,x}$ such that

$$u_i(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_i(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v_i(s)}{s} ds,$$

$i = 1, 2,\ t \in [1, e]$. For any $\theta \in [0, 1]$, we have

$$\begin{aligned}
&\theta u_1(t) + (1-\theta)u_2(t) \\
&= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{[\theta v_1(s) + (1-\theta)v_2(s)]}{s} ds \\
&\quad -(\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{[\theta v_1(s) + (1-\theta)v_2(s)]}{s} ds.
\end{aligned}$$

Since $\theta v_1(t) + (1-\theta)v_2(t) \in F(t, x(t))$ for all $t \in [1, e]$, $\theta u_1(t) + (1-\theta)u_2(t) \in \mathscr{B}x$ and consequently $\mathscr{B}x$ is convex for each $x \in \mathscr{C}_1$. As a result $\mathscr{B}$ defines a multivalued operator $\mathscr{B} : \mathscr{E}_1 \to \mathscr{P}_{cv}(\mathscr{E}_1)$.

Next, we show that $\mathscr{B}$ maps bounded sets into bounded sets in $\mathscr{E}_1$. For that, let Q be a bounded set in $\mathscr{E}_1$. Then there exists a real number $r > 0$ such that $\|x\| \le r, \forall x \in Q$.

Now, for each $h \in \mathscr{B}x$, there exists a $v \in S_{F,x}$ such that

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds.$$

Then for each $t \in [1, e]$, using (9.5.2), we have

$$\begin{aligned}
|h(t)| &= \left| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \right| \\
&\le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{p(s)}{s} ds + (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{p(s)}{s} ds \\
&\le \frac{2}{\Gamma(\alpha+1)} \|p\|.
\end{aligned}$$

This further implies that

$$\|h\| \le \frac{2}{\Gamma(\alpha+1)} \|p\|,$$

and so $\mathscr{B}(\mathscr{E}_1)$ is uniformly bounded.

Next, we show that $\mathscr{B}$ maps bounded sets into equicontinuous sets. Let Q be, as above, a bounded set and $h \in \mathscr{B}x$ for some $x \in Q$. Then there exists a $v \in S_{F,x}$ such that

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds,$$

$t \in [1, e]$. Then, for any $\tau_1, \tau_2 \in [1, e]$ with $\tau_1 < \tau_2$, we have

$$\begin{aligned} |h(\tau_2) - h(\tau_1)| &\leq \frac{\|p\|}{\Gamma(\alpha)} \left| \int_1^{\tau_1} \left(\log \frac{\tau_1}{s}\right)^{\alpha-1} \frac{1}{s} ds - \int_1^{\tau_2} \left(\log \frac{\tau_2}{s}\right)^{\alpha-1} \frac{1}{s} ds \right| \\ &\quad + \frac{\|p\| |(\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1}|}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{1}{s} ds \\ &\leq \frac{\|p\|}{\Gamma(\alpha+1)} \Big[2(\log(\tau_2/\tau_1))^{\alpha} + |(\log \tau_2)^{\alpha} - (\log \tau_1)^{\alpha}| \Big] \\ &\quad + \frac{\|p\| |(\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1}|}{\Gamma(\alpha+1)}. \end{aligned}$$

Obviously the right hand side of the above inequality tends to zero, independently of $x \in Q$ as $\tau_2 - \tau_1 \to 0$. Therefore it follows by the Arzelá-Ascoli Theorem that $\mathscr{B} : \mathscr{E}_1 \to \mathscr{P}(\mathscr{E}_1)$ is completely continuous.

In our next step, we show that $\mathscr{B}_1$ is upper semicontinuous. By Lemma 1.1, $\mathscr{B}_1$ will be upper semicontinuous if we prove that it has a closed graph, since $\mathscr{B}_1$ is already shown to be completely continuous.

Thus in our next step, we show that *$\mathscr{B}$ has a closed graph.* Let $x_n \to x_*, h_n \in \mathscr{B}(x_n)$ and $h_n \to h_*$. Then, we need to show that $h_* \in \mathscr{B}$. Associated with $h_n \in \mathscr{B}(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in [1, e]$,

$$h_n(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_n(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v_n(s)}{s} ds.$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [1, e]$,

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_*(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v_*(s)}{s} ds.$$

Let us consider the linear operator $\Theta : L^1([1, e], \mathbb{R}) \to \mathscr{E}_1$ given by

$$\begin{aligned} f \mapsto \Theta(v)(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \\ &\quad - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds. \end{aligned}$$

Observe that

$$\|h_n(t) - h_*(t)\| = \Bigg\| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{(v_n(s) - v_*(s))}{s} ds$$
$$- (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{(v_n(s) - v_*(s))}{s} ds \Bigg\| \to 0,$$

as $n \to \infty$. Thus, it follows by Lemma 1.2 that $\Theta \circ S_{F,x}$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, we have

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_*(s)}{s} ds - (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v_*(s)}{s} ds.$$

for some $v_* \in S_{F,x_*}$.

As a result, we have that $\mathscr{B}$ is compact and upper semicontinuous operator on $\mathscr{E}_1$.

Step 3. *Now, we show that* $2Mk < 1$, *i.e., (c) of Theorem 1.8 holds.*

This is obvious by (9.5.3) as $M = \|B(\mathscr{E}_1)\| = \sup\{|\mathscr{B}x : x \in \mathscr{E}_1\} \le \dfrac{2}{\Gamma(\alpha)}\|p\|$ and $k = \|\phi\|$.

Thus all the conditions of Theorem 1.8 are satisfied and hence its direct application implies that either the conclusion (i) or the conclusion (ii) holds. We show that the conclusion (ii) is not possible.

Let $\bar{\mathscr{E}} = \{u \in \mathscr{C}_1 | \lambda u \in \mathscr{A} u \mathscr{B} u,\ \lambda > 1\}$ and $u \in \bar{\mathscr{E}}$ be arbitrary. Then, we have for $\lambda > 1,\ \lambda u \in \mathscr{A} u \mathscr{B} u$. Then there exists $v \in S_{F,x}$ such that for any $\lambda > 1$, we have

$$u(t) = \lambda^{-1}[f(t, u(t)]\Bigg(\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds$$
$$- (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \Bigg),$$

for all $t \in [1, e]$. Then, we have

$$|u(t)| \le \lambda^{-1}|f(t, u(t)|\Bigg(\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{|v(s)|}{s} ds$$
$$+ (\log t)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{|v(s)|}{s} ds \Bigg)$$

$$\leq [|f(t,u(t)-f(t,0)| + |f(t,0)|]\left(\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|v(s)|}{s}ds\right.$$

$$\left.+(\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{|v(s)|}{s}ds\right)$$

$$\leq [\|\phi\|\|u\| + F_0]\frac{2}{\Gamma(\alpha+1)}\|p\|, \quad F_0 = \sup_{t\in[1,e]} |F(t,0)|,$$

which yields

$$\|u\| \leq \frac{\dfrac{2F_0}{\Gamma(\alpha+1)}\|p\|}{1-\dfrac{2\|\phi\|}{\Gamma(\alpha+1)}\|p\|} := M.$$

Thus the condition (ii) of Theorem 1.8 does not hold since $\dfrac{2\|\phi\|}{\Gamma(\alpha+1)}\|p\| < \dfrac{1}{2}$. Therefore the operator equation $x = \mathscr{A}x\mathscr{B}x$ and consequently the problem (9.23) has at least one solution on $[1,e]$. This completes the proof. □

Example 9.5 Consider the boundary value problem

$$\begin{cases} D^{3/2}\left[\dfrac{x(t)}{\frac{1}{12}e^{1-t}|\arctan x|+2}\right] \in F(t,x(t)), \quad 1<t<e, \\ x(1)=x(e)=0, \end{cases} \tag{9.26}$$

where $F : [1,e]\times\mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map given by

$$t \to F(t,x) = \left[\frac{|x|^3}{10(|x|^3+3)}, \frac{|\sin x|}{9(|\sin x|+1)} + \frac{8}{9}\right].$$

By the condition (9.5.1), $\phi(t) = e^{1-t}/12$ with $\|\phi\| = 1/12$. For $\tilde{f} \in F$, we have

$$|\tilde{f}| \leq \max\left(\frac{|x|^3}{10(|x|^3+3)}, \frac{|\sin x|}{9(|\sin x|+1)} + \frac{8}{9}\right) \leq 1, \ x \in \mathbb{R}$$

and

$$\|F(t,x)\| = \sup\{|y| : y \in F(t,x)\} \leq 1 = p(t), \quad x \in \mathbb{R}.$$

Clearly

$$\frac{2\|\phi\|\|p\|}{\Gamma(\alpha+1)} = \frac{2}{9\sqrt{\pi}} < 1/2.$$

Hence all the conditions of Theorem 9.5 are satisfied and accordingly, the problem (9.26) has a solution on $[1, e]$.

9.6 Boundary Value Problems for Hybrid Hadamard Fractional Differential Equations and Inclusions with Nonlocal Conditions

In this section, we study the existence of solutions for boundary value problems of hybrid fractional differential equations and inclusions of Hadamard type with nonlocal conditions. As a first problem, we consider

$$\begin{cases} D^{\alpha}\left(\dfrac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \quad 1 \leq t \leq e, \quad 1 < \alpha \leq 2, \\ \qquad x(1) = 0, \quad x(e) = m(x), \end{cases} \tag{9.27}$$

where D^{α} is the Hadamard fractional derivative, $f \in C([1,e] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $g : C([1,e] \times \mathbb{R}, \mathbb{R})$ and $m : C([1,e], \mathbb{R}) \to \mathbb{R}$.

In the second problem, we study the multivalued case of the problem (9.27) given by

$$\begin{cases} D^{\alpha}\left(\dfrac{x(t)}{f(t,x(t))}\right) \in F(t,x(t)), \quad 1 \leq t \leq e, \quad 1 < \alpha \leq 2, \\ \qquad x(1) = 0, \quad x(e) = m(x), \end{cases} \tag{9.28}$$

where $F : [1,e] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map, $\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

9.6.1 Existence Results: The Single Valued Case

Lemma 9.3 *Given $y \in C([1,e], \mathbb{R})$, x is a solution of the boundary value problem*

$$\begin{cases} D^{\alpha}\left(\dfrac{x(t)}{f(t,x(t))}\right) = y(t), \quad 0 < t < 1, \\ x(1) = 0, \quad x(e) = m(x) \end{cases} \tag{9.29}$$

if and only if

$$x(t) = f(t, x(t))\left(\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds\right.$$
$$\left.+(\log t)^{\alpha-1}\left[\frac{m(x)}{f(e, m(x))} - \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds\right]\right), \quad t\in[1,e].$$

Proof As before, the solution of Hadamard differential equation in (9.29) can be written as

$$x(t) = f(t, x(t))\left(\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds + c_1(\log t)^{\alpha-1} + c_2(\log t)^{\alpha-2}\right), \tag{9.30}$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants. Using the boundary conditions given in (9.29), we find that

$$c_2 = 0, \quad c_1 = \frac{m(x)}{f(e, m(x))} - \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds.$$

Substituting the values of c_1, c_2 in (9.30), we get

$$x(t) = f(t, x(t))\left(\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds\right.$$
$$\left.+(\log t)^{\alpha-1}\left[\frac{m(x)}{f(e, m(x))} - \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{y(s)}{s}ds\right]\right), \quad t\in[1,e].$$

The converse follows by direct computation. The proof is completed. □

Theorem 9.6 *Assume that (9.3.1) and (9.3.2) hold. In addition, we suppose that:*

(9.6.1) *there exists a constant* $M_1 > 0$ *such that* $\left|\frac{m(x)}{f(e, m(x))}\right| \le M_1$;

(9.6.2) *there exists a number* $r > 0$ *such that*

$$r \ge \frac{F_0\left[\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r) + M_1\right]}{1 - \|\phi\|\left[\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r) + M_1\right]}, \tag{9.31}$$

where

$$\|\phi\|\left[\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r)+M_1\right]<1.$$

Then the problem (9.27) has at least one solution on $[1,e]$.

Proof Set $\mathscr{E}_1 = C([1,e],\mathbb{R})$ and define a subset S of $\mathscr{E}_1$ as follows:

$$S=\{x\in\mathscr{E}_1 : \|x\|\le r\},$$

where r satisfies the inequality (9.31).

Clearly S is closed, convex and bounded subset of the Banach space $\mathscr{E}_1$. By Lemma 9.3, the problem (9.27) is equivalent to the integral equation

$$\begin{aligned}x(t) &= f(t,x(t))\left(\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s,x(s))}{s}ds\right.\\ &\left.+(\log t)^{\alpha-1}\left[\frac{m(x)}{f(e,m(x))}-\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{g(s,x(s))}{s}ds\right]\right),\quad t\in[1,e].\end{aligned} \tag{9.32}$$

Define two operators $\mathscr{A}:\mathscr{E}_1\to\mathscr{E}_1$ by

$$\mathscr{A}x(t)=f(t,x(t)),\ \ t\in[1,e], \tag{9.33}$$

and $\mathscr{B}:S\to\mathscr{E}_1$ by

$$\begin{aligned}\mathscr{B}x(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s,x(s))}{s}ds\\ &+(\log t)^{\alpha-1}\left[\frac{m(x)}{f(e,m(x))}-\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{g(s,x(s))}{s}ds\right],\quad t\in[1,e].\end{aligned} \tag{9.34}$$

Then $x=\mathscr{A}x\mathscr{B}x$. We complete the proof in a series of steps.

Step 1. *We first show that $\mathscr{A}$ is Lipschitz on $\mathscr{E}_1$, i.e., (a) of Theorem 1.7 holds.*

This was proved in Step 1 of Theorem 9.3.

Step 2. *The operator $\mathscr{B}$ is completely continuous on S, i.e., (b) of Theorem 1.7 holds.*

First, we show that $\mathscr{B}$ is continuous on S. Let $\{x_n\}$ be a sequence in S converging to a point $x\in S$. Then by Lebesgue dominated convergence theorem,

$$\begin{aligned}\lim_{n\to\infty}\mathscr{B}x_n(t) &= \lim_{n\to\infty}\left(\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\frac{g(s,x_n(s))}{s}ds\right.\\ &\left.+(\log t)^{\alpha-1}\left[\frac{m(x)}{f(e,m(x))}-\frac{1}{\Gamma(\alpha)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-1}\frac{g(s,x_n(s))}{s}ds\right]\right)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{\lim\limits_{n\to\infty} g(s, x_n(s))}{s} ds \\
&\quad + (\log t)^{\alpha-1} \left[\frac{m(x)}{f(e, m(x))} - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{\lim\limits_{n\to\infty} g(s, x_n(s))}{s} ds \right] \\
&= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s, x(s))}{s} ds \\
&\quad + (\log t)^{\alpha-1} \left[\frac{m(x)}{f(e, m(x))} - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{g(s, x(s))}{s} ds \right] \\
&= \mathscr{B}x(t),
\end{aligned}
$$

for all $t \in [1, e]$. This shows that $\mathscr{B}$ is continuous os S. It is enough to show that $\mathscr{B}(S)$ is a uniformly bounded and equicontinuous set in $\mathscr{C}_1$. First, we note that

$$
\begin{aligned}
|\mathscr{B}x(t)| &= \left| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{g(s, x(s))}{s} ds \right. \\
&\quad \left. + (\log t)^{\alpha-1} \left[\frac{m(x)}{f(e, m(x))} - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{g(s, x(s))}{s} ds \right] \right| \\
&\le \frac{\|p\| \Omega(r)}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{1}{s} ds + M_1 + \frac{\|p\| \Omega(r)}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{1}{s} ds \\
&= \frac{2}{\Gamma(\alpha+1)} \|p\| \Omega(r) + M_1,
\end{aligned}
$$

for all $t \in [1, e]$. Taking supremum over the interval $[1, e]$, the above inequality becomes

$$
\|\mathscr{B}x\| \le \frac{2}{\Gamma(\alpha+1)} \|p\| \Omega(r) + M_1,
$$

for all $x \in S$. This shows that $\mathscr{B}$ is uniformly bounded on S.

Next, we show that $\mathscr{B}$ is an equicontinuous set in $\mathscr{C}_1$. Let $\tau_1, \tau_2 \in [1, e]$ with $\tau_1 < \tau_2$ and $x \in S$. Then, we have

$$
\begin{aligned}
&|(\mathscr{B}x)(\tau_2) - (\mathscr{B}x)(\tau_1)| \\
&\le \frac{\|p\| \Omega(r)}{\Gamma(\alpha)} \left| \int_1^{\tau_2} \left(\log \frac{\tau_2}{s}\right)^{\alpha-1} \frac{1}{s} ds - \int_1^{\tau_1} \left(\log \frac{\tau_1}{s}\right)^{\alpha-1} \frac{1}{s} ds \right| \\
&\quad + \frac{\|p\| \Omega(r) |(\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1}|}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{1}{s} ds
\end{aligned}
$$

$$\leq \frac{\|p\|\Omega(r)}{\Gamma(\alpha+1)}\Big[2(\log(\tau_2/\tau_1))^{\alpha} + |(\log \tau_2)^{\alpha} - (\log \tau_1)^{\alpha}|\Big] + \frac{\|p\|\Omega(r)|(\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1}|}{\Gamma(\alpha+1)}.$$

Obviously the right hand side of the above inequality tends to zero, independently of $x \in S$ as $\tau_2 - \tau_1 \to 0$. Therefore, it follows from the Arzelá-Ascoli Theorem that $\mathscr{B}$ is a completely continuous operator on S.

Step 3. *Here we show that hypothesis (c) of Theorem 1.7 is satisfied*. Let $x \in \mathscr{E}_1$ and $y \in S$ be arbitrary elements such that $x = \mathscr{A}x\mathscr{B}y$. Then, we have

$$\begin{aligned}
|x(t)| &= |\mathscr{A}x(t)||\mathscr{B}y(t)| \\
&= |f(t,x(t))|\Bigg|\Bigg(\frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{g(s,y(s))}{s}ds \\
&\quad +(\log t)^{\alpha-1}\Bigg[\frac{m(y)}{f(e,m(y))} - \frac{1}{\Gamma(\alpha)}\int_1^e \Big(\log\frac{e}{s}\Big)^{\alpha-1}\frac{g(s,y(s))}{s}ds\Bigg]\Bigg)\Bigg| \\
&\leq [|f(t,x(t)) - f(t,0)| + |f(t,0)|] \times \Bigg|\Bigg(\frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{g(s,y(s))}{s}ds \\
&\quad +(\log t)^{\alpha-1}\Bigg[\frac{m(y)}{f(e,m(y))} - \frac{1}{\Gamma(\alpha)}\int_1^e \Big(\log\frac{e}{s}\Big)^{\alpha-1}\frac{g(s,y(s))}{s}ds\Bigg]\Bigg)\Bigg| \\
&\leq [\phi(t)|x(t)| + F_0]\Bigg[M_1 + \frac{\|p\|\Omega(r)}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{1}{s}ds \\
&\quad +\frac{\|p\|\Omega(r)}{\Gamma(\alpha)}\int_1^e \Big(\log\frac{e}{s}\Big)^{\alpha-1}\frac{1}{s}ds\Bigg] \\
&\leq [\|\phi\||x(t)| + F_0]\Bigg[\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r) + M_1\Bigg], \quad F_0 = \sup_{t\in[1,e]}|f(t,0)|.
\end{aligned}$$

Thus

$$|x(t)| \leq \|\phi\||x(t)|\Bigg[\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r) + M_1\Bigg] + F_0\Bigg[\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r) + M_1\Bigg],$$

which, on taking supremum for $t \in [1, e]$, yields

$$\|x\| \leq \frac{F_0\left[\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r) + M_1\right]}{1 - \|\phi\|\left[\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r) + M_1\right]} \leq r,$$

that is, $x \in S$.

Step 4. *Now, we show that $Mk < 1$, i.e., (d) of Theorem 1.7 holds.*

This is obvious by (9.6.3) in view of $k = \|\phi\|$ and $M = \|B(S)\| = \sup\{\|\mathscr{B}x\| : x \in S\} \leq \frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r) + M_1$.

Thus all the conditions of Theorem 1.7 are satisfied and hence the operator equation $x = \mathscr{A}x\mathscr{B}x$ has a solution in S. In consequence, the problem (9.27) has a solution on $[1, e]$. This completes the proof. □

Example 9.6 Consider the boundary value problem

$$\begin{cases} D^{3/2}\left(\dfrac{x(t)}{|x(t)\sin t| + 1}\right) = \dfrac{1}{4}\cos x(t), \quad 1 < t < e, \\ \\ x(1) = 0, \quad x(e) = \dfrac{1}{16}\sin x(\eta), \quad \eta \in (0, 1). \end{cases} \tag{9.35}$$

Let $f(t, x) = |x \sin t| + 1, g(t, x) = \frac{1}{4}\cos x$. Then (9.3.1) and (9.3.2) hold with $\phi(t) = 1$ and $p(t) = \frac{1}{4}, \Omega(r) = 1$ respectively. Since $\frac{2}{\Gamma(\alpha+1)}\|p\|\Omega(r) + M_1 = \frac{2}{3\sqrt{\pi}} + \frac{1}{16} < 1$, the problem (9.35) has a solution on $[1, e]$ by Theorem 9.6.

9.6.2 Existence Result: The Multivalued Case

Definition 9.2 A function $x \in \mathscr{C}^2([1, e], \mathbb{R})$ is called a solution of the problem (9.28) if there exists a function $v \in L^1([1, e], \mathbb{R})$ with $v(t) \in F(t, x(t))$, a.e. on $[1, e]$ such that $D^\alpha\left(\frac{x(t)}{f(t, x(t))}\right) = v(t)$, a.e. on $[1, e]$ and $x(1) = 0,\ x(e) = m(x)$.

Theorem 9.7 *Assume that (9.3.1), (9.6.1) and the following conditions hold:*

(9.7.1) $F : [1, e] \times \mathbb{R} \to \mathscr{P}_{cp,cv}(\mathbb{R})$ *is L^1-Carathéodory multivalued map;*

(9.7.2) there exists a continuous function $\zeta \in C([1,e],\mathbb{R}^+)$ *such that*

$$\|F(t,x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t,x)\} \le \zeta(t) \ \text{for each} \ (t,x) \in [1,e]\times\mathbb{R};$$

(9.7.3) $\|\phi\|\left[\dfrac{2}{\Gamma(\alpha+1)}\|\zeta\| + M_1\right] < \dfrac{1}{2}.$

Then the problem (9.28) has at least one solution on $[1,e]$.

Proof To transform the problem (9.28) into a fixed point problem, define an operator $\mathscr{F} : \mathscr{E}_1 \to \mathscr{P}(\mathscr{E}_1)$ as

$$\mathscr{F}(x) = \left\{ \begin{array}{l} h \in \mathscr{E}_1 : \\ h(t) = \left\{ \begin{array}{l} f(t,x(t))\left(\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds \right. \\ \left. +(\log t)^{\alpha-1}\left[\dfrac{m(x)}{f(e,m(x))} - \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds\right]\right), \end{array} \right. \end{array} \right\}$$

for $v \in S_{F,x}$. Now, we define two operators $\mathscr{A} : \mathscr{E}_1 \to \mathscr{E}_1$ by

$$\mathscr{A}x(t) = f(t,x(t)), \ t \in [1,e], \tag{9.36}$$

and $\mathscr{B} : \mathscr{E}_1 \to \mathscr{P}(\mathscr{E}_1)$ by

$$\mathscr{B}(x) = \left\{ \begin{array}{l} h \in \mathscr{E}_1 : \\ h(t) = \left\{ \begin{array}{l} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds \\ +(\log t)^{\alpha-1}\left[\dfrac{m(x)}{f(e,m(x))} - \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds\right]. \end{array} \right. \end{array} \right\} \tag{9.37}$$

Observe that $\mathscr{F}(x) = \mathscr{A}x\mathscr{B}x$. For the sake of clarity, we split the proof into different steps.

Step 1. *$\mathscr{A}$ is Lipschitz on $\mathscr{E}_1$ (see Step 1 of Theorem 9.6), so (a) of Theorem 1.8 holds.*

Step 2. *The multivalued operator $\mathscr{B}$ is compact and upper semicontinuous on $\mathscr{E}_1$, i.e., (b) of Theorem 1.8 holds.*

First, we show that $\mathscr{B}$ has convex values. Let $u_1, u_2 \in \mathscr{B}x$. Then there are $v_1, v_2 \in S_{F,x}$ such that

$$u_i(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v_i(s)}{s}ds$$
$$+(\log t)^{\alpha-1}\left[\frac{m(x)}{f(e,m(x))} - \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v_i(s)}{s}ds\right],$$

$i = 1, 2,\ t \in [1, e]$. For any $\theta \in [0, 1]$, we have

$$\begin{aligned} &\theta u_1(t) + (1-\theta)u_2(t) \\ &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{[\theta v_1(s) + (1-\theta)v_2(s)]}{s} ds \\ &\quad + (\log t)^{\alpha-1}\left[\frac{m(x)}{f(e, m(x))} - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{[\theta v_1(s) + (1-\theta)v_2(s)]}{s} ds\right]. \end{aligned}$$

Since $\theta v_1(t) + (1-\theta)v_2(t) \in F(t, x(t))$ for all $t \in [1, e]$, therefore $\theta u_1(t) + (1-\theta)u_2(t) \in \mathscr{B}x$ and consequently $\mathscr{B}x$ is convex for each $x \in \mathscr{C}_1$. As a result $\mathscr{B}$ defines a multivalued operator $\mathscr{B} : \mathscr{E}_1 \to \mathscr{P}_{cv}(\mathscr{E}_1)$.

Next, we show that $\mathscr{B}$ maps bounded sets into bounded sets in $\mathscr{E}_1$. To do so, let Q be a bounded set in $\mathscr{E}_1$. Then there exists a real number $r > 0$ such that $\|x\| \le r, \forall x \in Q$.

Now for each $h \in \mathscr{B}x$, there exists a $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) = &\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \\ &+ (\log t)^{\alpha-1}\left[\frac{m(x)}{f(e, m(x))} - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds\right]. \end{aligned}$$

Then, for each $t \in [1, e]$, we have

$$\begin{aligned} |h(t)| = &\left| \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \right. \\ &\left. + (\log t)^{\alpha-1}\left[\frac{m(x)}{f(e, m(x))} - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds\right] \right| \\ \le &\frac{2}{\Gamma(\alpha+1)} \|\zeta\| + M_1. \end{aligned}$$

This further implies that

$$\|h\| \le \frac{2}{\Gamma(\alpha+1)} \|\zeta\| + M_1,$$

and so $\mathscr{B}(\mathscr{E}_1)$ is uniformly bounded.

Next, we show that $\mathscr{B}$ maps bounded sets into equicontinuous sets. Let Q be, as above, a bounded set and $h \in \mathscr{B}x$ for some $x \in Q$. Then there exists a $v \in S_{F,x}$ such that

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds$$
$$+ (\log t)^{\alpha-1} \left[\frac{m(x)}{f(e, m(x))} - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v(s)}{s} ds \right], \ t \in [1, e].$$

Then, for any $\tau_1, \tau_2 \in [1, e]$ with $\tau_1 < \tau_2$, we have

$$|h(\tau_2) - h(\tau_1)| \leq \frac{\|\zeta\|}{\Gamma(\alpha)} \left| \int_1^{\tau_2} \left(\log \frac{\tau_2}{s}\right)^{\alpha-1} \frac{1}{s} ds - \int_1^{\tau_1} \left(\log \frac{\tau_1}{s}\right)^{\alpha-1} \frac{1}{s} ds \right|$$
$$+ \frac{\|\zeta\| |(\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1}|}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{1}{s} ds$$
$$\leq \frac{\|\zeta\|}{\Gamma(\alpha+1)} \left[2(\log(\tau_2/\tau_1))^{\alpha} + |(\log \tau_2)^{\alpha} - (\log \tau_1)^{\alpha}| \right]$$
$$+ \frac{\|\zeta\| |(\log \tau_2)^{\alpha-1} - (\log \tau_1)^{\alpha-1}|}{\Gamma(\alpha+1)}.$$

Obviously the right hand side of the above inequality tends to zero, independently of $x \in Q$ as $\tau_2 - \tau_1 \to 0$. Therefore it follows by the Arzelá-Ascoli Theorem that $\mathscr{B} : \mathscr{E}_1 \to \mathscr{P}(\mathscr{E}_1)$ is completely continuous.

In our next step, we show that $\mathscr{B}$ is upper semicontinuous. By Lemma 1.1, $\mathscr{B}$ will be upper semicontinuous if we establish that it has a closed graph. Since $\mathscr{B}$ is already shown to be completely continuous, we just need to prove that $\mathscr{B}$ *has a closed graph.*

Let $x_n \to x_*, h_n \in \mathscr{B}(x_n)$ and $h_n \to h_*$. Then, we need to show that $h_* \in \mathscr{B}$. Associated with $h_n \in \mathscr{B}(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in [1, e]$,

$$h_n(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_n(s)}{s} ds$$
$$+ (\log t)^{\alpha-1} \left[\frac{m(x)}{f(e, m(x))} - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v_n(s)}{s} ds \right].$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [1, e]$,

$$h_*(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{v_*(s)}{s} ds$$
$$+ (\log t)^{\alpha-1} \left[\frac{m(x)}{f(e, m(x))} - \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{v_*(s)}{s} ds \right].$$

Let us consider the linear operator $\Theta : L^1([1,e],\mathbb{R}) \to \mathscr{E}_1$ given by

$$f \mapsto \Theta(v)(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds + (\log t)^{\alpha-1}\left[\frac{m(x)}{f(e,m(x))} - \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds\right].$$

Observe that

$$\|h_n(t) - h_*(t)\| = \left\| \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{(v_n(s)-v_*(s))}{s}ds - (\log t)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{(v_n(s)-v_*(s))}{s}ds \right\| \to 0,$$

as $n \to \infty$. Thus, it follows by Lemma 1.2 that $\Theta \circ S_{F,x}$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v_*(s)}{s}ds + (\log t)^{\alpha-1}\left[\frac{m(x)}{f(e,m(x))} - \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v_*(s)}{s}ds\right],$$

for some $v_* \in S_{F,x_*}$.

In consequence, we have that the operator $\mathscr{B}$ is compact and upper semicontinuous operator on $\mathscr{E}_1$.

Step 3. *Now, we show that* $2Mk < 1$, *i.e., (c) of Theorem 1.8 holds.*

This is obvious by (9.7.3) as $k = \|\phi\|$ and $M = \|B(\mathscr{E}_1)\| = \sup\{|\mathscr{B}x : x \in \mathscr{E}_1\} \leq \dfrac{2}{\Gamma(\alpha+1)}\|\zeta\| + M_1$.

Thus all the conditions of Theorem 1.8 are satisfied and hence its direct application implies that either the conclusion (i) or the conclusion (ii) holds. We show that the conclusion (ii) is not possible.

Let $\bar{\mathscr{E}} = \{u \in \mathscr{C}_1 | \lambda u \in \mathscr{A}u\mathscr{B}u,\ \lambda > 1\}$ and $u \in \bar{\mathscr{E}}$. Then, for $\lambda > 1$, we have $\lambda u \in \mathscr{A}u\mathscr{B}u$. Then there exists $v \in S_{F,x}$ such that for any $\lambda > 1$, we have

$$u(t) = \lambda^{-1}[f(t,u(t))]\Bigg(\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds + (\log t)^{\alpha-1}\left[\frac{m(u)}{f(e,m(u))} - \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{v(s)}{s}ds\right],$$

for all $t \in [1, e]$. Then, we have

$$\begin{aligned}
|u(t)| &\le \lambda^{-1}|f(t,u(t))|\Bigg(\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|v(s)|}{s}ds \\
&\quad +(\log t)^{\alpha-1}\left[\left|\frac{m(u)}{f(e,m(u))}\right| + \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{|v(s)|}{s}ds\right]\Bigg) \\
&\le [|f(t,u(t) - f(t,0)| + |f(t,0)|]\Bigg(\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\frac{|v(s)|}{s}ds \\
&\quad +(\log t)^{\alpha-1}\left[\left|\frac{m(u)}{f(e,m(u))}\right| + \frac{1}{\Gamma(\alpha)}\int_1^e \left(\log\frac{e}{s}\right)^{\alpha-1}\frac{|v(s)|}{s}ds\right]\Bigg) \\
&\le [\|\phi\|\|u\| + F_0]\left[\frac{2}{\Gamma(\alpha+1)}\|\zeta\| + M_1\right], \quad F_0 = \sup_{t\in[1,e]}|F(t,0)|.
\end{aligned}$$

Thus we obtain

$$\|u\| \le \frac{F_0\left[\dfrac{2}{\Gamma(\alpha+1)}\|\zeta\| + M_1\right]}{1 - \|\phi\|\left[\dfrac{2}{\Gamma(\alpha+1)}\|\zeta\| + M_1\right]} := M,$$

which implies that the condition (ii) of Theorem 1.8 does not hold as $\|\phi\|\left[\frac{2}{\Gamma(\alpha+1)}\|\zeta\| + M_1\right] < \frac{1}{2}$. Therefore the operator equation $x = \mathscr{A}x\mathscr{B}x$ and consequently problem (9.28) has a solution on $[1, e]$. This completes the proof. □

Example 9.7 Consider the boundary value problem

$$\begin{cases} D^{3/2}\left[\dfrac{x(t)}{\frac{1}{12}e^{1-t}\tan^{-1}x + 2}\right] \in F(t,x(t)), \quad 1 < t < e, \\ x(1) = 0, \quad x(e) = \dfrac{1}{16}\sin x(\eta), \quad 0 < \eta < 1, \end{cases} \tag{9.38}$$

where $F : [1, e] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map given by

$$t \to F(t,x) = \left[\frac{|x|^3}{10(|x|^3+3)}, \frac{|\sin x|}{3(|\sin x|+1)} + \frac{1}{3}\right].$$

By the condition (9.3.1), $\phi(t) = e^{1-t}/12$ with $\|\phi\| = 1/12$. For $\tilde{f} \in F$, we have

$$|\tilde{f}| \le \max\left(\frac{|x|^3}{10(|x|^3+3)}, \frac{|\sin x|}{3(|\sin x|+1)} + \frac{1}{3}\right) \le \frac{2}{3}, \;\; x \in \mathbb{R}$$

and

$$\|F(t,x)\| = \sup\{|y| : y \in F(t,x)\} \le \frac{2}{3} = \zeta(t), \;\; x \in \mathbb{R}.$$

Clearly $\|\phi\|\left[\frac{2}{\Gamma(\alpha+1)}\|\zeta\| + M_1\right] = \frac{1}{12}\left[\frac{16}{9\sqrt{\pi}} + \frac{1}{16}\right] \simeq 0.088131 < 1/2$. Hence all the conditions of Theorem 9.7 are satisfied and accordingly, the problem (9.38) has a solution on $[1, e]$.

9.7 Notes and Remarks

Existence results for initial and boundary value problems of hybrid fractional differential equations and inclusions of Hadamard type were studied in this chapter, via fixed point theorems involving the product of two operators. The content of this chapter is based on the papers [10, 14, 16, 21, 22] and [18].

Chapter 10
Positive Solutions for Hadamard Fractional Differential Equations on Infinite Domain

10.1 Introduction

Boundary value problems on semi-infinite/infinite intervals often appear in applied mathematics and physics. Examples include unsteady flow of gas through a semi-infinite porous medium, the drain flow problems, etc. More details and works concerning the existence of solutions for boundary value problems on infinite intervals for differential, difference and integral equations may be found in the monographs [2, 132]. For details on fractional order boundary value problems on infinite domain, we refer the reader to a series of papers [110, 163, 180–182, 184, 185].

10.2 Positive Solutions for Hadamard Fractional Differential Equations on Semi-Infinite Domain

In this section, we aim to investigate the existence criteria for positive solutions of fractional differential equations of Hadamard type with integral boundary condition on semi-infinite intervals. Precisely, we consider the following boundary value problem:

$$D^{\alpha}u(t) + a(t)f(u(t)) = 0, \quad 1 < \alpha \leq 2, \ t \in (1, \infty), \tag{10.1}$$

$$u(1) = 0, \qquad D^{\alpha-1}u(\infty) = \sum_{i=1}^{m} \lambda_i I^{\beta_i} u(\eta), \tag{10.2}$$

where D^{α} denotes the Hadamard fractional derivative of order α, $\eta \in (1, \infty)$ and I^{β_i} is the Hadamard fractional integral of order $\beta_i > 0$, $i = 1, 2, \ldots, m$ and $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$ are given constants.

B. Ahmad et al., *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, DOI 10.1007/978-3-319-52141-1_10

10.3 Auxiliary Results

Lemma 10.1 *Let* $h \in C[1, \infty)$ *with* $0 < \int_1^\infty h(s)\frac{ds}{s} < \infty$, *and*

$$\Omega = \Gamma(\alpha) - \sum_{i=1}^{m} \frac{\lambda_i \Gamma(\alpha)}{\Gamma(\alpha + \beta_i)} (\log \eta)^{\alpha+\beta_i-1} > 0. \tag{10.3}$$

Then the unique solution of the following fractional differential equation

$$D^\alpha u(t) + h(t) = 0, \quad t \in (1, \infty), \quad \alpha \in (1, 2), \tag{10.4}$$

subject to the boundary conditions

$$u(1) = 0, \qquad D^{\alpha-1} u(\infty) = \sum_{i=1}^{m} \lambda_i I^{\beta_i} u(\eta), \tag{10.5}$$

is given by the integral equation

$$u(t) = \int_1^\infty G(t, s) h(s) \frac{ds}{s}, \tag{10.6}$$

where

$$G(t, s) = g(t, s) + \sum_{i=1}^{m} \frac{\lambda_i (\log t)^{\alpha-1}}{\Omega \Gamma(\alpha + \beta_i)} g_i(\eta, s), \tag{10.7}$$

and

$$g(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (\log t)^{\alpha-1} - \left(\log \frac{t}{s}\right)^{\alpha-1}, & 1 \le s \le t < \infty, \\ (\log t)^{\alpha-1}, & 1 \le t \le s < \infty, \end{cases} \tag{10.8}$$

$$g_i(\eta, s) = \begin{cases} (\log \eta)^{\alpha+\beta_i-1} - \left(\log \frac{\eta}{s}\right)^{\alpha+\beta_i-1}, & 1 \le s \le \eta < \infty, \\ (\log \eta)^{\alpha+\beta_i-1}, & 1 \le \eta \le s < \infty. \end{cases} \tag{10.9}$$

Proof Applying the Hadamard fractional integral of order α on both sides of (10.4), we get

$$u(t) = c_1 (\log t)^{\alpha-1} + c_2 (\log t)^{\alpha-2} - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} h(s) \frac{ds}{s}, \tag{10.10}$$

where $c_1, \ c_2 \in \mathbb{R}$.

The first condition of (10.5) implies that $c_2 = 0$. Therefore,

$$u(t) = c_1(\log t)^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} h(s)\frac{ds}{s}. \tag{10.11}$$

In accordance with Lemma 1.6, we have

$$D^{\alpha-1}u(t) = c_1\Gamma(\alpha) - \int_1^t h(s)\frac{ds}{s},$$

which, together with the second condition of (10.5), leads to

$$c_1 = \frac{1}{\Omega}\left(\int_1^\infty h(s)\frac{ds}{s} - \sum_{i=1}^m \frac{\lambda_i}{\Gamma(\alpha+\beta_i)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha+\beta_i-1} h(s)\frac{ds}{s}\right), \tag{10.12}$$

where Ω is defined by (10.3). Inserting the value of c_1 in (10.11), the solution of the problem (10.4)–(10.5) is

$$\begin{aligned}
u(t) &= (\log t)^{\alpha-1}\int_1^\infty \frac{h(s)}{\Gamma(\alpha)}\frac{ds}{s} + \sum_{i=1}^m \frac{\lambda_i(\log t)^{\alpha-1}}{\Omega\Gamma(\alpha+\beta_i)}(\log\eta)^{\alpha+\beta_i-1}\int_1^\infty h(s)\frac{ds}{s} \\
&\quad - \sum_{i=1}^m \frac{\lambda_i(\log t)^{\alpha-1}}{\Omega\Gamma(\alpha+\beta_i)}\int_1^\eta \left(\log\frac{\eta}{s}\right)^{\alpha+\beta_i-1} h(s)\frac{ds}{s} \\
&\quad - \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} h(s)\frac{ds}{s} \\
&= \frac{1}{\Gamma(\alpha)}\int_1^t \left[(\log t)^{\alpha-1} - \left(\log\frac{t}{s}\right)^{\alpha-1}\right] h(s)\frac{ds}{s} + \frac{1}{\Gamma(\alpha)}\int_t^\infty (\log t)^{\alpha-1}h(s)\frac{ds}{s} \\
&\quad + \sum_{i=1}^m \frac{\lambda_i(\log t)^{\alpha-1}}{\Omega\Gamma(\alpha+\beta_i)}\int_1^\eta \left[(\log\eta)^{\alpha+\beta_i-1} - \left(\log\frac{\eta}{s}\right)^{\alpha+\beta_i-1}\right] h(s)\frac{ds}{s} \\
&\quad + \sum_{i=1}^m \frac{\lambda_i(\log t)^{\alpha-1}}{\Omega\Gamma(\alpha+\beta_i)}\int_\eta^\infty (\log\eta)^{\alpha+\beta_i-1}h(s)\frac{ds}{s} \\
&= \int_1^\infty g(t,s)h(s)\frac{ds}{s} + \sum_{i=1}^m \frac{\lambda_i(\log t)^{\alpha-1}}{\Omega\Gamma(\alpha+\beta_i)}\int_1^\infty g_i(\eta,s)h(s)\frac{ds}{s} \\
&= \int_1^\infty G(t,s)h(s)\frac{ds}{s}.
\end{aligned}$$

The proof is completed. □

Lemma 10.2 *The Green's function $G(t,s)$ defined by (10.7) satisfies the following properties:*

(C_1) $G(t,s)$ *is a continuous function for* $(t,s) \in [1,\infty) \times [1,\infty)$*;*
(C_2) $G(t,s) \geq 0$ *for all* $s,t \in [1,\infty)$*;*
(C_3) $\dfrac{G(t,s)}{1+(\log t)^{\alpha-1}} \leq \dfrac{1}{\Gamma(\alpha)} + \sum_{i=1}^{m} \dfrac{\lambda_i g_i(\eta,s)}{\Omega\Gamma(\alpha+\beta_i)}$ *for all* $s,t \in [1,\infty)$*;*
(C_4) $\min_{\eta\leq t\leq k\eta} \dfrac{G(t,s)}{1+(\log t)^{\alpha-1}} \geq \sum_{i=1}^{m} \dfrac{\lambda_i(\log\eta)^{\alpha-1} g_i(\eta,s)}{\Omega\Gamma(\alpha+\beta_i)\,(1+(\log\eta)^{\alpha-1})}$ *for* $k > 1$ *and* $s \in [1,\infty)$.

Proof It is easy to check that (C_1) and (C_2) hold.

To prove (C_3), for $s,t \in [1,\infty)$, we have

$$\begin{aligned}\frac{G(t,s)}{1+(\log t)^{\alpha-1}} &= \frac{g(t,s)}{1+(\log t)^{\alpha-1}} + \sum_{i=1}^{m} \frac{\lambda_i(\log t)^{\alpha-1} g_i(\eta,s)}{\Omega\Gamma(\alpha+\beta_i)\,(1+(\log t)^{\alpha-1})}\\ &\leq \frac{1}{\Gamma(\alpha)} \cdot \frac{(\log t)^{\alpha-1}}{1+(\log t)^{\alpha-1}} + \sum_{i=1}^{m} \frac{\lambda_i(\log t)^{\alpha-1} g_i(\eta,s)}{\Omega\Gamma(\alpha+\beta_i)\,(1+(\log t)^{\alpha-1})}\\ &\leq \frac{1}{\Gamma(\alpha)} + \sum_{i=1}^{m} \frac{\lambda_i g_i(\eta,s)}{\Omega\Gamma(\alpha+\beta_i)}.\end{aligned}$$

To prove (C_4), from $g(t,s) \geq 0$ and $g_i(\eta,s) \geq 0$, $i = 1,2,\ldots,m$ for all $s,t \in [1,\infty)$, $k > 1$, we have

$$\begin{aligned}&\min_{\eta\leq t\leq k\eta} \frac{G(t,s)}{1+(\log t)^{\alpha-1}}\\ &= \min_{\eta\leq t\leq k\eta} \left[\frac{g(t,s)}{1+(\log t)^{\alpha-1}} + \sum_{i=1}^{m} \frac{\lambda_i(\log t)^{\alpha-1} g_i(\eta,s)}{\Omega\Gamma(\alpha+\beta_i)\,(1+(\log t)^{\alpha-1})}\right]\\ &\geq \min_{\eta\leq t\leq k\eta} \frac{g(t,s)}{1+(\log t)^{\alpha-1}} + \min_{\eta\leq t\leq k\eta} \sum_{i=1}^{m} \frac{\lambda_i(\log t)^{\alpha-1} g_i(\eta,s)}{\Omega\Gamma(\alpha+\beta_i)\,(1+(\log t)^{\alpha-1})}\\ &\geq \min_{\eta\leq t\leq k\eta} \sum_{i=1}^{m} \frac{\lambda_i(\log t)^{\alpha-1} g_i(\eta,s)}{\Omega\Gamma(\alpha+\beta_i)\,(1+(\log t)^{\alpha-1})}\\ &\geq \sum_{i=1}^{m} \frac{\lambda_i(\log\eta)^{\alpha-1} g_i(\eta,s)}{\Omega\Gamma(\alpha+\beta_i)\,(1+(\log\eta)^{\alpha-1})},\end{aligned}$$

for $s \in [1,\infty)$. The proof is completed. □

In the forthcoming analysis, we will use the space E defined by

$$E = \left\{ u \in C([1,\infty),\mathbb{R}) : \sup_{t\in[1,\infty)} \frac{|u(t)|}{1+(\log t)^{\alpha-1}} < \infty \right\}$$

equipped with the norm

$$\|u\|_E = \sup_{t\in[1,\infty)} \frac{|u(t)|}{1+(\log t)^{\alpha-1}}.$$

Lemma 10.3 *$(E, \|\cdot\|_E)$ is a Banach space.*

Proof Let $\{u_n\}_{n=1}^{\infty}$ be any Cauchy sequence in the space $(E, \|\cdot\|_E)$. Then, $\forall \varepsilon > 0$, $\exists N > 0$ such that

$$\|u_n - u_m\|_E = \sup_{t\in[1,\infty)} \frac{|u_n(t) - u_m(t)|}{1+(\log t)^{\alpha-1}} < \varepsilon,$$

for $n, m > N$. Therefore, for any fixed $t_0 \in [1,\infty)$, $\{u_n(t_0)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. In this way, we can associate to each $t \in [1,\infty)$ a unique $u(t)$. Letting $n \to \infty$, we have $|u(t) - u_m(t)| \le \varepsilon$ for all $m > N$ and $t \in [1,\infty)$. It is easy to show that $u_m \to u$ in E as $m \to \infty$. Therefore, we deduce that $(E, \|\cdot\|_E)$ is a Banach space. □

Lemma 10.4 *Let $U \subset E$ be a bounded set. Then U is relatively compact in E if the following conditions hold:*

(i) for any $u \in U$, $\dfrac{u(t)}{1+(\log t)^{\alpha-1}}$ is equicontinuous on any compact interval of $[1,\infty)$;
(ii) for any $\varepsilon > 0$, there exists a constant $T = T(\varepsilon) > 0$ such that

$$\left| \frac{u(t_1)}{1+(\log t_1)^{\alpha-1}} - \frac{u(t_2)}{1+(\log t_2)^{\alpha-1}} \right| < \varepsilon, \tag{10.13}$$

for any $t_1, t_2 \ge T$ and $u \in U$.

Proof Evidently, it is sufficient to prove that U is totally bounded. In what follows, we divide the proof into two steps.

Step 1. Let us consider the case $t \in [1, T]$.

Define

$$U_{[1,T]} = \{u(t) : u(t) \in U, t \in [1,T]\}.$$

Clearly $U_{[1,T]}$ equipped with the norm $\|u\|_\infty = \sup_{t\in[1,T]} \dfrac{|u(t)|}{1+(\log t)^{\alpha-1}}$ is a Banach space. The condition (i), combined with the Arzelá-Ascoli Theorem, indicates that

$U_{[1,T]}$ is relatively compact. Hence $U_{[1,T]}$ is totally bounded, namely, for any $\varepsilon > 0$, there exist finitely many balls $B_\varepsilon(u_i)$ such that

$$U_{[1,T]} \subset \bigcup_{i=1}^{n} B_\varepsilon(u_i),$$

where

$$B_\varepsilon(u_i) = \left\{ u(t) \in U_{[1,T]} : \|u - u_i\|_\infty = \sup_{t\in[1,T]} \left| \frac{u(t)}{1 + (\log t)^{\alpha-1}} - \frac{u_i(t)}{1 + (\log t)^{\alpha-1}} \right| < \varepsilon \right\}.$$

Step 2. Define

$$U_i = \left\{ u(t) \in U : u_{[1,T]} \in B_\varepsilon(u_i) \right\}.$$

It is clear that $U_{[1,T]} \subset \bigcup_{1\le i\le n} U_{i[1,T]}$. Now, let us take $u_i \in U_i$ so that U can be covered by the balls $B_{3\varepsilon}(u_i)$, $i = 1, 2, \ldots, n$, where

$$B_{3\varepsilon}(u_i) = \{u(t) \in U : \|u - u_i\|_E < 3\varepsilon\}.$$

In fact, for $u(t) \in U$, the arguments in Step 1 imply that there exist i such that $u_{[1,T]} \in B_\varepsilon(u_i)$. Hence, for $t \in [1, T]$, we have

$$\left| \frac{u(t)}{1 + (\log t)^{\alpha-1}} - \frac{u_i(t)}{1 + (\log t)^{\alpha-1}} \right| < \varepsilon. \tag{10.14}$$

For $t \in [T, +\infty)$, (10.13) and (10.14) yields

$$\begin{aligned}
&\left| \frac{u(t)}{1 + (\log t)^{\alpha-1}} - \frac{u_i(t)}{1 + (\log t)^{\alpha-1}} \right| \\
&\le \left| \frac{u(t)}{1 + (\log t)^{\alpha-1}} - \frac{u(T)}{1 + (\log T)^{\alpha-1}} \right| + \left| \frac{u(T)}{1 + (\log T)^{\alpha-1}} - \frac{u_i(T)}{1 + (\log T)^{\alpha-1}} \right| \\
&\quad + \left| \frac{u_i(T)}{1 + (\log T)^{\alpha-1}} - \frac{u_i(t)}{1 + (\log t)^{\alpha-1}} \right| \\
&< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.
\end{aligned} \tag{10.15}$$

The relations (10.14) and (10.15) show that $\|u(t) - u_i(t)\|_E < 3\varepsilon$. Therefore, U is totally bounded and Lemma 10.4 is proved. □

We define the cone $P \subset E$ by

$$P = \{u \in E : u(t) \ge 0 \text{ on } [1, \infty)\},$$

and the operator $T : P \to E$ by

$$Tu(t) = \int_1^\infty G(t,s)a(s)f(u(s))\frac{ds}{s}, \quad t \in [1,\infty), \tag{10.16}$$

where $G(t,s)$ is defined by (10.7).

Lemma 10.5 *Assume that:*

(10.5.1) $f \in C([0,\infty),[0,\infty))$ *such that* $f(u) \neq 0$ *on any subinterval of* $(0,\infty)$ *and* $f\big((1+(\log t)^{\alpha-1})u\big)$ *is bounded on* $[0,\infty)$;

(10.5.2) $a : [1,\infty) \to [0,\infty)$ *is not identically zero on any closed subinterval of* $[1,\infty)$ *and*

$$0 < \int_1^\infty a(s)\frac{ds}{s} < \infty.$$

Then $T : P \to P$ *is completely continuous.*

Proof We divide the proof into four steps.

Step 1: *We show that* T *is uniformly bounded on* P.

From the definition of E, we can choose r_0 such that $\sup_{n\in\mathbb{N}} \|u_n\|_E < r_0$. If $B_{r_0} = \sup\big\{f((1+(\log t)^{\alpha-1})u), u \in [1,r_0]\big\}$ and Φ is any bounded subset of P, then there exists $r > 0$ such that $\|u\|_E \le r$ for all $u \in \Phi$. From (C_3), it follows that

$$\begin{aligned}
\|Tu\|_E &= \sup_{t\in[1,\infty)} \frac{1}{1+(\log t)^{\alpha-1}}\left|\int_1^\infty G(t,s)a(s)f(u(s))\frac{ds}{s}\right| \\
&\le \int_1^\infty \left(\frac{1}{\Gamma(\alpha)} + \sum_{i=1}^m \frac{\lambda_i g_i(\eta,s)}{\Omega\Gamma(\alpha+\beta_i)}\right) a(s)f(u(s))\frac{ds}{s} \\
&\le \left(\frac{1}{\Gamma(\alpha)} + \sum_{i=1}^m \frac{\lambda_i (\log \eta)^{\alpha+\beta_i-1}}{\Omega\Gamma(\alpha+\beta_i)}\right) B_r \int_1^\infty a(s)\frac{ds}{s} < \infty,
\end{aligned}$$

for $u \in \Phi$. Therefore $T\Phi$ is uniformly bounded.

Step 2: *We show that* T *is equicontinuous on any compact interval of* $[1,\infty)$.

For any $S > 1$, $t_1, t_2 \in [1,S]$ with $t_1 < t_2$ and $u \in \Phi$, we have

$$\begin{aligned}
&\left|\frac{Tu(t_2)}{1+(\log t_2)^{\alpha-1}} - \frac{Tu(t_1)}{1+(\log t_1)^{\alpha-1}}\right| \\
&= \left|\int_1^\infty \frac{G(t_2,s)}{1+(\log t_2)^{\alpha-1}} a(s)f(u(s))\frac{ds}{s} - \int_1^\infty \frac{G(t_1,s)}{1+(\log t_1)^{\alpha-1}} a(s)f(u(s))\frac{ds}{s}\right| \\
&\le \left|\int_1^\infty \left(\frac{g(t_2,s)}{1+(\log t_2)^{\alpha-1}} - \frac{g(t_1,s)}{1+(\log t_1)^{\alpha-1}}\right) a(s)f(u(s))\frac{ds}{s}\right|
\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_1^\infty \left(\frac{(\log t_2)^{\alpha-1}}{1+(\log t_2)^{\alpha-1}}-\frac{(\log t_1)^{\alpha-1}}{1+(\log t_1)^{\alpha-1}}\right)\sum_{i=1}^m \frac{\lambda_i g_i(\eta,s)}{\Omega\Gamma(\alpha+\beta_i)}a(s)f(u(s))\frac{ds}{s}\right|\\
\le &\int_1^\infty \left|\frac{g(t_2,s)}{1+(\log t_2)^{\alpha-1}}-\frac{g(t_1,s)}{1+(\log t_2)^{\alpha-1}}\right|a(s)f(u(s))\frac{ds}{s}\\
&+\int_1^\infty \left|\frac{g(t_1,s)}{1+(\log t_2)^{\alpha-1}}-\frac{g(t_1,s)}{1+(\log t_1)^{\alpha-1}}\right|a(s)f(u(s))\frac{ds}{s}\\
&+\int_1^\infty \left|\frac{(\log t_2)^{\alpha-1}}{1+(\log t_2)^{\alpha-1}}-\frac{(\log t_1)^{\alpha-1}}{1+(\log t_1)^{\alpha-1}}\right|\sum_{i=1}^m \frac{\lambda_i g_i(\eta,s)}{\Omega\Gamma(\alpha+\beta_i)}a(s)f(u(s))\frac{ds}{s}\\
= &\int_1^\infty \left|\frac{g(t_2,s)}{1+(\log t_2)^{\alpha-1}}-\frac{g(t_1,s)}{1+(\log t_2)^{\alpha-1}}\right|a(s)f(u(s))\frac{ds}{s}\\
&+\int_1^\infty \frac{(\log t_2)^{\alpha-1}-(\log t_1)^{\alpha-1}}{(1+(\log t_2)^{\alpha-1})(1+(\log t_1)^{\alpha-1})}g(t_1,s)a(s)f(u(s))\frac{ds}{s}\\
&+\int_1^\infty \frac{(\log t_2)^{\alpha-1}-(\log t_1)^{\alpha-1}}{(1+(\log t_2)^{\alpha-1})(1+(\log t_1)^{\alpha-1})}\sum_{i=1}^m \frac{\lambda_i g_i(\eta,s)}{\Omega\Gamma(\alpha+\beta_i)}a(s)f(u(s))\frac{ds}{s}.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\int_1^\infty \left|\frac{g(t_2,s)}{1+(\log t_2)^{\alpha-1}}-\frac{g(t_1,s)}{1+(\log t_2)^{\alpha-1}}\right|a(s)f(u(s))\frac{ds}{s}\\
= &\int_1^{t_1} \left|\frac{g(t_2,s)}{1+(\log t_2)^{\alpha-1}}-\frac{g(t_1,s)}{1+(\log t_2)^{\alpha-1}}\right|a(s)f(u(s))\frac{ds}{s}\\
&+\int_{t_1}^{t_2} \left|\frac{g(t_2,s)}{1+(\log t_2)^{\alpha-1}}-\frac{g(t_1,s)}{1+(\log t_2)^{\alpha-1}}\right|a(s)f(u(s))\frac{ds}{s}\\
&+\int_{t_2}^\infty \left|\frac{g(t_2,s)}{1+(\log t_2)^{\alpha-1}}-\frac{g(t_1,s)}{1+(\log t_2)^{\alpha-1}}\right|a(s)f(u(s))\frac{ds}{s}\\
\le &\frac{1}{\Gamma(\alpha)}\int_1^{t_1} \frac{(\log t_2)^{\alpha-1}-(\log t_1)^{\alpha-1}+(\log \frac{t_2}{s})^{\alpha-1}-(\log \frac{t_1}{s})^{\alpha-1}}{1+(\log t_2)^{\alpha-1}}a(s)f(u(s))\frac{ds}{s}\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} \frac{(\log t_2)^{\alpha-1}-(\log t_1)^{\alpha-1}+(\log \frac{t_2}{s})^{\alpha-1}}{1+(\log t_2)^{\alpha-1}}a(s)f(u(s))\frac{ds}{s}\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_2}^\infty \frac{(\log t_2)^{\alpha-1}-(\log t_1)^{\alpha-1}}{1+(\log t_2)^{\alpha-1}}a(s)f(u(s))\frac{ds}{s}\\
\to &\ 0 \ \text{ uniformly as } \ t_1\to t_2. \qquad (10.17)
\end{aligned}
$$

Similarly, we have

$$\int_1^{\infty} \frac{(\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}}{(1+(\log t_2)^{\alpha-1})(1+(\log t_1)^{\alpha-1})} g(t_1,s)a(s)f(u(s))\frac{ds}{s} \to 0, \quad (10.18)$$

uniformly as $t_1 \to t_2$, and

$$\int_1^{\infty} \frac{(\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}}{(1+(\log t_2)^{\alpha-1})(1+(\log t_1)^{\alpha-1})} \sum_{i=1}^{m} \frac{\lambda_i g_i(\eta,s)}{\Omega\Gamma(\alpha+\beta_i)} a(s)f(u(s))\frac{ds}{s} \to 0, \quad (10.19)$$

uniformly as $t_1 \to t_2$.

Hence, from (10.17), (10.18) and (10.19), we get

$$\left| \frac{Tu(t_2)}{1+(\log t_2)^{\alpha-1}} - \frac{Tu(t_1)}{1+(\log t_1)^{\alpha-1}} \right| \to 0 \text{ uniformly as } t_1 \to t_2.$$

Thus $T\Phi$ is equicontinuous on any compact interval of $[1,\infty)$.

Step 3: *We show that T is equiconvergent at ∞.*

For any $u \in \Phi$, we have

$$\int_1^{\infty} a(s)f(u(s))\frac{ds}{s} \le B_r \int_1^{\infty} a(s)\frac{ds}{s} < \infty,$$

and

$$\begin{aligned}
&\lim_{t\to\infty} \left| \frac{(Tu)(t)}{1+(\log t)^{\alpha-1}} \right| \\
&= \lim_{t\to\infty} \left| \frac{1}{1+(\log t)^{\alpha-1}} \int_1^{\infty} G(t,s)a(s)f(u(s))\frac{ds}{s} \right| \\
&\le \lim_{t\to\infty} \int_1^{\infty} \left(\frac{1}{\Gamma(\alpha)} \cdot \frac{(\log t)^{\alpha-1}}{1+(\log t)^{\alpha-1}} + \frac{(\log t)^{\alpha-1}}{1+(\log t)^{\alpha-1}} \cdot \sum_{i=1}^{m} \frac{\lambda_i g_i(\eta,s)}{\Omega\Gamma(\alpha+\beta_i)} \right) a(s)f(u(s))\frac{ds}{s} \\
&\le \int_1^{\infty} \left(\frac{1}{\Gamma(\alpha)} + \sum_{i=1}^{m} \frac{\lambda_i g_i(\eta,s)}{\Omega\Gamma(\alpha+\beta_i)} \right) a(s)f(u(s))\frac{ds}{s} \\
&\le \left(\frac{1}{\Gamma(\alpha)} + \sum_{i=1}^{m} \frac{\lambda_i (\log\eta)^{\alpha+\beta_i-1}}{\Omega\Gamma(\alpha+\beta_i)} \right) B_r \int_1^{\infty} a(s)\frac{ds}{s} \\
&< \infty.
\end{aligned}$$

Hence, $T\Phi$ is equiconvergent at infinity.

Step 4: *We show that T is continuous.*

Let $u_n \to u$ as $n \to \infty$ in P. Recall that

$$\int_1^\infty a(s)f(u(s))\frac{ds}{s} < \infty.$$

Hence Lebesgue's dominated convergence theorem and the continuity of f guarantee that

$$\int_1^\infty a(s)f(u_n(s))\frac{ds}{s} \to \int_1^\infty a(s)f(u(s))\frac{ds}{s}, \text{ as } n \to \infty.$$

Therefore, we get

$$\begin{aligned}
&\|Tu_n - Tu\|_E \\
&= \sup_{t\in[1,\infty)} \frac{1}{1+(\log t)^{\alpha-1}} |Tu_n - Tu| \\
&= \sup_{t\in[1,\infty)} \left| \int_1^\infty \frac{G(t,s)}{1+(\log t)^{\alpha-1}} a(s)\left[f(u_n(s)) - f(u(s))\right]\frac{ds}{s} \right| \\
&\le \left(\frac{1}{\Gamma(\alpha)} + \sum_{i=1}^m \frac{\lambda_i (\log \eta)^{\alpha+\beta_i-1}}{\Omega\Gamma(\alpha+\beta_i)} \right) \left| \int_1^\infty a(s)f(u_n(s))\frac{ds}{s} - \int_1^\infty a(s)f(u(s))\frac{ds}{s} \right| \\
&\to 0, \quad \text{as} \quad n \to \infty.
\end{aligned}$$

So, T is continuous.

Using Lemma 10.4, we deduce that $T : P \to P$ is completely continuous. The proof is completed. □

10.4 Existence of at Least Three Positive Solutions

In this section, we use the Leggett-Williams fixed point theorem to prove the existence of at least three positive solutions for the problem (10.1)–(10.2).

For convenience, we denote

$$M = \left(\frac{1}{\Gamma(\alpha)} + \sum_{i=1}^m \frac{\lambda_i (\log \eta)^{\alpha+\beta_i-1}}{\Omega\Gamma(\alpha+\beta_i)} \right) \int_1^\infty a(s)\frac{ds}{s} > 0,$$

$$m = \sum_{i=1}^m \frac{\lambda_i (\log \eta)^{2\alpha+\beta_i-2}}{\Omega\Gamma(\alpha+\beta_i)(1+(\log \eta)^{\alpha-1})} \int_\eta^{k\eta} a(s)\frac{ds}{s} > 0.$$

Theorem 10.1 *Suppose that the conditions (10.5.1) and (10.5.2) hold. Let* $0 < a < b < d \leq c$ *and suppose that f satisfies the following conditions:*

(10.1.1) $f((1+(\log t)^{\alpha-1})u) < \frac{c}{M}$ *for all* $(t,u) \in [1,\infty) \times [0,c]$*;*

(10.1.2) $f((1+(\log t)^{\alpha-1})u) > \frac{b}{m}$ *for all* $(t,u) \in [\eta, k\eta] \times [b,c]$*;*

(10.1.3) $f((1+(\log t)^{\alpha-1})u) < \frac{a}{M}$ *for all* $(t,u) \in [1,\infty) \times [0,a]$.

Then the problem (10.1)–(10.2) has at least three positive solutions u_1, u_2 *and* u_3 *satisfying*

$$\|u_1\|_E < a, \quad b < \theta(u_2)$$

and

$$a < \|u_3\|_E \quad \textit{with} \quad \theta(u_3) < b.$$

Proof We will show that the conditions of the Leggett-Williams fixed point theorem are satisfied for the operator T defined by (10.16). We define a nonnegative functional on E by

$$\theta(u) = \min_{\eta \leq t \leq k\eta} \frac{u(t)}{1+(\log t)^{\alpha-1}}.$$

For $u \in \overline{P}_c$, we have $\|u\|_E \leq c$, that is,

$$0 \leq \frac{u(t)}{1+(\log t)^{\alpha-1}} \leq c \ \text{ for } \ t \in [1,\infty).$$

Then, assumption (10.1.1) implies that

$$f(u) < \frac{c}{M} \ \text{ for } \ (t,u) \in [1,\infty) \times [0,c].$$

Therefore,

$$\begin{aligned} \|Tu\|_E &= \sup_{t\in[1,\infty)} \frac{1}{1+(\log t)^{\alpha-1}} \left| \int_1^\infty G(t,s)a(s)f(u(s))\frac{ds}{s} \right| \\ &< \left(\frac{1}{\Gamma(\alpha)} + \sum_{i=1}^m \frac{\lambda_i (\log \eta)^{\alpha+\beta_i-1}}{\Omega\Gamma(\alpha+\beta_i)} \right) \frac{c}{M} \int_1^\infty a(s)\frac{ds}{s} = c. \end{aligned}$$

Hence, $T : P_c \to P_c$ and by Lemma 10.5, T is completely continuous. Using the preceding arguments, it follows from condition (10.1.3) that if $u \in \overline{P}_a$, then $\|Tu\|_E < a$. Therefore, condition (*ii*) of Theorem 1.20 holds.

Let

$$u^*(t) = \frac{b+c}{2}(1 + (\log t)^{\alpha-1}), \ \ t \in [1, \infty).$$

It obvious that $u^*(t) \in P$ and $\|u^*\| = \dfrac{b+c}{2} < c$. From the definition of $\theta(u)$, we have

$$\theta(u^*) = \frac{b+c}{2} > b.$$

Hence, we get

$$u^* \in \{x \in P(\theta, b, d) : \theta(x) > b\} \neq \emptyset.$$

Moreover, for $u \in P(\theta, b, d)$, it follow that

$$b \leq \frac{u(t)}{1 + (\log t)^{\alpha-1}} \leq c \quad \text{for} \quad t \in [\eta, k\eta].$$

Then, assumption (10.1.2) implies that

$$f(u) > \frac{b}{m} \quad \text{for} \quad (t, u) \in [\eta, k\eta] \times [b, c].$$

So, we have

$$\begin{aligned}
\theta(Tu) &= \min_{\eta \leq t \leq k\eta} \frac{Tu(t)}{1 + (\log t)^{\alpha-1}} \\
&\geq \int_1^\infty \min_{\eta \leq t \leq k\eta} \frac{G(t, s)}{1 + (\log t)^{\alpha-1}} a(s) f(u(s)) \frac{ds}{s} \\
&\geq \sum_{i=1}^m \frac{\lambda_i (\log \eta)^{\alpha-1}}{\Omega \Gamma(\alpha + \beta_i)(1 + (\log \eta)^{\alpha-1})} \int_1^\infty g_i(\eta, s) a(s) f(u(s)) \frac{ds}{s} \\
&\geq \sum_{i=1}^m \frac{\lambda_i (\log \eta)^{\alpha-1}}{\Omega \Gamma(\alpha + \beta_i)(1 + (\log \eta)^{\alpha-1})} \int_\eta^{k\eta} g_i(\eta, s) a(s) f(u(s)) \frac{ds}{s} \\
&> \sum_{i=1}^m \frac{\lambda_i (\log \eta)^{2\alpha+\beta_i-2}}{\Omega \Gamma(\alpha + \beta_i)(1 + (\log \eta)^{\alpha-1})} \frac{b}{m} \int_\eta^{k\eta} a(s) \frac{ds}{s} \\
&= b.
\end{aligned}$$

Thus $\theta(Tu) > b$ for all $u \in P(\theta, b, d)$. This shows that condition (i) of Theorem 1.20 holds.

Finally, we assume that $u \in P(\theta, b, c)$ with $\|Tu\|_E > d$. Then $\|u\|_E \le c$ and $b \le \dfrac{u(t)}{1 + (\log t)^{\alpha-1}} \le c$ and from assumption (10.1.2), we can have $\theta(Tu) > b$. So, condition (*iii*) of Theorem 1.20 is satisfied. As a consequence of Theorem 1.20, the problem (10.1)–(10.2) has at least three positive solutions u_1, u_2 and u_3 such that

$$\|u_1\|_E < a, \quad b < \theta(u_2) \quad \text{and} \quad a < \|u_3\|_E \quad \text{with} \quad \theta(u_3) < b.$$

The proof is completed. □

Example 10.1 Consider the following Hadamard fractional differential equation with nonlocal boundary conditions on an unbounded domain:

$$\begin{cases} D^{3/2}u(t) + e^{-t}f(u(t)) = 0, \quad t \in (1, \infty), \\ u(1) = 0, \quad D^{1/2}u(\infty) = \dfrac{1}{10}I^{3/4}u\left(\dfrac{5}{2}\right) + \pi I^{5/2}u\left(\dfrac{5}{2}\right) + \dfrac{1}{3}I^{7/2}u\left(\dfrac{5}{2}\right), \end{cases} \tag{10.20}$$

where

$$f(u(t)) = \begin{cases} \cos^2\left(\left(\dfrac{4}{5} - u\right)\pi\right) + \dfrac{1}{2}, \quad u \in \left[0, \dfrac{4}{5}\right], \\ \cos^2\left(\left(\dfrac{4}{5} - u\right)\pi\right) + \dfrac{1}{2} + 200 \arctan\left(u - \dfrac{4}{5}\right), \quad u \in \left[\dfrac{4}{5}, \dfrac{7}{5}\right], \\ \cos^2\left(\left(\dfrac{4}{5} - u\right)\pi\right) + \dfrac{1}{2} + 200 \arctan\left(u - \dfrac{4}{5}\right) \\ \qquad\qquad + \dfrac{1}{3}\sin^2\left(u - \dfrac{7}{5}\right), u \in \left[\dfrac{7}{5}, \infty\right). \end{cases}$$

Set $m = 3$, $\alpha = 3/2$, $\eta = 5/2$, $k = 6$, $a(t) = e^{-t}$, $\lambda_1 = 1/10$, $\lambda_2 = \pi$, $\lambda_3 = 1/3$, $\beta_1 = 3/4$, $\beta_2 = 5/2$, $\beta_3 = 7/2$. Using the given data, it is found that $\Omega \approx 0.450449$, $M \approx 0.487034$ and $m \approx 0.013302$. Choose $a = 4/5$, $b = 7/5$ and $c = 160$. Then f satisfies

$$f((1 + (\log t)^{\frac{1}{2}})u) \le 1.5 < 1.6426 \approx \frac{a}{M}, \quad (t, u) \in [1, \infty) \times \left[0, \frac{4}{5}\right],$$

$$f((1 + (\log t)^{\frac{1}{2}})u) \ge 219.4760 > 105.2492 \approx \frac{b}{m}, \quad (t, u) \in \left[\frac{5}{2}, 15\right] \times \left[\frac{7}{5}, 160\right],$$

$$f((1 + (\log t)^{\frac{1}{2}})u) \le 315.9926 < 328.5189 \approx \frac{c}{M}, \quad (t, u) \in [1, \infty) \times [0, 160].$$

Thus, by an application of Theorem 10.1, the problem (10.20) has at least three positive solutions u_1, u_2, u_3 such that

$$\sup_{t \in [1,\infty)} \frac{|u_1(t)|}{1 + (\log t)^{\frac{1}{2}}} < \frac{4}{5}, \qquad \frac{7}{5} < \min_{t \in [\frac{5}{2}, 15]} \frac{u_2(t)}{1 + (\log t)^{\frac{1}{2}}}$$

and

$$\frac{4}{5} < \sup_{t\in[1,\infty)} \frac{|u_3(t)|}{1+(\log t)^{\frac{1}{2}}} \quad \text{with} \quad \min_{t\in[\frac{5}{2},15]} \frac{u_3(t)}{1+(\log t)^{\frac{1}{2}}} < \frac{7}{5}.$$

10.5 Existence of at Least One Positive Solution

In this section, we use the Guo-Krasnoselskii fixed point theorem (Theorem 1.21) to prove the existence of at least one positive solution.

Theorem 10.2 *Suppose that the conditions (10.5.1) and (10.5.2) hold. Let* $r_2 > r_1 > 0$, $\rho_1 \in (m^{-1}, \infty)$, $\rho_2 \in (0, M^{-1})$ *and that* f *satisfies the following conditions:*

(10.2.1) $f((1+(\log t)^{\alpha-1})u) \geq \rho_1 r_1$ *for all* $(t,u) \in [1,\infty) \times [0, r_1]$*;*
(10.2.2) $f((1+(\log t)^{\alpha-1})u) \leq \rho_2 r_2$ *for all* $(t,u) \in [1,\infty) \times [0, r_2]$.

Then the problem (10.1)–(10.2) has at least one positive solution u, *such that*

$$r_1 < \|u\|_E < r_2.$$

Proof We will show that the condition (i) of Theorem 1.21 is satisfied. By Lemma 10.5, the operator $T : P \to P$ is completely continuous.

Let $\Phi_1 = \{u \in E : \|u\|_E < r_1\}$. Then, for any $u \in P \cap \partial\Phi_1$, we have

$$0 \leq \frac{u(t)}{1+(\log t)^{\alpha-1}} \leq r_1, \quad \text{for all} \quad t \in [1,\infty).$$

Then assumption (10.2.1) implies that

$$f(u) \geq \rho_1 r_1 \ \text{ for } \ (t,u) \in [1,\infty) \times [0, r_1].$$

Therefore, for $t \in [1,\infty)$, we get

$$\begin{aligned} \|Tu\|_E &= \sup_{t\in[1,\infty)} \frac{1}{1+(\log t)^{\alpha-1}} \left| \int_1^\infty G(t,s)a(s)f(u(s))\frac{ds}{s} \right| \\ &\geq \min_{t\in[\eta,k\eta]} \int_1^\infty \frac{G(t,s)}{1+(\log t)^{\alpha-1}} a(s)f(u(s))\frac{ds}{s} \\ &\geq \int_1^\infty \min_{t\in[\eta,k\eta]} \frac{G(t,s)}{1+(\log t)^{\alpha-1}} a(s)f(u(s))\frac{ds}{s} \\ &\geq \sum_{i=1}^m \frac{\lambda_i (\log\eta)^{2\alpha+\beta_i-2}}{\Omega\Gamma(\alpha+\beta_i)(1+(\log\eta)^{\alpha-1})} \rho_1 r_1 \int_\eta^{k\eta} a(s)\frac{ds}{s} \\ &\geq r_1 = \|u\|_E, \quad \text{for} \quad u \in P \cap \partial\Phi_1. \end{aligned} \tag{10.21}$$

Let $\Phi_2 = \{u \in E : \|u\|_E < r_2\}$. Then, for any $u \in P \cap \partial\Phi_2$, it follows that

$$0 \leq \frac{u(t)}{1 + (\log t)^{\alpha-1}} \leq r_2, \quad \text{for all} \quad t \in [1, \infty).$$

For $t \in [1, \infty)$, the assumption (10.2.2) yields

$$\begin{aligned}
\|Tu\|_E &= \sup_{t\in[1,\infty)} \frac{1}{1 + (\log t)^{\alpha-1}} \left| \int_1^\infty G(t, s)a(s)f(u(s))\frac{ds}{s} \right| \\
&\leq \int_1^\infty \left(\frac{1}{\Gamma(\alpha)} + \sum_{i=1}^{m} \frac{\lambda_i g_i(\eta, s)}{\Omega\Gamma(\alpha + \beta_i)} \right) a(s)f(u(s))\frac{ds}{s} \\
&\leq \left(\frac{1}{\Gamma(\alpha)} + \sum_{i=1}^{m} \frac{\lambda_i(\log \eta)^{\alpha+\beta_i-1}}{\Omega\Gamma(\alpha + \beta_i)} \right) \rho_2 r_2 \int_1^\infty a(s)\frac{ds}{s} \\
&\leq r_2 = \|u\|_E, \quad \text{for} \quad u \in P \cap \partial\Phi_2.
\end{aligned} \tag{10.22}$$

Hence, from (10.21), (10.22) and condition (*i*) of Theorem 1.21, it follows that T has a fixed point in $P \cap (\overline{\Phi}_2 \setminus \Phi_1)$. Therefore, the problem (10.1)–(10.2) has at least one positive solution such that

$$r_1 < \|u\|_E < r_2.$$

The proof is completed. □

Similar to the previous theorem, we can establish the following result.

Theorem 10.3 *Assume that the conditions (10.5.1) and (10.5.2) hold. Let* $r_2 > r_1 > 0$, $\rho_1 \in (m^{-1}, \infty)$, $\rho_2 \in (0, M^{-1})$ *and that* f *satisfies the following conditions:*

(10.3.1) $f((1 + (\log t)^{\alpha-1})u) \leq \rho_2 r_1$ *for all* $(t, u) \in [1, \infty) \times [0, r_1]$*;*
(10.3.2) $f((1 + (\log t)^{\alpha-1})u) \geq \rho_1 r_2$ *for all* $(t, u) \in [1, \infty) \times [0, r_2]$.

Then the problem (10.1)–(10.2) has at least one positive solution u, *such that*

$$r_1 < \|u\|_E < r_2.$$

Example 10.2 Consider the following Hadamard fractional differential equation with nonlocal boundary conditions on unbounded domain:

$$\begin{cases} D^{4/3}u(t) + t^{-2}f(u(t)) = 0, \quad t \in (1, \infty), \\ u(0) = 0, \\ D^{1/3}u(\infty) = 2I^{\sqrt{\pi}}u(3) + e^{-1}I^{1/2}u(3) + \sin(11)I^{9/7}u(3) + \dfrac{\sqrt{\pi}}{2}I^{11/3}u(3), \end{cases} \tag{10.23}$$

where

$$f(u(t)) = \begin{cases} 2e^{-u} + \frac{1}{5}\cos^2\left(\frac{u\pi}{2}\right) + 22, & u \in [0,2], \\ 2e^{-u} + \frac{1}{5}\cos^2\left(\frac{u\pi}{2}\right) + 19 + \frac{8}{\pi}\arctan(u-2) \\ \qquad + 3\sin^2\left(\frac{(u+1)\pi}{6}\right), u \in [2,\infty). \end{cases}$$

Letting $m = 4$, $\alpha = 4/3$, $\eta = 3$, $k = 4$, $a(t) = t^{-2}$, $\lambda_1 = 2$, $\lambda_2 = e^{-1}$, $\lambda_3 = \sin(11)$, $\lambda_4 = \sqrt{\pi}/2$, $\beta_1 = \sqrt{\pi}$, $\beta_2 = 1/2$, $\beta_3 = 9/7$, $\beta_4 = 11/3$, we obtain $\Omega \approx 0.199034$. By direct calculation, we can get $M \approx 2.512136$, $m \approx 0.103271$. Choose $r_1 = 2$, $r_2 = 100$, $\rho_1 = 10$ and $\rho_2 = 3/10$ and note that

$$f((1 + (\log t)^{\frac{1}{3}})u) \geq 22 \geq 20 = \rho_1 r_1, \quad (t, u) \in [1, \infty) \times [0, 2],$$

$$f((1 + (\log t)^{\frac{1}{3}})u) \leq 28.2 \leq 30 = \rho_2 r_2, \quad (t, u) \in [1, \infty) \times [0, 100].$$

Thus, the conclusion of Theorem 10.2 applies and hence the problem (10.23) has at least one positive solution u such that

$$2 \leq \sup_{t\in[1,\infty)} \frac{|u(t)|}{1 + (\log t)^{\frac{1}{3}}} \leq 100.$$

10.6 Notes and Remarks

In this chapter, we studied the existence of positive solutions for Hadamard fractional differential equations on infinite intervals. As we know, $[1, \infty)$ is noncompact, so a special Banach space was introduced. However, this Banach space was found to be unsuitable for Hadamard fractional differential equations. In order to overcome this difficulty, a variant of Banach space was introduced in [164] so that we may establish some inequalities, which guarantee that the functionals defined on $[1, \infty)$ have better properties. Applying first the well-known Leggett-Williams fixed point theorem, we obtained the existence of at least three distinct nonnegative solutions under appropriate conditions. As a second result, we proved the existence of at least one positive solution for the given problem by using Guo-Krasnoselskii's fixed point theorem. The content of this chapter is taken from the paper [164].

Chapter 11
Fractional Integral Inequalities via Hadamard's Fractional Integral

11.1 Introduction

Inequalities have emerged as one of the most powerful and far-reaching tools for the development of many branches of mathematics. The topic of mathematical inequalities plays quite an important role in the study of classical differential and integral equations [113, 134–137]. Fractional inequalities are also important in studying the existence, uniqueness and other properties of fractional differential equations. Recently many authors have studied integral inequalities by using Riemann-Liouville and Caputo derivative, for instance, see [35, 44, 65–67, 72] and the references therein. More recently, some results on fractional integral inequalities involving Hadamard fractional integral have also appeared [59, 60, 147].

In this chapter, we derive some new fractional integral inequalities via Hadamard fractional integral with the aid of Young and weighted AM-GM inequalities. The obtained results correspond to several interesting situations and produce some important new inequalities as special cases. A Grüss type Hadamard fractional integral inequality is also established. Moreover, we obtain some new integral inequalities involving Hadamard integral with "maxima".

11.2 Hadamard Fractional Integral Inequalities

In this subsection, we discuss some inequalities involving an integrable function bounded by integrable functions.

Theorem 11.1 *Let f be an integrable function on $[1,\infty)$. Assume that:*

(11.1.1) there exist two integrable functions φ_1, φ_2 on $[1,\infty)$ such that

$$\varphi_1(t) \le f(t) \le \varphi_2(t), \quad \textit{for all} \quad t \in [1,\infty).$$

B. Ahmad et al., *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, DOI 10.1007/978-3-319-52141-1_11

Then, for $t > 1$, $\alpha, \beta > 0$, *we have*

$$
\begin{aligned}
&{}_HJ^{\beta}\varphi_1(t)\,{}_HJ^{\alpha}f(t) + {}_HJ^{\alpha}\varphi_2(t)\,{}_HJ^{\beta}f(t) \\
&\qquad\qquad \geq {}_HJ^{\alpha}\varphi_2(t)\,{}_HJ^{\beta}\varphi_1(t) + {}_HJ^{\alpha}f(t)\,{}_HJ^{\beta}f(t).
\end{aligned}
\tag{11.1}
$$

Proof From (11.1.1), for all $\tau \geq 1$, $\rho \geq 1$, we have

$$(\varphi_2(\tau) - f(\tau))\,(f(\rho) - \varphi_1(\rho)) \geq 0,$$

which can alternatively be written as

$$\varphi_2(\tau)f(\rho) + \varphi_1(\rho)f(\tau) \geq \varphi_1(\rho)\varphi_2(\tau) + f(\tau)f(\rho). \tag{11.2}$$

Multiplying both sides of (11.2) by $\left(\log\frac{t}{\tau}\right)^{\alpha-1}/(\tau\Gamma(\alpha))$, $\tau \in (1,t)$, and then integrating with respect to τ on $(1,t)$, we obtain

$$
\begin{aligned}
&f(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\varphi_2(\tau)\frac{d\tau}{\tau} + \varphi_1(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}f(\tau)\frac{d\tau}{\tau} \\
&\geq \varphi_1(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\varphi_2(\tau)\frac{d\tau}{\tau} + f(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}f(\tau)\frac{d\tau}{\tau},
\end{aligned}
$$

which yields

$$f(\rho)\,{}_HJ^{\alpha}\varphi_2(t) + \varphi_1(\rho)\,{}_HJ^{\alpha}f(t) \geq \varphi_1(\rho)\,{}_HJ^{\alpha}\varphi_2(t) + f(\rho)\,{}_HJ^{\alpha}f(t). \tag{11.3}$$

Multiplying both sides of (11.3) by $\left(\log\frac{t}{\rho}\right)^{\beta-1}/(\rho\Gamma(\beta))$, $\rho \in (1,t)$, and then integrating with respect to ρ on $(1,t)$, we get

$$
\begin{aligned}
&{}_HJ^{\alpha}\varphi_2(t)\frac{1}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{\rho}\right)^{\beta-1}f(\rho)\frac{d\rho}{\rho} \\
&\qquad + {}_HJ^{\alpha}f(t)\frac{1}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{\rho}\right)^{\beta-1}\varphi_1(\rho)\frac{d\rho}{\rho} \\
&\geq {}_HJ^{\alpha}\varphi_2(t)\frac{1}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{\rho}\right)^{\beta-1}\varphi_1(\rho)\frac{d\rho}{\rho} \\
&\qquad + {}_HJ^{\alpha}f(t)\frac{1}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{\rho}\right)^{\beta-1}f(\rho)\frac{d\rho}{\rho}.
\end{aligned}
$$

Hence, we deduce the inequality (11.1). This completes the proof. □

As special cases of Theorems 11.1, we obtain the following results.

Corollary 11.1 *Let f be an integrable function on $[1,\infty)$ satisfying $m \le f(t) \le M$, for all $t \in [1,\infty)$ and $m, M \in \mathbb{R}$. Then, for $t > 1$ and $\alpha, \beta > 0$, we have*

$$m\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}f(t) + M\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}f(t) \\ \ge mM\frac{(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} + {}_HJ^{\alpha}f(t){}_HJ^{\beta}f(t).$$

Corollary 11.2 *Let f be an integrable function on $[1,\infty)$. Assume that there exists an integrable function $\varphi(t)$ on $[1,\infty)$ and a constant $M > 0$ such that*

$$\varphi(t) - M \le f(t) \le \varphi(t) + M,$$

for all $t \in [1,\infty)$. Then, for $t > 1$ and $\alpha, \beta > 0$, we have

$${}_HJ^{\beta}\varphi(t){}_HJ^{\alpha}f(t) + {}_HJ^{\alpha}\varphi(t){}_HJ^{\beta}f(t) + \frac{M(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}\varphi(t) \\ + \frac{M(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}f(t) + \frac{M^2(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} \\ \ge {}_HJ^{\alpha}\varphi(t){}_HJ^{\beta}\varphi(t) + {}_HJ^{\alpha}f(t){}_HJ^{\beta}f(t) \\ + \frac{M(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}f(t) + \frac{M(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}\varphi(t).$$

Theorem 11.2 *Let f be an integrable function on $[1,\infty)$ and $\theta_1, \theta_2 > 0$ are such that $1/\theta_1 + 1/\theta_2 = 1$. Suppose that (11.1.1) holds. Then, for $t > 1$, $\alpha, \beta > 0$, we have*

$$\frac{1}{\theta_1}\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}\left((\varphi_2 - f)^{\theta_1}\right)(t) + \frac{1}{\theta_2}\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}\left((f-\varphi_1)^{\theta_2}\right)(t) \\ + {}_HJ^{\alpha}\varphi_2(t){}_HJ^{\beta}\varphi_1(t) + {}_HJ^{\alpha}f(t){}_HJ^{\beta}f(t) \\ \ge {}_HJ^{\alpha}\varphi_2(t){}_HJ^{\beta}f(t) + {}_HJ^{\alpha}f(t){}_HJ^{\beta}\varphi_1(t). \tag{11.4}$$

Proof Setting $x = \varphi_2(\tau) - f(\tau)$ and $y = f(\rho) - \varphi_1(\rho)$, $\tau, \rho > 1$, in the well-known Young's inequality:

$$\frac{1}{\theta_1}x^{\theta_1} + \frac{1}{\theta_2}y^{\theta_2} \ge xy, \qquad \forall x, y \ge 0, \ \ \theta_1, \theta_2 > 0, \ \ \frac{1}{\theta_1} + \frac{1}{\theta_2} = 1,$$

we get

$$\frac{1}{\theta_1}(\varphi_2(\tau)-f(\tau))^{\theta_1}+\frac{1}{\theta_2}(f(\rho)-\varphi_1(\rho))^{\theta_2} \geq (\varphi_2(\tau)-f(\tau))(f(\rho)-\varphi_1(\rho)). \tag{11.5}$$

Multiplying both sides of (11.5) by $\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\beta-1}/(\tau\rho\Gamma(\alpha)\Gamma(\beta))$, $\tau,\rho\in(1,t)$, and then integrating with respect to τ and ρ from 1 to t, we have

$$\frac{1}{\theta_1}{}_HJ^{\beta}(1)(t){}_HJ^{\alpha}(\varphi_2-f)^{\theta_1}(t)+\frac{1}{\theta_2}{}_HJ^{\alpha}(1)(t){}_HJ^{\beta}(f-\varphi_1)^{\theta_2}(t) \geq {}_HJ^{\alpha}(\varphi_2-f)(t){}_HJ^{\beta}(f-\varphi_1)(t),$$

which implies (11.4). □

Corollary 11.3 *Let f be an integrable function on $[1,\infty)$ satisfying $m\leq f(t)\leq M$, for all $t\in[1,\infty)$ and $m,M\in\mathbb{R}$. Then, for $t>1$ and $\alpha,\beta>0$, we have*

$$(m+M)^2\frac{(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}f^2(t) + \frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}f^2(t)+2{}_HJ^{\alpha}f(t){}_HJ^{\beta}f(t) \geq 2(m+M)\left(\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}f(t)+\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}f(t)\right).$$

Theorem 11.3 *Let f be an integrable function on $[1,\infty)$ and $\theta_1,\theta_2>0$ are such that $\theta_1+\theta_2=1$. In addition, suppose that (11.1.1) holds. Then, for $t>1$, $\alpha,\beta>0$, we have*

$$\theta_1\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}\varphi_2(t)+\theta_2\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}f(t) \geq {}_HJ^{\alpha}(\varphi_2-f)^{\theta_1}(t){}_HJ^{\beta}(f-\varphi_1)^{\theta_2}(t)+\theta_1\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}f(t) + \theta_2\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}\varphi_1(t). \tag{11.6}$$

Proof Letting $x=\varphi_2(\tau)-f(\tau)$ and $y=f(\rho)-\varphi_1(\rho)$, $\tau,\rho>1$ in the well-known Weighted AM-GM inequality:

$$\theta_1x+\theta_2y\geq x^{\theta_1}y^{\theta_2},\qquad \forall x,y\geq 0,\;\; \theta_1,\theta_2>0,\;\; \theta_1+\theta_2=1,$$

we have

$$\theta_1(\varphi_2(\tau) - f(\tau)) + \theta_2(f(\rho) - \varphi_1(\rho)) \\ \geq (\varphi_2(\tau) - f(\tau))^{\theta_1}(f(\rho) - \varphi_1(\rho))^{\theta_2}. \tag{11.7}$$

Multiplying both sides of (11.7) by $\left(\log \frac{t}{\tau}\right)^{\alpha-1} \left(\log \frac{t}{\rho}\right)^{\beta-1} /(\tau\rho\Gamma(\alpha)\Gamma(\beta))$, $\tau, \rho \in (1, t)$, and then integrating with respect to τ and ρ from 1 to t, we obtain

$$\theta_{1H}J^\beta(1)(t)_H J^\alpha(\varphi_2 - f)(t) + \theta_{2H}J^\alpha(1)(t)_H J^\beta(f - \varphi_1)(t) \\ \geq {}_HJ^\alpha(\varphi_2 - f)^{\theta_1}(t)_H J^\beta(f - \varphi_1)^{\theta_2}(t).$$

Therefore, we deduce inequality (11.6). □

Corollary 11.4 *Let f be an integrable function on $[1, \infty)$ satisfying $m \leq f(t) \leq M$, for all $t \in [1, \infty)$ and $m, M \in \mathbb{R}$. Then, for $t > 1$ and $\alpha, \beta > 0$, we have*

$$M \frac{(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} {}_HJ^\beta f(t) \\ \geq m \frac{(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{(\log t)^\beta}{\Gamma(\beta+1)} {}_HJ^\alpha f(t) \\ + 2_H J^\alpha (M - f)^{\frac{1}{2}}(t)_H J^\beta (f - m)^{\frac{1}{2}}(t).$$

Lemma 11.1 ([95]) *Assume that $a \geq 0$, $p \geq q \geq 0$, and $p \neq 0$. Then*

$$a^{\frac{q}{p}} \leq \left(\frac{q}{p} k^{\frac{q-p}{p}} a + \frac{p-q}{p} k^{\frac{q}{p}}\right), \text{ for any } k > 0.$$

Theorem 11.4 *Let f be an integrable function on $[1, \infty)$ and there exist constants $p \geq q \geq 0$, $p \neq 0$. In addition, assume that (11.1.1) holds. Then, for any $k > 0$, $t > 1$, α, $\beta > 0$, the following two inequalities hold:*

$$(A_1)\ {}_HJ^\alpha(\varphi_2 - f)^{\frac{q}{p}}(t) + \frac{q}{p} k^{\frac{q-p}{p}} {}_HJ^\alpha f(t) \leq \frac{q}{p} k^{\frac{q-p}{p}} {}_HJ^\alpha \varphi_2(t) + \frac{p-q}{p} k^{\frac{q}{p}} \frac{(\log t)^\alpha}{\Gamma(\alpha+1)}.$$

$$(B_1)\ {}_HJ^\alpha(f - \varphi_1)^{\frac{q}{p}}(t) + \frac{q}{p} k^{\frac{q-p}{p}} {}_HJ^\alpha \varphi_1(t) \leq \frac{q}{p} k^{\frac{q-p}{p}} {}_HJ^\alpha f(t) + \frac{p-q}{p} k^{\frac{q}{p}} \frac{(\log t)^\alpha}{\Gamma(\alpha+1)}.$$

Proof By condition (11.1.1) and Lemma 11.1, for $p \geq q \geq 0$, $p \neq 0$, it follows that

$$(\varphi_2(\tau) - f(\tau))^{\frac{q}{p}} \leq \frac{q}{p} k^{\frac{q-p}{p}} (\varphi_2(\tau) - f(\tau)) + \frac{p-q}{p} k^{\frac{q}{p}}, \tag{11.8}$$

for any $k > 0$. Multiplying both sides of (11.8) by $\left(\log \frac{t}{\tau}\right)^{\alpha-1} /(\tau\Gamma(\alpha))$, $\tau \in (1, t)$, and integrating the resulting identity with respect to τ from 1 to t, we find that

$$_HJ^\alpha(\varphi_2 - f)^{\frac{q}{p}} \le \frac{q}{p}k^{\frac{q-p}{p}}\left(_HJ^\alpha\varphi_2(t) - {_HJ^\alpha}f(t)\right) + \frac{p-q}{p}k^{\frac{q}{p}}\frac{(\log t)^\alpha}{\Gamma(\alpha+1)},$$

which leads to inequality (A_1). Inequality (B_1) can be proved by employing similar arguments. □

Corollary 11.5 *Let f be an integrable function on* $[1,\infty)$ *satisfying* $m \le f(t) \le M$, *for all* $t \in [1,\infty)$ *and* $m, M \in \mathbb{R}$. *Then, for* $t > 1$ *and* $\alpha > 0$, *we have*

$$(A_2) \quad 2{_HJ^\alpha}(M-f)^{1/2}(t) + {_HJ^\alpha}f(t) \le (M+1)\frac{(\log t)^\alpha}{\Gamma(\alpha+1)}.$$

$$(B_2) \quad 2{_HJ^\alpha}(f-m)^{1/2}(t) + m\frac{(\log t)^\alpha}{\Gamma(\alpha+1)} \le {_HJ^\alpha}f(t) + \frac{(\log t)^\alpha}{\Gamma(\alpha+1)}.$$

Theorem 11.5 *Let f and g be two integrable functions on* $[1,\infty)$. *Suppose that (11.1.1) holds and moreover, we assume that:*

(11.5.1) there exist integrable functions ψ_1 *and* ψ_2 *on* $[1,\infty)$ *such that*

$$\psi_1(t) \le g(t) \le \psi_2(t), \quad \textit{for all} \quad t \in [1,\infty).$$

Then, for $t > 0$, α, $\beta > 0$, *the following inequalities hold:*

(A_3) $\quad {_HJ^\beta}\psi_1(t){_HJ^\alpha}f(t) + {_HJ^\alpha}\varphi_2(t){_HJ^\beta}g(t) \ge {_HJ^\beta}\psi_1(t){_HJ^\alpha}\varphi_2(t) + {_HJ^\alpha}f(t){_HJ^\beta}g(t).$
(B_3) $\quad {_HJ^\beta}\varphi_1(t){_HJ^\alpha}g(t) + {_HJ^\alpha}\psi_2(t){_HJ^\beta}f(t) \ge {_HJ^\beta}\varphi_1(t){_HJ^\alpha}\psi_2(t) + {_HJ^\beta}f(t){_HJ^\alpha}g(t).$
(C_3) $\quad {_HJ^\beta}\psi_2(t){_HJ^\alpha}\varphi_2(t) + {_HJ^\alpha}f(t){_HJ^\beta}g(t) \ge {_HJ^\alpha}\varphi_2(t){_HJ^\beta}g(t) + {_HJ^\beta}\psi_2(t){_HJ^\alpha}f(t).$
(D_3) $\quad {_HJ^\alpha}\varphi_1(t){_HJ^\beta}\psi_1(t) + {_HJ^\alpha}f(t){_HJ^\beta}g(t) \ge {_HJ^\alpha}\varphi_1(t){_HJ^\beta}g(t) + {_HJ^\beta}\psi_1(t){_HJ^\alpha}f(t).$

Proof To establish (A_3), from (11.1.1) and (11.5.1) for $t \in [1,\infty)$, we have

$$(\varphi_2(\tau) - f(\tau))(g(\rho) - \psi_1(\rho)) \ge 0.$$

Therefore

$$\varphi_2(\tau)g(\rho) + \psi_1(\rho)f(\tau) \ge \psi_1(\rho)\varphi_2(\tau) + f(\tau)g(\rho). \tag{11.9}$$

Multiplying both sides of (11.9) by $\left(\log\frac{t}{\tau}\right)^{\alpha-1}/(\tau\Gamma(\alpha))$, $\tau \in (1,t)$, we get

$$\begin{aligned} g(\rho)\frac{\left(\log\frac{t}{\tau}\right)^{\alpha-1}}{\tau\Gamma(\alpha)}\varphi_2(\tau) + \psi_1(\rho)\frac{\left(\log\frac{t}{\tau}\right)^{\alpha-1}}{\tau\Gamma(\alpha)}f(\tau) \\ \ge \psi_1(\rho)\frac{\left(\log\frac{t}{\tau}\right)^{\alpha-1}}{\tau\Gamma(\alpha)}\varphi_2(\tau) + g(\rho)\frac{\left(\log\frac{t}{\tau}\right)^{\alpha-1}}{\tau\Gamma(\alpha)}f(\tau). \end{aligned} \tag{11.10}$$

Integrating both sides of (11.10) with respect to τ on $(1, t)$, we obtain

$$g(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\varphi_2(\tau)\frac{d\tau}{\tau}+\psi_1(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}f(\tau)\frac{d\tau}{\tau}$$
$$\geq\psi_1(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\varphi_2(\tau)\frac{d\tau}{\tau}+g(\rho)\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}f(\tau)\frac{d\tau}{\tau}.$$

Thus we have

$$g(\rho)_HJ^\alpha\varphi_2(t)+\psi_1(\rho)_HJ^\alpha f(t)\geq\psi_1(\rho)_HJ^\alpha\varphi_2(t)+g(\rho)_HJ^\alpha f(t). \quad (11.11)$$

Multiplying both sides of (11.11) by $\left(\log\frac{t}{\rho}\right)^{\beta-1}/(\rho\Gamma(\beta))$, $\rho\in(1,t)$, we have

$$_HJ^\alpha\varphi_2(t)\frac{\left(\log\frac{t}{\rho}\right)^{\beta-1}}{\rho\Gamma(\beta)}g(\rho)+{}_HJ^\alpha f(t)\frac{\left(\log\frac{t}{\rho}\right)^{\beta-1}}{\rho\Gamma(\beta)}\psi_1(\rho)$$
$$\geq{}_HJ^\alpha\varphi_2(t)\frac{\left(\log\frac{t}{\rho}\right)^{\beta-1}}{\rho\Gamma(\beta)}\psi_1(\rho)+{}_HJ^\alpha f(t)\frac{\left(\log\frac{t}{\rho}\right)^{\beta-1}}{\rho\Gamma(\beta)}g(\rho). \quad (11.12)$$

Integrating (11.12) with respect to ρ on $(1, t)$, we get the desired inequality (A_3).

We can establish (B_3)–(D_3) in a similar manner by using the following inequalities:

(B_3) $(\psi_2(\tau)-g(\tau))\,(f(\rho)-\varphi_1(\rho))\geq 0.$
(C_3) $(\varphi_2(\tau)-f(\tau))\,(g(\rho)-\psi_2(\rho))\leq 0.$
(D_3) $(\varphi_1(\tau)-f(\tau))\,(g(\rho)-\psi_1(\rho))\leq 0.$

□

As a special case of Theorem 11.5, we have the following corollary.

Corollary 11.6 *Let f and g be two integrable functions on $[1,\infty)$. Assume that:*

(11.6.1) there exist real constants m, M, n, N such that

$$m\leq f(t)\leq M \quad \text{and} \quad n\leq g(t)\leq N, \quad \text{for all} \quad t\in[1,\infty).$$

Then, for $t>1$, α, $\beta>0$, we have

(A_4) $\dfrac{n(\log t)^\beta}{\Gamma(\beta+1)}{}_HJ^\alpha f(t)+\dfrac{M(\log t)^\alpha}{\Gamma(\alpha+1)}{}_HJ^\beta g(t)\geq\dfrac{nM(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+{}_HJ^\alpha f(t)_HJ^\beta g(t).$

(B_4) $\dfrac{m(\log t)^\beta}{\Gamma(\beta+1)}{}_HJ^\alpha g(t)+\dfrac{N(\log t)^\alpha}{\Gamma(\alpha+1)}{}_HJ^\beta f(t)\geq\dfrac{mN(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+{}_HJ^\beta f(t)_HJ^\alpha g(t).$

(C_4) $\frac{MN(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} + {}_HJ^{\alpha}f(t){}_HJ^{\beta}g(t) \geq \frac{M(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}g(t) + \frac{N(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}f(t).$

(D_4) $\frac{mn(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} + {}_HJ^{\alpha}f(t){}_HJ^{\beta}g(t) \geq \frac{m(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}g(t) + \frac{n(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}f(t).$

Theorem 11.6 *Let f and g be two integrable functions on* $[1,\infty)$ *and* $\theta_1, \theta_2 > 0$ *satisfy the relation* $1/\theta_1 + 1/\theta_2 = 1$. *Suppose that (11.1.1) and (11.5.1) hold. Then, for* $t > 1$, α, $\beta > 0$, *the following inequalities hold:*

(A_5) $\frac{1}{\theta_1}\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}(\varphi_2-f)^{\theta_1}(t) + \frac{1}{\theta_2}\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}(\psi_2-g)^{\theta_2}(t)$
$\geq {}_HJ^{\alpha}(\varphi_2-f)(t){}_HJ^{\beta}(\psi_2-g)(t).$

(B_5) $\frac{1}{\theta_1}{}_HJ^{\alpha}(\varphi_2-f)^{\theta_1}(t){}_HJ^{\beta}(\psi_2-g)^{\theta_1}(t) + \frac{1}{\theta_2}{}_HJ^{\alpha}(\psi_2-g)^{\theta_2}(t){}_HJ^{\beta}(\varphi_2-f)^{\theta_2}(t)$
$\geq {}_HJ^{\alpha}(\varphi_2-f)(\psi_2-g)(t){}_HJ^{\beta}(\varphi_2-f)(\psi_2-g)(t).$

(C_5) $\frac{1}{\theta_1}\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}(f-\varphi_1)^{\theta_1}(t) + \frac{1}{\theta_2}\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}(g-\psi_1)^{\theta_2}(t)$
$\geq {}_HJ^{\alpha}(f-\varphi_1)(t){}_HJ^{\beta}(g-\psi_1)(t).$

(D_5) $\frac{1}{\theta_1}{}_HJ^{\alpha}(f-\varphi_1)^{\theta_1}(t){}_HJ^{\beta}(g-\psi_1)^{\theta_1}(t) + \frac{1}{\theta_2}{}_HJ^{\alpha}(g-\psi_1)^{\theta_2}(t){}_HJ^{\beta}(f-\varphi_1)^{\theta_2}(t)$
$\geq {}_HJ^{\alpha}(f-\varphi_1)(g-\psi_1)(t){}_HJ^{\beta}(f-\varphi_1)(g-\psi_1)(t).$

Proof The inequalities (A_5)–(D_5) can be obtained by choosing the following parameters in the Young's inequality:

(A_5) $x = \varphi_2(\tau) - f(\tau), \quad y = \psi_2(\rho) - g(\rho).$
(B_5) $x = (\varphi_2(\tau) - f(\tau))(\psi_2(\rho) - g(\rho)), \quad y = (\psi_2(\tau) - g(\tau))(\varphi_2(\rho) - f(\rho)).$
(C_5) $x = f(\tau) - \varphi_1(\tau), \quad y = g(\rho) - \psi_1(\rho).$
(D_5) $x = (f(\tau) - \varphi_1(\tau))(g(\rho) - \psi_1(\rho)), \quad y = (g(\tau) - \psi_1(\tau))(f(\rho) - \varphi_1(\rho)).$

□

Theorem 11.7 *Let f and g be two integrable functions on* $[1,\infty)$ *and* $\theta_1, \theta_2 > 0$ *are such that* $\theta_1 + \theta_2 = 1$. *Suppose that (11.1.1) and (11.5.1) hold. Then, for* $t > 1$, α, $\beta > 0$, *the following inequalities hold:*

(A_6) $\theta_1\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}\varphi_2(t) + \theta_2\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}\psi_2(t)$
$\geq {}_HJ^{\alpha}(\varphi_2-f)^{\theta_1}(t){}_HJ^{\beta}(\psi_2-g)^{\theta_2}(t) + \theta_1\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}f(t) + \theta_2\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}g(t).$

(B_6) $\theta_1{}_HJ^{\alpha}\varphi_2(t){}_HJ^{\beta}\psi_2(t) + \theta_1{}_HJ^{\alpha}f(t){}_HJ^{\beta}g(t)$
$+\theta_2{}_HJ^{\alpha}\psi_2(t){}_HJ^{\beta}\varphi_2(t) + \theta_2{}_HJ^{\alpha}g(t){}_HJ^{\beta}f(t)$

$$\geq {}_HJ^{\alpha}(\varphi_2-f)^{\theta_1}(\psi_2-g)^{\theta_2}(t)\,{}_HJ^{\beta}(\psi_2-g)^{\theta_1}(\varphi_2-f)^{\theta_2}(t)$$
$$+\theta_1\,{}_HJ^{\alpha}\varphi_2(t)\,{}_HJ^{\beta}g(t)+\theta_1\,{}_HJ^{\alpha}f(t)\,{}_HJ^{\beta}\psi_2(t)$$
$$+\theta_2\,{}_HJ^{\alpha}\psi_2(t)\,{}_HJ^{\beta}f(t)+\theta_2\,{}_HJ^{\alpha}g(t)\,{}_HJ^{\beta}\varphi_2(t)$$

(C_6)
$$\theta_1\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}f(t)+\theta_2\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}g(t)$$
$$\geq {}_HJ^{\alpha}(f-\varphi_1)^{\theta_1}(t)\,{}_HJ^{\beta}(g-\psi_1)^{\theta_2}(t)+\theta_1\frac{(\log t)^{\beta}}{\Gamma(\beta+1)}{}_HJ^{\alpha}\varphi_1(t)+\theta_2\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\beta}\psi_1(t).$$

(D_6)
$$\theta_1\,{}_HJ^{\alpha}f(t)\,{}_HJ^{\beta}g(t)+\theta_1\,{}_HJ^{\alpha}\varphi_1(t)\,{}_HJ^{\beta}\psi_1(t)$$
$$+\theta_2\,{}_HJ^{\alpha}g(t)\,{}_HJ^{\beta}f(t)+\theta_2\,{}_HJ^{\alpha}\psi_1(t)\,{}_HJ^{\beta}\varphi_1(t)$$
$$\geq {}_HJ^{\alpha}(f-\varphi_1)^{\theta_1}(g-\psi_1)^{\theta_2}(t)\,{}_HJ^{\beta}(g-\psi_1)^{\theta_1}(f-\varphi_1)^{\theta_2}(t)$$
$$+\theta_1\,{}_HJ^{\alpha}f(t)\,{}_HJ^{\beta}\psi_1(t)+\theta_1\,{}_HJ^{\alpha}\varphi_1(t)\,{}_HJ^{\beta}g(t)$$
$$+\theta_2\,{}_HJ^{\alpha}g(t)\,{}_HJ^{\beta}\varphi_1(t)+\theta_2\,{}_HJ^{\alpha}\psi_1(t)\,{}_HJ^{\beta}f(t).$$

Proof The inequalities (A_6)-(D_6) can be established by choosing the following parameters in the Weighted AM-GM:

(A_6) $x=\varphi_2(\tau)-f(\tau),\quad y=\psi_2(\rho)-g(\rho).$
(B_6) $x=(\varphi_2(\tau)-f(\tau))(\psi_2(\rho)-g(\rho)),\quad y=(\psi_2(\tau)-g(\tau))(\varphi_2(\rho)-f(\rho)).$
(C_6) $x=f(\tau)-\varphi_1(\tau),\quad y=g(\rho)-\psi_1(\rho).$
(D_6) $x=(f(\tau)-\varphi_1(\tau))(g(\rho)-\psi_1(\rho)),\quad y=(g(\tau)-\psi_1(\tau))(f(\rho)-\varphi_1(\rho)).$

□

Theorem 11.8 *Let f and g be two integrable functions on $[1,\infty)$ and there exist constants $p\geq q\geq 0$, $p\neq 0$. Assume that (11.1.1) and (11.5.1) hold. Then, for any $k>0$, $t>1$, α, $\beta>0$, the following inequalities hold:*

(A_7)
$${}_HJ^{\alpha}(\varphi_2-f)^{\frac{q}{p}}(\psi_2-g)^{\frac{q}{p}}(t)+\frac{q}{p}k^{\frac{q-p}{p}}\,{}_HJ^{\alpha}\varphi_2g(t)+\frac{q}{p}k^{\frac{q-p}{p}}\,{}_HJ^{\alpha}f\psi_2(t)$$
$$\leq\frac{q}{p}k^{\frac{q-p}{p}}\,{}_HJ^{\alpha}\varphi_2\psi_2(t)+\frac{q}{p}k^{\frac{q-p}{p}}\,{}_HJ^{\alpha}fg(t)+\frac{p-q}{p}k^{\frac{q}{p}}\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}.$$

(B_7)
$${}_HJ^{\alpha}(\varphi_2-f)^{\frac{q}{p}}(t)\,{}_HJ^{\beta}(\psi_2-g)^{\frac{q}{p}}(t)$$
$$+\frac{q}{p}k^{\frac{q-p}{p}}\,{}_HJ^{\alpha}\varphi_2(t)\,{}_HJ^{\beta}g(t)+\frac{q}{p}k^{\frac{q-p}{p}}\,{}_HJ^{\alpha}f(t)\,{}_HJ^{\beta}\psi_2(t)$$
$$\leq\frac{q}{p}k^{\frac{q-p}{p}}\,{}_HJ^{\alpha}\varphi_2(t)\,{}_HJ^{\beta}\psi_2(t)+\frac{q}{p}k^{\frac{q-p}{p}}\,{}_HJ^{\alpha}f(t)\,{}_HJ^{\beta}g(t)$$
$$+\frac{p-q}{p}k^{\frac{q}{p}}\frac{(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}.$$

(C_7)
$${}_HJ^{\alpha}(f-\varphi_1)^{\frac{q}{p}}(g-\psi_1)^{\frac{q}{p}}(t)+\frac{q}{p}k^{\frac{q-p}{p}}\,{}_HJ^{\alpha}f\psi_1(t)+\frac{q}{p}k^{\frac{q-p}{p}}\,{}_HJ^{\alpha}\varphi_1g(t)$$
$$\leq\frac{q}{p}k^{\frac{q-p}{p}}\,{}_HJ^{\alpha}fg(t)+\frac{q}{p}k^{\frac{q-p}{p}}\,{}_HJ^{\alpha}\varphi_1\psi_1(t)+\frac{p-q}{p}k^{\frac{q}{p}}\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}.$$

$$(D_7)\quad {}_HJ^{\alpha}(f-\varphi_1)^{\frac{q}{p}}(t){}_HJ^{\beta}(g-\psi_1)^{\frac{q}{p}}(t)$$
$$+\frac{q}{p}k^{\frac{q-p}{p}}{}_HJ^{\alpha}f(t){}_HJ^{\beta}\psi_1(t)+\frac{q}{p}k^{\frac{q-p}{p}}{}_HJ^{\alpha}\varphi_1(t){}_HJ^{\beta}g(t)$$
$$\leq\frac{q}{p}k^{\frac{q-p}{p}}{}_HJ^{\alpha}f(t){}_HJ^{\beta}g(t)+\frac{q}{p}k^{\frac{q-p}{p}}{}_HJ^{\alpha}\varphi_1(t){}_HJ^{\beta}\psi_1(t)$$
$$+\frac{p-q}{p}k^{\frac{q}{p}}\frac{(\log t)^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}.$$

Proof The inequalities (A_7)–(D_7) can be proved by choosing the following parameters in Lemma 11.1:

(A_7) $a=(\varphi_2(\tau)-f(\tau))(\psi_2(\tau)-g(\tau))$.
(B_7) $a=(\varphi_2(\tau)-f(\tau))(\psi_2(\rho)-g(\rho))$.
(C_7) $a=(f(\tau)-\varphi_1(\tau))(g(\tau)-\psi_1(\tau))$.
(D_7) $a=(f(\tau)-\varphi_1(\tau))(g(\rho)-\psi_1(\rho))$.

□

Lemma 11.2 *Let f be an integrable function on $[1,\infty)$ and φ_1, φ_2 be two integrable functions on $[1,\infty)$. Assume that the condition (11.1.1) holds. Then, for $t>1$, $\alpha>0$, we have*

$$\begin{aligned}\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}f^2(t)-({}_HJ^{\alpha}f(t))^2&=({}_HJ^{\alpha}\varphi_2(t)-{}_HJ^{\alpha}f(t))({}_HJ^{\alpha}f(t)-{}_HJ^{\alpha}\varphi_1(t))\\&-\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}((\varphi_2-f)(f-\varphi_1)(t))\\&+\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}\varphi_1f(t)-{}_HJ^{\alpha}\varphi_1(t){}_HJ^{\alpha}f(t)\\&+\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}\varphi_2f(t)-{}_HJ^{\alpha}\varphi_2(t){}_HJ^{\alpha}f(t)\\&+{}_HJ^{\alpha}\varphi_1(t){}_HJ^{\alpha}\varphi_2(t)-\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}\varphi_1\varphi_2(t).\end{aligned}\tag{11.13}$$

Proof For any $\tau>1$ and $\rho>1$, we have

$$\begin{aligned}&(\varphi_2(\rho)-f(\rho))(f(\tau)-\varphi_1(\tau))+(\varphi_2(\tau)-f(\tau))(f(\rho)-\varphi_1(\rho))\\&\qquad-(\varphi_2(\tau)-f(\tau))(f(\tau)-\varphi_1(\tau))-(\varphi_2(\rho)-f(\rho))(f(\rho)-\varphi_1(\rho))\\&=f^2(\tau)+f^2(\rho)-2f(\tau)f(\rho)+\varphi_2(\rho)f(\tau)+\varphi_1(\tau)f(\rho)-\varphi_1(\tau)\varphi_2(\rho)\\&\qquad+\varphi_2(\tau)f(\rho)+\varphi_1(\rho)f(\tau)-\varphi_1(\rho)\varphi_2(\tau)-\varphi_2(\tau)f(\tau)+\varphi_1(\tau)\varphi_2(\tau)\\&\qquad-\varphi_1(\tau)f(\tau)-\varphi_2(\rho)f(\rho)+\varphi_1(\rho)\varphi_2(\rho)-\varphi_1(\rho)f(\rho).\end{aligned}\tag{11.14}$$

Multiplying (11.14) by $\left(\log \frac{t}{\tau}\right)^{\alpha-1} / (\tau\Gamma(\alpha))$, $\tau \in (1, t)$, $t > 1$ and integrating the resulting identity with respect to τ from 1 to t, we get

$$\begin{aligned}
&(\varphi_2(\rho) - f(\rho)) \left({}_HJ^\alpha f(t) - {}_HJ^\alpha \varphi_1(t)\right) + \left({}_HJ^\alpha \varphi_2(t) - {}_HJ^\alpha f(t)\right) (f(\rho) - \varphi_1(\rho)) \\
&\quad - {}_HJ^\alpha \left((\varphi_2 - f)(f - \varphi_1)(t)\right) - (\varphi_2(\rho) - f(\rho))(f(\rho) - \varphi_1(\rho)) \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} \\
&= {}_HJ^\alpha f^2(t) + f^2(\rho) \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} - 2f(\rho)\,{}_HJ^\alpha f(t) + \varphi_2(\rho)\,{}_HJ^\alpha f(t) + f(\rho)\,{}_HJ^\alpha \varphi_1(t) \\
&\quad - \varphi_2(\rho)\,{}_HJ^\alpha \varphi_1(t) + f(\rho)\,{}_HJ^\alpha \varphi_2(t) + \varphi_1(\rho)\,{}_HJ^\alpha f(t) - \varphi_1(\rho)\,{}_HJ^\alpha \varphi_2(t) \\
&\quad - {}_HJ^\alpha \varphi_2 f(t) + {}_HJ^\alpha \varphi_1\varphi_2(t) - {}_HJ^\alpha \varphi_1 f(t) - \varphi_2(\rho) f(\rho) \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} \\
&\quad + \varphi_1(\rho)\varphi_2(\rho) \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} - \varphi_1(\rho) f(\rho) \frac{(\log t)^\alpha}{\Gamma(\alpha+1)}.
\end{aligned} \tag{11.15}$$

Multiplying (11.15) by $\left(\log \frac{t}{\rho}\right)^{\alpha-1} / (\rho\Gamma(\alpha))$, $\rho \in (1, t)$, $t > 1$ and then integrating with respect to ρ from 1 to t, we obtain

$$\begin{aligned}
&\left({}_HJ^\alpha \varphi_2(t) - {}_HJ^\alpha f(t)\right) \left({}_HJ^\alpha f(t) - {}_HJ^\alpha \varphi_1(t)\right) \\
&\quad + \left({}_HJ^\alpha \varphi_2(t) - {}_HJ^\alpha f(t)\right) \left({}_HJ^\alpha f(t) - {}_HJ^\alpha \varphi_1(t)\right) \\
&\quad\quad - {}_HJ^\alpha \left((\varphi_2 - f)(f - \varphi_1)(t)\right) \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} \\
&\quad\quad\quad - {}_HJ^\alpha \left((\varphi_2 - f)(f - \varphi_1)(t)\right) \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} \\
&= \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} {}_HJ^\alpha f^2(t) + \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} {}_HJ^\alpha f^2(t) - 2\,{}_HJ^\alpha f(t)\,{}_HJ^\alpha f(t) \\
&\quad + {}_HJ^\alpha \varphi_2(t)\,{}_HJ^\alpha f(t) + {}_HJ^\alpha \varphi_1(t)\,{}_HJ^\alpha f(t) - {}_HJ^\alpha \varphi_1(t)\,{}_HJ^\alpha \varphi_2(t) \\
&\quad + {}_HJ^\alpha \varphi_2(t)\,{}_HJ^\alpha f(t) + {}_HJ^\alpha \varphi_1(t)\,{}_HJ^\alpha f(t) - {}_HJ^\alpha \varphi_1(t)\,{}_HJ^\alpha \varphi_2(t) \\
&\quad - \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} {}_HJ^\alpha \varphi_2 f(t) + \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} {}_HJ^\alpha \varphi_1\varphi_2(t) - \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} {}_HJ^\alpha \varphi_1 f(t) \\
&\quad - \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} {}_HJ^\alpha \varphi_2 f(t) + \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} {}_HJ^\alpha \varphi_1\varphi_2(t) \\
&- \frac{(\log t)^\alpha}{\Gamma(\alpha+1)} {}_HJ^\alpha \varphi_1 f(t),
\end{aligned} \tag{11.16}$$

which implies (11.13). □

Corollary 11.7 *Let f be an integrable function on* $[1,\infty)$ *satisfying* $m \le f(t) \le M$, *for all* $t \in [1,\infty)$. *Then, for all* $t > 1,\ \alpha > 0$, *we have*

$$\begin{aligned}\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}f^2(t) - ({}_HJ^{\alpha}f(t))^2 \\ = \left(M\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)} - {}_HJ^{\alpha}f(t)\right)\left({}_HJ^{\alpha}f(t) - m\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}\right) \\ - \frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}\left((M-f(t))(f(t)-m)\right).\end{aligned} \tag{11.17}$$

Theorem 11.9 *Let f and g be two integrable functions on* $[1,\infty)$ *and* φ_1, φ_2, ψ_1 *and* ψ_2 *are integrable functions on* $[1,\infty)$ *satisfying the conditions (11.1.1) and (11.5.1) on* $[1,\infty)$. *Then, for all* $t > 1$, $\alpha > 0$, *we have*

$$\left|\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}fg(t) - {}_HJ^{\alpha}f(t){}_HJ^{\alpha}g(t)\right| \le |T(f,\varphi_1,\varphi_2)|^{1/2}\,|T(g,\psi_1,\psi_2)|^{1/2}, \tag{11.18}$$

where $T(u, v, w)$ *is defined by*

$$\begin{aligned}T(u,v,w) = ({}_HJ^{\alpha}w(t) - {}_HJ^{\alpha}u(t))\,({}_HJ^{\alpha}u(t) - {}_HJ^{\alpha}v(t)) \\ + \frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}vu(t) - {}_HJ^{\alpha}v(t){}_HJ^{\alpha}u(t) \\ + \frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}wu(t) - {}_HJ^{\alpha}w(t){}_HJ^{\alpha}u(t) \\ + {}_HJ^{\alpha}v(t){}_HJ^{\alpha}w(t) - \frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}vw(t).\end{aligned} \tag{11.19}$$

Proof Let f and g be two integrable functions defined on $[1,\infty)$ satisfying (11.1.1) and (11.5.1). Define

$$H(\tau,\rho) := (f(\tau) - f(\rho))\,(g(\tau) - g(\rho)), \quad \tau,\rho \in (1,t),\ t > 1. \tag{11.20}$$

Multiplying both sides of (11.20) by $\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\alpha-1}/(\tau\rho\Gamma^2(\alpha))$, $\tau,\rho \in (1,t)$ and integrating the resulting identity with respect to τ and ρ from 1 to t, we get

$$\begin{aligned}\frac{1}{2\Gamma^2(\alpha)}\int_1^t\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\alpha-1}H(\tau,\rho)\frac{d\tau}{\tau}\frac{d\rho}{\rho} \\ = \frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}fg(t) - {}_HJ^{\alpha}f(t){}_HJ^{\alpha}g(t).\end{aligned} \tag{11.21}$$

Applying the Cauchy-Schwarz inequality to (11.21), we obtain

$$\begin{aligned}
&\left(\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}fg(t) - {}_HJ^{\alpha}f(t){}_HJ^{\alpha}g(t)\right)^2 \\
&= \left(\frac{1}{2\Gamma^2(\alpha)}\int_1^t\int_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\alpha-1} H(\tau,\rho)\frac{d\tau}{\tau}\frac{d\rho}{\rho}\right)^2 \\
&\le \left(\frac{1}{2\Gamma^2(\alpha)}\int_1^t\int_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\alpha-1} (f(\tau)-f(\rho))^2\frac{d\tau}{\tau}\frac{d\rho}{\rho}\right) \qquad (11.22)\\
&\quad\times\left(\frac{1}{2\Gamma^2(\alpha)}\int_1^t\int_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\alpha-1} (g(\tau)-g(\rho))^2\frac{d\tau}{\tau}\frac{d\rho}{\rho}\right) \\
&= \left(\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}f^2(t) - ({}_HJ^{\alpha}f(t))^2\right)\left(\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}g^2(t) - ({}_HJ^{\alpha}g(t))^2\right).
\end{aligned}$$

Since $(\varphi_2(t)-f(t))(f(t)-\varphi_1(t)) \ge 0$ and $(\psi_2(t)-g(t))(g(t)-\psi_1(t)) \ge 0$ for $t\in[1,\infty)$, we have

$$\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}(\varphi_2-f)(f-\varphi_1)(t) \ge 0,$$

and

$$\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}(\psi_2-g)(g-\psi_1)(t) \ge 0.$$

Then, from Lemma 11.2, we get

$$\begin{aligned}
&\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}f^2(t) - ({}_HJ^{\alpha}f(t))^2 \\
&\le ({}_HJ^{\alpha}\varphi_2(t) - {}_HJ^{\alpha}f(t))\,({}_HJ^{\alpha}f(t) - {}_HJ^{\alpha}\varphi_1(t)) \\
&\quad+\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}\varphi_1 f(t) - {}_HJ^{\alpha}\varphi_1(t){}_HJ^{\alpha}f(t) \qquad (11.23)\\
&\quad+\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}\varphi_2 f(t) - {}_HJ^{\alpha}\varphi_2(t){}_HJ^{\alpha}f(t) \\
&\quad+{}_HJ^{\alpha}\varphi_1(t){}_HJ^{\alpha}\varphi_2(t) - \frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ^{\alpha}\varphi_1\varphi_2(t) \\
&= T(f,\varphi_1,\varphi_2),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)} {}_{H}J^{\alpha} g^{2}(t) - ({}_{H}J^{\alpha} g(t))^{2} \\
&\leq ({}_{H}J^{\alpha}\psi_{2}(t) - {}_{H}J^{\alpha} g(t))\,({}_{H}J^{\alpha} g(t) - {}_{H}J^{\alpha}\psi_{1}(t)) \\
&\quad + \frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)} {}_{H}J^{\alpha}\psi_{1} g(t) - {}_{H}J^{\alpha}\psi_{1}(t) {}_{H}J^{\alpha} g(t) \qquad (11.24)\\
&\quad + \frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)} {}_{H}J^{\alpha}\psi_{2} g(t) - {}_{H}J^{\alpha}\psi_{2}(t) {}_{H}J^{\alpha} g(t) \\
&\quad + {}_{H}J^{\alpha}\psi_{1}(t) {}_{H}J^{\alpha}\psi_{2}(t) - \frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)} {}_{H}J^{\alpha}\psi_{1}\psi_{2}(t) \\
&= T(g,\psi_{1},\psi_{2}).
\end{aligned}$$

From (11.22)–(11.24), we obtain (11.18). □

Remark 11.1 If $T(f,\varphi_1,\varphi_2) = T(f,m,M)$ and $T(g,\psi_1,\psi_2) = T(g,p,P)$, $m, M, p, P \in \mathbb{R}$, then the inequality (11.18) reduces to the following Grüss type Hadamard fractional integral inequality:

$$\left|\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)} {}_{H}J^{\alpha} fg(t) - {}_{H}J^{\alpha} f(t) {}_{H}J^{\alpha} g(t)\right| \leq \left(\frac{1}{2}\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)}\right)^{2} (M-m)(P-p).$$

11.3 On Mixed Type Riemann-Liouville and Hadamard Fractional Integral Inequalities

In this section, we obtain some new inequalities of mixed type for Riemann-Liouville and Hadamard fractional integrals for the functions, which are bounded by integrable functions and are not necessarily increasing or decreasing like the synchronous functions.

Theorem 11.10 *Let f be an integrable function on $[a,\infty)$, $a > 0$. Assume that:*

(11.10.1) there exist two integrable functions φ_1, φ_2 on $[a,\infty)$ such that

$$\varphi_1(t) \leq f(t) \leq \varphi_2(t), \quad \textit{for all } t \in [a,\infty),\ a > 0. \qquad (11.25)$$

Then, for $0 < a < t < \infty$ and $\alpha, \beta > 0$, the following two inequalities hold:

(E_1) $\quad {}_{H}J_a^{\alpha}\varphi_2(t) I_a^{\beta} f(t) + {}_{H}J_a^{\alpha} f(t) I_a^{\beta}\varphi_1(t) \geq {}_{H}J_a^{\alpha}\varphi_2(t) I_a^{\beta}\varphi_1(t) + {}_{H}J_a^{\alpha} f(t) I_a^{\beta} f(t),$

(F_1) $\quad I_a^{\alpha}\varphi_2(t) {}_{H}J_a^{\beta} f(t) + I_a^{\alpha} f(t) {}_{H}J_a^{\beta}\varphi_1(t) \geq I_a^{\alpha}\varphi_2(t) {}_{H}J_a^{\beta}\varphi_1(t) + I_a^{\alpha} f(t) {}_{H}J_a^{\beta} f(t).$

Proof From (11.10.1), for all $\tau, \rho > a$, we have

$$(\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho)) \geq 0,$$

which implies that

$$\varphi_2(\tau)f(\rho) + \varphi_1(\rho)f(\tau) \geq \varphi_1(\rho)\varphi_2(\tau) + f(\tau)f(\rho). \tag{11.26}$$

Multiplying both sides of (11.26) by $(\log(t/\tau))^{\alpha-1}/(\tau\Gamma(\alpha))$, $\tau \in (a,t)$, and then integrating with respect to τ on (a,t), we obtain

$$f(\rho)\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\varphi_2(\tau)\frac{d\tau}{\tau} + \varphi_1(\rho)\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}f(\tau)\frac{d\tau}{\tau}$$

$$\geq \varphi_1(\rho)\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\varphi_2(\tau)\frac{d\tau}{\tau} + f(\rho)\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}f(\tau)\frac{d\tau}{\tau},$$

which yields

$$f(\rho)\,{}_HJ_a^\alpha\varphi_2(t) + \varphi_1(\rho)\,{}_HJ_a^\alpha f(t) \geq \varphi_1(\rho)\,{}_HJ_a^\alpha\varphi_2(t) + f(\rho)\,{}_HJ_a^\alpha f(t). \tag{11.27}$$

Multiplying both sides of (11.27) by $(t-\rho)^{\beta-1}/\Gamma(\beta)$, $\rho \in (a,t)$, and then integrating with respect to ρ on (a,t), we get

$${}_HJ_a^\alpha\varphi_2(t)\frac{1}{\Gamma(\beta)}\int_a^t (t-\rho)^{\beta-1}f(\rho)d\rho + {}_HJ_a^\alpha f(t)\frac{1}{\Gamma(\beta)}\int_a^t (t-\rho)^{\beta-1}\varphi_1(\rho)d\rho$$

$$\geq {}_HJ_a^\alpha\varphi_2(t)\frac{1}{\Gamma(\beta)}\int_a^t (t-\rho)^{\beta-1}\varphi_1(\rho)d\rho + {}_HJ_a^\alpha f(t)\frac{1}{\Gamma(\beta)}\int_a^t (t-\rho)^{\beta-1}f(\rho)d\rho. \tag{11.28}$$

Hence, we get the desired inequality in (E_1). The inequality (F_1) can be obtained by using similar arguments. □

Corollary 11.8 *Let f be an integrable function on $[a,\infty)$, $a > 0$ satisfying $m \leq f(t) \leq M$, for all $t \in [a,\infty)$ and $m, M \in \mathbb{R}$. Then, for $0 < a < t < \infty$ and $\alpha, \beta > 0$, the following two inequalities hold:*

(E_2) $M\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\beta f(t) + m\frac{(t-a)^\beta}{\Gamma(\beta+1)}{}_HJ_a^\alpha f(t) \geq mM\frac{(\log\frac{t}{a})^\alpha(t-a)^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} + {}_HJ_a^\alpha f(t)I_a^\beta f(t),$

(F_2) $M\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}{}_HJ_a^\beta f(t) + m\frac{(\log\frac{t}{a})^\beta}{\Gamma(\beta+1)}I_a^\alpha f(t) \geq mM\frac{(t-a)^\alpha(\log\frac{t}{a})^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} + I_a^\alpha f(t)\,{}_HJ_a^\beta f(t).$

Theorem 11.11 *Let f be an integrable function on $[a,\infty)$, $a > 0$ and there exist $\theta_1, \theta_2 > 0$ such that $1/\theta_1 + 1/\theta_2 = 1$. In addition, suppose that the condition (11.10.1) holds. Then, for $0 < a < t < \infty$ and α, $\beta > 0$, the following two inequalities hold:*

$$(E_3)\quad {}_HJ_a^\alpha\varphi_2(t)I_a^\beta\varphi_1(t) + {}_HJ_a^\alpha f(t)I_a^\beta f(t) + \frac{1}{\theta_1}\frac{(t-a)^\beta}{\Gamma(\beta+1)}{}_HJ_a^\alpha(\varphi_2 - f)^{\theta_1}(t)$$
$$+ \frac{1}{\theta_2}\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\beta(f-\varphi_1)^{\theta_2}(t)$$
$$\geq {}_HJ_a^\alpha\varphi_2(t)I_a^\beta f(t) + {}_HJ_a^\alpha f(t)I_a^\beta\varphi_1(t),$$
$$(F_3)\quad I_a^\alpha\varphi_2(t){}_HJ_a^\beta\varphi_1(t) + I_a^\alpha f(t){}_HJ_a^\beta f(t) + \frac{1}{\theta_1}\frac{(\log\frac{t}{a})^\beta}{\Gamma(\beta+1)}I_a^\alpha(\varphi_2 - f)^{\theta_1}(t)$$
$$+ \frac{1}{\theta_2}\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}{}_HJ_a^\beta(f-\varphi_1)^{\theta_2}(t)$$
$$\geq I_a^\alpha\varphi_2(t){}_HJ_a^\beta f(t) + I_a^\alpha f(t){}_HJ_a^\beta\varphi_1(t).$$

Proof Firstly, we recall the well-known Young's inequality:

$$\frac{1}{\theta_1}x^{\theta_1} + \frac{1}{\theta_2}y^{\theta_2} \geq xy, \quad \forall x, y \geq 0, \quad \theta_1, \theta_2 > 0,$$

where $1/\theta_1 + 1/\theta_2 = 1$. By setting $x = \varphi_2(\tau) - f(\tau)$ and $y = f(\rho) - \varphi_1(\rho)$, $\tau, \rho > a$, in the above inequality, we have

$$\frac{1}{\theta_1}(\varphi_2(\tau) - f(\tau))^{\theta_1} + \frac{1}{\theta_2}(f(\rho) - \varphi_1(\rho))^{\theta_2} \geq (\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho)). \tag{11.29}$$

Multiplying both sides of (11.29) by $(\log(t/\tau))^{\alpha-1}(t-\rho)^{\beta-1}/\tau\Gamma(\alpha)\Gamma(\beta)$, $\tau, \rho \in (a,t)$, we get

$$\begin{aligned}
&\frac{1}{\theta_1}\frac{(\log t/\tau)^{\alpha-1}(t-\rho)^{\beta-1}}{\tau\Gamma(\alpha)\Gamma(\beta)}(\varphi_2(\tau) - f(\tau))^{\theta_1} \\
&+ \frac{1}{\theta_2}\frac{(\log t/\tau)^{\alpha-1}(t-\rho)^{\beta-1}}{\tau\Gamma(\alpha)\Gamma(\beta)}(f(\rho) - \varphi_1(\rho))^{\theta_2} \\
&\geq \frac{(\log t/\tau)^{\alpha-1}}{\tau\Gamma(\alpha)}(\varphi_2(\tau) - f(\tau))\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}(f(\rho) - \varphi_1(\rho)).
\end{aligned} \tag{11.30}$$

Double integrating (11.30) with respect to τ and ρ from a to t, we obtain

$$\frac{1}{\theta_1}{}_HJ_a^\alpha(\varphi_2 - f)^{\theta_1}(t)I_a^\beta(1)(t) + \frac{1}{\theta_2}{}_HJ_a^\alpha(1)(t)I_a^\beta(f-\varphi_1)^{\theta_2}(t)$$
$$\geq {}_HJ_a^\alpha(\varphi_2 - f)(t)I_a^\beta(f-\varphi_1)(t),$$

which proves the result in (E_3). By using the similar method, we obtain the inequality in (F_3). □

Corollary 11.9 *Let f be an integrable function on $[a,\infty)$, $a>0$ satisfying $m \le f(t) \le M, \theta_1 = \theta_2 = 2$ for all $t \in [a,\infty)$ and $m, M \in \mathbb{R}$. Then, for $0<a<t<\infty$ and $\alpha, \beta > 0$, the following two inequalities hold:*

$$(E_4) \quad (m+M)^2 \frac{(\log \frac{t}{a})^{\alpha}(t-a)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{(t-a)^{\beta}}{\Gamma(\beta+1)} {}_HJ_a^{\alpha} f^2(t) + \frac{(\log \frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)} I_a^{\beta} f^2(t) + 2{}_HJ_a^{\alpha} f(t) I_a^{\beta} f(t)$$
$$\ge 2(m+M)\left(\frac{(\log \frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)} I_a^{\beta} f(t) + \frac{(t-a)^{\beta}}{\Gamma(\beta+1)} {}_HJ_a^{\alpha} f(t)\right),$$

$$(F_4) \quad (m+M)^2 \frac{(t-a)^{\alpha}(\log \frac{t}{a})^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} {}_HJ_a^{\beta} f^2(t) + \frac{(\log \frac{t}{a})^{\beta}}{\Gamma(\beta+1)} I_a^{\alpha} f^2(t) + 2{}_HJ_a^{\beta} f(t) I_a^{\alpha} f(t)$$
$$\ge 2(m+M)\left(\frac{(\log \frac{t}{a})^{\beta}}{\Gamma(\beta+1)} I_a^{\alpha} f(t) + \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} {}_HJ_a^{\beta} f(t)\right).$$

Theorem 11.12 *Let f be an integrable function on $[a,\infty)$, $a>0$ and $\theta_1, \theta_2 > 0$ satisfying $\theta_1 + \theta_2 = 1$. In addition, suppose that the condition (11.10.1) holds. Then, for $0<a<t<\infty$, and α, $\beta>0$, the following two inequalities hold:*

$$(E_5) \quad \theta_1 \frac{(t-a)^{\beta}}{\Gamma(\beta+1)} {}_HJ_a^{\alpha} \varphi_2(t) + \theta_2 \frac{(\log \frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)} I_a^{\beta} f(t)$$
$$\ge {}_HJ_a^{\alpha}(\varphi_2 - f)^{\theta_1}(t) I_a^{\beta}(f-\varphi_1)^{\theta_2}(t) + \theta_1 \frac{(t-a)^{\beta}}{\Gamma(\beta+1)} {}_HJ_a^{\alpha} f(t) + \theta_2 \frac{(\log \frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)} I_a^{\beta} \varphi_1(t),$$

$$(F_5) \quad \theta_1 \frac{(\log \frac{t}{a})^{\beta}}{\Gamma(\beta+1)} I_a^{\alpha} \varphi_2(t) + \theta_2 \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} {}_HJ_a^{\beta} f(t)$$
$$\ge I_a^{\alpha}(\varphi_2 - f)^{\theta_1}(t) {}_HJ_a^{\beta}(f-\varphi_1)^{\theta_2}(t) + \theta_1 \frac{(\log \frac{t}{a})^{\beta}}{\Gamma(\beta+1)} I_a^{\alpha} f(t) + \theta_2 \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} {}_HJ_a^{\beta} \varphi_1(t).$$

Proof Let us consider the well-known Weighted AM-GM inequality

$$\theta_1 x + \theta_2 y \ge x^{\theta_1} y^{\theta_2}, \quad \forall x, y \ge 0, \quad \theta_1, \theta_2 > 0,$$

where $\theta_1 + \theta_2 = 1$. Setting $x = \varphi_2(\tau) - f(\tau)$ and $y = f(\rho) - \varphi_1(\rho)$, $\tau, \rho > a$, in the above inequality, we have

$$\theta_1(\varphi_2(\tau) - f(\tau)) + \theta_2(f(\rho) - \varphi_1(\rho)) \ge (\varphi_2(\tau) - f(\tau))^{\theta_1}(f(\rho) - \varphi_1(\rho))^{\theta_2}. \quad (11.31)$$

Multiplying both sides of (11.31) by $(\log(t/\tau))^{\alpha-1}(t-\rho)^{\beta-1}/(\tau\Gamma(\alpha)\Gamma(\beta))$, $\tau, \rho \in (a,t)$, we obtain

$$\theta_1 \frac{(\log(t/\tau))^{\alpha-1}(t-\rho)^{\beta-1}}{\tau\Gamma(\alpha)\Gamma(\beta)}(\varphi_2(\tau) - f(\tau))$$

$$+\,\theta_2\frac{(\log(t/\tau))^{\alpha-1}(t-\rho)^{\beta-1}}{\tau\Gamma(\alpha)\Gamma(\beta)}(f(\rho)-\varphi_1(\rho)) \tag{11.32}$$
$$\geq \frac{(\log t/\tau)^{\alpha-1}}{\tau\Gamma(\alpha)}(\varphi_2(\tau)-f(\tau))^{\theta_1}\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}(f(\rho)-\varphi_1(\rho))^{\theta_2}.$$

Double integration of (11.32) with respect to τ and ρ from a to t, we obtain

$$\theta_{1H}J_a^{\alpha}(\varphi_2-f)(t)I_a^{\beta}(1)(t)+\theta_{2H}J_a^{\alpha}(1)(t)I_a^{\beta}(f-\varphi_1)(t)$$
$$\geq {}_HJ_a^{\alpha}(\varphi_2-f)^{\theta_1}(t)I_a^{\beta}(f-\varphi_1)^{\theta_2}(t).$$

Therefore, we deduce the inequality in (E_5). By using the similar method, we obtain the inequality in (F_5). □

Corollary 11.10 *Let f be an integrable function on $[a,\infty)$, $a>0$ satisfying $m\leq f(t)\leq M$, $\theta_1=\theta_2=1/2$ for all $0<a<t<\infty$ and $m,M\in\mathbb{R}$. Then, for $0<a<t<\infty$ and $\alpha,\beta>0$, the following two inequalities hold:*

$$(E_6)\quad M\frac{(\log\frac{t}{a})^{\alpha}(t-a)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}I_a^{\beta}f(t)$$
$$\geq m\frac{(\log\frac{t}{a})^{\alpha}(t-a)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}{}_HJ_a^{\alpha}f(t)+2{}_HJ_a^{\alpha}(M-f)^{1/2}(t)I_a^{\beta}(f-m)^{1/2}(t),$$
$$(F_6)\quad M\frac{(t-a)^{\alpha}(\log\frac{t}{a})^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ_a^{\beta}f(t)$$
$$\geq m\frac{(t-a)^{\alpha}(\log\frac{t}{a})^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{(\log\frac{t}{a})^{\beta}}{\Gamma(\beta+1)}I_a^{\alpha}f(t)+2I_a^{\alpha}(M-f)^{1/2}(t){}_HJ_a^{\beta}(f-m)^{1/2}(t).$$

Theorem 11.13 *Let f be an integrable function on $[a,\infty)$, $a>0$ and there exist constants $p\geq q\geq 0$, $p\neq 0$. In addition, assume that the condition (11.10.1) holds. Then, for any $k>0$, $0<a<t<\infty$, $\alpha>0$, the following two inequalities hold:*

$$(E_7)\quad {}_HJ_a^{\alpha}(\varphi_2-f)^{q/p}(t)I_a^{\alpha}(f-\varphi_1)^{q/p}(t)+\frac{q}{p}k^{(q-p)/p}\left({}_HJ_a^{\alpha}\varphi_2(t)I_a^{\alpha}\varphi_1(t)+{}_HJ_a^{\alpha}f(t)I_a^{\alpha}f(t)\right)$$
$$\leq\frac{q}{p}k^{(q-p)/p}\left({}_HJ_a^{\alpha}\varphi_2(t)I_a^{\alpha}f(t)+{}_HJ_a^{\alpha}f(t)I_a^{\alpha}\varphi_1(t)\right)+\frac{p-q}{p}k^{q/p}\frac{(t-a)^{\alpha}(\log\frac{t}{a})^{\alpha}}{\Gamma^2(\alpha+1)},$$
$$(F_7)\quad I_a^{\alpha}(\varphi_2-f)^{q/p}(t){}_HJ_a^{\alpha}(f-\varphi_1)^{q/p}(t)+\frac{q}{p}k^{(q-p)/p}\left(I_a^{\alpha}\varphi_2(t){}_HJ_a^{\alpha}\varphi_1(t)+I_a^{\alpha}f(t){}_HJ_a^{\alpha}f(t)\right)$$
$$\leq\frac{q}{p}k^{(q-p)/p}\left(I_a^{\alpha}\varphi_2(t){}_HJ_a^{\alpha}f(t)+I_a^{\alpha}f(t){}_HJ_a^{\alpha}\varphi_1(t)\right)+\frac{p-q}{p}k^{q/p}\frac{(t-a)^{\alpha}(\log\frac{t}{a})^{\alpha}}{\Gamma^2(\alpha+1)}.$$

Proof From the assumption (11.10.1) and Lemma 11.1, for $p\geq q\geq 0$, $p\neq 0$, it follows that

$$((\varphi_2(\tau)-f(\tau))(f(\rho)-\varphi_1(\rho)))^{q/p}$$
$$\leq\frac{q}{p}k^{(q-p)/p}(\varphi_2(\tau)-f(\tau))(f(\rho)-\varphi_1(\rho))+\frac{p-q}{p}k^{q/p}, \tag{11.33}$$

for any $k > 0$. Multiplying both sides of (11.33) by $(\log(t/\tau))^{\alpha-1}/(\tau\Gamma(\alpha))$, $\tau \in (a, t)$, and integrating the resulting identity with respect to τ from a to t, we have

$$(f(\rho) - \varphi_1(\rho))^{q/p}\,{}_H J_a^\alpha(\varphi_2 - f)^{q/p}(t) \\ \leq \frac{q}{p} k^{(q-p)/p}(f(\rho) - \varphi_1(\rho))\,{}_H J_a^\alpha(\varphi_2 - f)(t) + \frac{p-q}{p} k^{q/p} \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)}. \tag{11.34}$$

Multiplying both sides of (11.34) by $(t-\rho)^{\alpha-1}/\Gamma(\alpha)$, $\rho \in (a, t)$, and integrating the resulting identity with respect to ρ from a to t, we obtain

$${}_H J_a^\alpha(\varphi_2 - f)^{q/p}(t) I_a^\alpha(f - \varphi_1)(t)^{q/p} \\ \leq \frac{q}{p} k^{(q-p)/p}\,{}_H J_a^\alpha(\varphi_2 - f)(t) I_a^\alpha(f - \varphi_1)(t) + \frac{p-q}{p} k^{q/p} \frac{(t-a)^\alpha (\log \frac{t}{a})^\alpha}{\Gamma^2(\alpha+1)},$$

which leads to the inequality in (E_6). Applying the similar arguments, we get the inequality in (F_6). □

Corollary 11.11 *Let f be an integrable function on $[a, \infty)$, $a > 0$ satisfying $m \leq f(t) \leq M$ for all $t \in [a, \infty)$, $q = 1$, $p = 2$, $k = 1$ and $m, M \in \mathbb{R}$. Then, for $0 < a < t < \infty$ and $\alpha > 0$, the following two inequalities hold:*

(E_8) $$2\,{}_H J_a^\alpha(M - f)^{1/2}(t) I_a^\alpha(f - m)^{1/2}(t) + {}_H J_a^\alpha f(t) I_a^\alpha f(t) \\ \leq \frac{M(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha f(t) + \frac{m(t-a)^\alpha}{\Gamma(\alpha+1)}\,{}_H J_a^\alpha f(t) + (1 - mM)\frac{(t-a)^\alpha (\log \frac{t}{a})^\alpha}{\Gamma^2(\alpha+1)},$$

(F_8) $$2 I_a^\alpha(M - f)^{1/2}(t)\,{}_H J_a^\alpha(f - m)^{1/2}(t) + I_a^\alpha f(t)\,{}_H J_a^\alpha f(t) \\ \leq \frac{M(t-a)^\alpha}{\Gamma(\alpha+1)}\,{}_H J_a^\alpha f(t) + \frac{m(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha f(t) + (1 - mM)\frac{(t-a)^\alpha (\log \frac{t}{a})^\alpha}{\Gamma^2(\alpha+1)}.$$

11.4 Chebyshev Type Inequalities for Riemann-Liouville and Hadamard Fractional Integrals

In this section, we establish fractional integral inequalities of Chebyshev type involving the integral of the product of two functions and the product of two integrals. We make use of the following lemma in the forthcoming results.

Lemma 11.3 *Let f be an integrable function on $[a, \infty)$, $a > 0$ and φ_1, φ_2 are two integrable functions on $[a, \infty)$. Assume that the condition (11.10.1) holds. Then, for $0 < a < t < \infty$, and $\alpha, \beta > 0$, we have*

$$(E_9)\quad \frac{(t-a)^\alpha}{\Gamma(\alpha+1)}{}_HJ_a^\alpha f^2(t)+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha f^2(t)-2{}_HJ_a^\alpha f(t)I_a^\alpha f(t)$$
$$={}_HJ_a^\alpha(f-\varphi_1)(t)I_a^\alpha(\varphi_2-f)(t)+{}_HJ_a^\alpha(\varphi_2-f)(t)I_a^\alpha(f-\varphi_1)(t)$$
$$+\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}\left({}_HJ_a^\alpha(\varphi_1 f+\varphi_2 f-\varphi_1\varphi_2)(t)-{}_HJ_a^\alpha((\varphi_2-f)(f-\varphi_1))(t)\right)$$
$$+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}\left(I_a^\alpha(\varphi_1 f+\varphi_2 f-\varphi_1\varphi_2)(t)-I_a^\alpha((\varphi_2-f)(f-\varphi_1))(t)\right)$$
$$+{}_HJ_a^\alpha\varphi_1(t)I_a^\alpha(\varphi_2-f)(t)+{}_HJ_a^\alpha\varphi_2(t)I_a^\alpha(\varphi_1-f)(t)-{}_HJ_a^\alpha f(t)I_a^\alpha(\varphi_1+\varphi_2)(t),$$

$$(F_9)\quad \frac{(t-a)^\beta}{\Gamma(\beta+1)}{}_HJ_a^\alpha f^2(t)+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\beta f^2(t)-2{}_HJ_a^\alpha f(t)I_a^\beta f(t)$$
$$={}_HJ_a^\alpha(f-\varphi_1)(t)I_a^\beta(\varphi_2-f)(t)+{}_HJ_a^\alpha(\varphi_2-f)(t)I_a^\beta(f-\varphi_1)(t)$$
$$+\frac{(t-a)^\beta}{\Gamma(\beta+1)}\left({}_HJ_a^\alpha(\varphi_1 f+\varphi_2 f-\varphi_1\varphi_2)(t)-{}_HJ_a^\alpha((\varphi_2-f)(f-\varphi_1))(t)\right)$$
$$+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}\left(I_a^\beta(\varphi_1 f+\varphi_2 f-\varphi_1\varphi_2)(t)-I_a^\beta((\varphi_2-f)(f-\varphi_1))(t)\right)$$
$$+{}_HJ_a^\alpha\varphi_1(t)I_a^\beta(\varphi_2-f)(t)+{}_HJ_a^\alpha\varphi_2(t)I_a^\beta(\varphi_1-f)(t)-{}_HJ_a^\alpha f(t)I_a^\beta(\varphi_1+\varphi_2)(t).$$

Proof For any $0<a<\tau,\rho<t<\infty$, we have

$$\begin{aligned}&(\varphi_2(\rho)-f(\rho))(f(\tau)-\varphi_1(\tau))+(\varphi_2(\tau)-f(\tau))(f(\rho)-\varphi_1(\rho))\\&-(\varphi_2(\tau)-f(\tau))(f(\tau)-\varphi_1(\tau))-(\varphi_2(\rho)-f(\rho))(f(\rho)-\varphi_1(\rho))\\=&f^2(\tau)+f^2(\rho)-2f(\tau)f(\rho)+\varphi_2(\rho)f(\tau)+\varphi_1(\tau)f(\rho)-\varphi_1(\tau)\varphi_2(\rho)\\&+\varphi_2(\tau)f(\rho)+\varphi_1(\rho)f(\tau)-\varphi_1(\rho)\varphi_2(\tau)-\varphi_2(\tau)f(\tau)+\varphi_1(\tau)\varphi_2(\tau)\\&-\varphi_1(\tau)f(\tau)-\varphi_2(\rho)f(\rho)+\varphi_1(\rho)\varphi_2(\rho)-\varphi_1(\rho)f(\rho).\end{aligned}\tag{11.35}$$

Multiplying (11.35) by $(\log(t/\tau))^{\alpha-1}/(\tau\Gamma(\alpha))$, $\tau\in(a,t)$, $0<a<t<\infty$, and integrating the resulting identity with respect to τ from a to t, we get

$$\begin{aligned}&(\varphi_2(\rho)-f(\rho))({}_HJ_a^\alpha f(t)-{}_HJ_a^\alpha\varphi_1(t))+({}_HJ_a^\alpha\varphi_2(t)-{}_HJ_a^\alpha f(t))(f(\rho)-\varphi_1(\rho))\\&-{}_HJ_a^\alpha((\varphi_2-f)(f-\varphi_1))(t)-(\varphi_2(\rho)-f(\rho))(f(\rho)-\varphi_1(\rho))\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}\\=&{}_HJ_a^\alpha f^2(t)+f^2(\rho)\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}-2f(\rho){}_HI_a^\alpha f(t)+\varphi_2(\rho){}_HI_a^\alpha f(t)\\&+f(\rho){}_HJ_a^\alpha\varphi_1(t)-\varphi_2(\rho){}_HJ_a^\alpha\varphi_1(t)+f(\rho){}_HJ_a^\alpha\varphi_2(t)+\varphi_1(\rho){}_HJ_a^\alpha f(t)-\varphi_1(\rho){}_HJ_a^\alpha\varphi_2(t)\\&-{}_HJ_a^\alpha\varphi_2 f(t)+{}_HJ_a^\alpha\varphi_1\varphi_2(t)-{}_HJ_a^\alpha\varphi_1 f(t)-\varphi_2(\rho)f(\rho)\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}\\&+\varphi_1(\rho)\varphi_2(\rho)\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}-\varphi_1(\rho)f(\rho)\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}.\end{aligned}\tag{11.36}$$

Multiplying (11.36) by $(t-\rho)^{\alpha-1}/\Gamma(\alpha)$, $\rho \in (a,t)$, $0 < a < t < \infty$, and integrating the resulting identity with respect to ρ from a to t, we obtain

$$\begin{aligned}
&({}_HJ_a^{\alpha}f(t) - {}_HJ_a^{\alpha}\varphi_1(t))(I_a^{\alpha}\varphi_2(t) - I_a^{\alpha}f(t)) + ({}_HJ_a^{\alpha}\varphi_2(t) - {}_HJ_a^{\alpha}f(t))(I_a^{\alpha}f(t) - I_a^{\alpha}\varphi_1(t))\\
&- {}_HJ_a^{\alpha}((\varphi_2 - f)(f - \varphi_1))(t)\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} - I_a^{\alpha}((\varphi_2 - f)(f - \varphi_1))(t)\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}\\
= &{}_HJ_a^{\alpha}f^2(t)\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} + I_a^{\alpha}f^2(t)\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)} - 2{}_HJ_a^{\alpha}f(t)I_a^{\alpha}f(t)\\
&+ {}_HJ_a^{\alpha}f(t)I_a^{\alpha}\varphi_2(t) + {}_HJ_a^{\alpha}\varphi_1(t)I_a^{\alpha}f(t) - {}_HJ_a^{\alpha}\varphi_1(t)I_a^{\alpha}\varphi_2(t)\\
&+ {}_HJ_a^{\alpha}\varphi_2(t)I_a^{\alpha}f(t) + {}_HJ_a^{\alpha}f(t)I_a^{\alpha}\varphi_1(t) - {}_HJ_a^{\alpha}\varphi_2(t)I_a^{\alpha}\varphi_1(t)\\
&- {}_HJ_a^{\alpha}\varphi_2 f(t)\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} + {}_HJ_a^{\alpha}\varphi_1\varphi_2(t)\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} - {}_HJ_a^{\alpha}\varphi_1 f(t)\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}\\
&- I_a^{\alpha}\varphi_2 f(t)\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)} + I_a^{\alpha}\varphi_1\varphi_2(t)\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)} - I_a^{\alpha}\varphi_1 f(t)\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}$$

Therefore, (E_9) is proved. (F_9) can be derived by using the similar arguments. □

Theorem 11.14 *Let f and g be two integrable functions on $[a,\infty)$, $a > 0$ and φ_1, φ_2, ψ_1 and ψ_2 are integrable functions on $[a,\infty)$ satisfying the conditions (11.10.1) and*

(11.14.1) there exist integrable functions ψ_1 and ψ_2 on $[a,\infty)$ such that

$$\psi_1(t) \le g(t) \le \psi_2(t) \quad \textit{for} \quad 0 < a < t < \infty.$$

Then, for all $0 < a < t < \infty$ and $\alpha > 0$, the following inequality holds:

$$\begin{aligned}
&\Big|\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ_a^{\alpha}fg(t) + \frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}I_a^{\alpha}fg(t) - {}_HJ_a^{\alpha}f(t)I_a^{\alpha}g(t) - I_a^{\alpha}f(t){}_HJ_a^{\alpha}g(t)\Big|\\
&\quad\le |K(f,\varphi_1,\varphi_2)|^{1/2}\,|K(g,\psi_1,\psi_2)|^{1/2}, \qquad (11.37)
\end{aligned}$$

where $K(u,v,w)$ is defined by

$$\begin{aligned}
K(u,v,w) = &\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ_a^{\alpha}(uw+uv-vw)(t)\\
&+\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}I_a^{\alpha}(uw+uv-vw)(t) - 2{}_HJ_a^{\alpha}u(t)I_a^{\alpha}u(t).
\end{aligned}$$

Proof Let f and g be two integrable functions defined on $[a,\infty)$ satisfying (11.10.1) and (11.14.1), respectively. We define a function H for $0<a<t<\infty$ as follows

$$H(\tau,\rho) := (f(\tau)-f(\rho))(g(\tau)-g(\rho)), \qquad \tau,\rho \in (a,t). \tag{11.38}$$

Multiplying both sides of (11.38) by $(\log(t/\tau))^{\alpha-1}(t-\rho)^{\alpha-1}/(\tau\Gamma^2(\alpha))$, $\tau,\rho \in (a,t)$, and double integrating the resulting identity with respect to τ and ρ from a to t, we get

$$\begin{aligned}&\frac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}(t-\rho)^{\alpha-1}H(\tau,\rho)d\rho\frac{d\tau}{\tau}\\ &=\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}{}_HJ_a^\alpha fg(t)+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha fg(t)-{}_HJ_a^\alpha f(t)I_a^\alpha g(t)-I_a^\alpha f(t){}_HJ_a^\alpha g(t).\end{aligned} \tag{11.39}$$

Applying the Cauchy-Schwarz inequality to (11.39), we have

$$\begin{aligned}&\left(\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}{}_HJ_a^\alpha fg(t)+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha fg(t)-{}_HJ_a^\alpha f(t)I_a^\alpha g(t)-I_a^\alpha f(t){}_HJ_a^\alpha g(t)\right)^2\\ &\le\left(\frac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}(t-\rho)^{\alpha-1}(f(\tau)-f(\rho))^2d\rho\frac{d\tau}{\tau}\right)\\ &\quad\times\left(\frac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}(t-\rho)^{\alpha-1}(g(\tau)-g(\rho))^2d\rho\frac{d\tau}{\tau}\right)\\ &=\left(\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}{}_HJ_a^\alpha f^2(t)+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha f^2(t)-2{}_HJ_a^\alpha f(t)I_a^\alpha f(t)\right)\\ &\quad\times\left(\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}{}_HJ_a^\alpha g^2(t)+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha g^2(t)-2{}_HJ_a^\alpha g(t)I_a^\alpha g(t)\right).\end{aligned} \tag{11.40}$$

Since $(\varphi_2(t)-f(t))(f(t)-\varphi_1(t))\ge 0$ and $(\psi_2(t)-f(t))(f(t)-\psi_1(t))\ge 0$ for $t\in[a,\infty)$, we get

$$\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}{}_HJ_a^\alpha((\varphi_2-f)(f-\varphi_1))(t)\ge 0,$$

$$\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha((\varphi_2-f)(f-\varphi_1))(t)\ge 0,$$

$$\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}{}_HJ_a^\alpha((\psi_2-g)(g-\psi_1))(t)\ge 0,$$

$$\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha((\psi_2-g)(g-\psi_1))(t)\ge 0.$$

Thus, from Lemma 11.3, we obtain

$$
\begin{aligned}
&\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} {}_HJ_a^\alpha f^2(t) + \frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha f^2(t) - 2\,{}_HJ_a^\alpha f(t) I_a^\alpha f(t)\\
&\le {}_HJ_a^\alpha (f-\varphi_1)(t) I_a^\alpha(\varphi_2 - f)(t) + {}_HJ_a^\alpha(\varphi_2 - f)(t) I_a^\alpha(f-\varphi_1)(t)\\
&\quad + \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} {}_HJ_a^\alpha(\varphi_2 f + \varphi_1 f - \varphi_1\varphi_2)(t)\\
&\quad + \frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha(\varphi_2 f + \varphi_1 f - \varphi_1\varphi_2)(t) \qquad (11.41)\\
&\quad + {}_HJ_a^\alpha\varphi_1(t) I_a^\alpha(\varphi_2 - f)(t) + {}_HJ_a^\alpha\varphi_2(t) I_a^\alpha(\varphi_1 - f)(t) - {}_HJ_a^\alpha f(t) I_a^\alpha(\varphi_1+\varphi_2)(t)\\
&= \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} {}_HJ_a^\alpha(\varphi_2 f + \varphi_1 f - \varphi_1\varphi_2)(t)\\
&\quad + \frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha(\varphi_2 f + \varphi_1 f - \varphi_1\varphi_2)(t) - 2\,{}_HJ_a^\alpha f(t) I_a^\alpha f(t)\\
&= K(f,\varphi_1,\varphi_2),
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} {}_HJ_a^\alpha g^2(t) + \frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha g^2(t) - 2\,{}_HJ_a^\alpha g(t) I_a^\alpha g(t)\\
&\le {}_HJ_a^\alpha (g-\psi_1)(t) I_a^\alpha(\psi_2 - g)(t) + {}_HJ_a^\alpha(\psi_2 - g)(t) I_a^\alpha(g-\psi_1)(t)\\
&\quad + \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} {}_HJ_a^\alpha(\psi_2 g + \psi_1 g - \psi_1\psi_2)(t)\\
&\quad + \frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha(\psi_2 g + \psi_1 g - \psi_1\psi_2)(t) \qquad (11.42)\\
&\quad + {}_HJ_a^\alpha\psi_1(t) I_a^\alpha(\psi_2 - g)(t) + {}_HJ_a^\alpha\psi_2(t) I_a^\alpha(\psi_1 - g)(t) - {}_HJ_a^\alpha g(t) I_a^\alpha(\psi_1+\psi_2)(t),\\
&= \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} {}_HJ_a^\alpha(\varphi_2 g + \varphi_1 g - \varphi_1\varphi_2)(t)\\
&\quad + \frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha(\varphi_2 g + \varphi_1 g - \varphi_1\varphi_2)(t) - 2\,{}_HJ_a^\alpha g(t) I_a^\alpha g(t)\\
&= K(g,\psi_1,\psi_2).
\end{aligned}
$$

From (11.40), (11.41) and (11.42), it follows that the required inequality in (11.37) holds true. □

Corollary 11.12 *If $K(f,\varphi_1,\varphi_2) = K(f,m,M)$ and $K(g,\psi_1,\psi_2) = K(g,p,P)$, m, M, p, $P \in \mathbb{R}$, then the inequality (11.37) reduces to the following fractional integral inequality:*

$$\begin{aligned}
&\left|\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}{}_HJ_a^\alpha fg(t) + \frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha fg(t) - {}_HJ_a^\alpha f(t)I_a^\alpha g(t) - I_a^\alpha f(t){}_HJ_a^\alpha g(t)\right| \\
\le &\frac{1}{4}\left\{\left[\left({}_HJ_a^\alpha f(t) - I_a^\alpha f(t) + M\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} - m\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}\right)^2\right.\right. \\
&+\left.\left(I_a^\alpha f(t) - {}_HJ_a^\alpha f(t) + M\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)} - m\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}\right)^2\right]^{1/2} \\
&\times\left[\left({}_HJ_a^\alpha g(t) - I_a^\alpha g(t) + P\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} - p\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}\right)^2\right. \\
&+\left.\left.\left({}_HJ_a^\alpha g(t) - I_a^\alpha g(t) + p\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} - P\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}\right)^2\right]^{1/2}\right\}.
\end{aligned}$$

Theorem 11.15 *Let f and g be two integrable functions on $[a,\infty)$, $a > 0$. Assume that there exist four integrable functions $\varphi_1, \varphi_2, \psi_1$ and ψ_2 satisfying the conditions (11.10.1) and (11.14.1) on $[a,\infty)$. Then, for all $0 < a < t < \infty$ and $\alpha, \beta > 0$, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{(t-a)^\beta}{\Gamma(\beta+1)}{}_HJ_a^\alpha fg(t) + \frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\beta fg(t) - {}_HJ_a^\alpha f(t)I_a^\beta g(t) - I_a^\beta f(t){}_HJ_a^\alpha g(t)\right| \\
&\quad\le |K_1(f,\varphi_1,\varphi_2)|^{1/2}\,|K_1(g,\psi_1,\psi_2)|^{1/2}, \qquad (11.43)
\end{aligned}$$

where $K_1(u,v,w)$ is defined by

$$\begin{aligned}
K_1(u,v,w) = &\frac{(t-a)^\beta}{\Gamma(\beta+1)}{}_HJ_a^\alpha(uw+uv-vw)(t) \\
&+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\beta(uw+uv-vw)(t) - 2{}_HJ_a^\alpha u(t)I_a^\beta u(t).
\end{aligned}$$

Proof Multiplying both sides of (11.38) by $(\log(t/\tau))^{\alpha-1}(t-\rho)^{\beta-1}/(\tau\Gamma(\alpha)\Gamma(\beta))$, $\tau, \rho \in (a,t)$, and then double integrating with respect to τ and ρ from a to t, we get

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}(t-\rho)^{\beta-1}H(\tau,\rho)d\rho\frac{d\tau}{\tau} \\
&= \frac{(t-a)^\beta}{\Gamma(\alpha+1)}{}_HJ_a^\alpha fg(t) + \frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\beta fg(t) - {}_HJ_a^\alpha f(t)I_a^\beta g(t) - I_a^\beta f(t){}_HJ_a^\alpha g(t).
\end{aligned} \qquad (11.44)$$

Applying the Cauchy-Schwarz inequality to double integrals, we have

$$\begin{aligned}
&\left|\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ_a^{\alpha}fg(t)+\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}I_a^{\alpha}fg(t)-{}_HJ_a^{\alpha}f(t)I_a^{\alpha}g(t)-I_a^{\alpha}f(t){}_HJ_a^{\alpha}g(t)\right|\\
\le&\Bigg[\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}(t-\rho)^{\beta-1}f^2(\tau)d\rho\frac{d\tau}{\tau}\\
&+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}(t-\rho)^{\beta-1}f^2(\rho)d\rho\frac{d\tau}{\tau}\\
&-\frac{2}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}(t-\rho)^{\beta-1}f(\tau)f(\rho)d\rho\frac{d\tau}{\tau}\Bigg]^{1/2}\\
&\times\Bigg[\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}(t-\rho)^{\beta-1}g^2(\tau)d\rho\frac{d\tau}{\tau}\\
&+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}(t-\rho)^{\beta-1}g^2(\rho)d\rho\frac{d\tau}{\tau}\\
&-\frac{2}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}(t-\rho)^{\beta-1}g(\tau)g(\rho)d\rho\frac{d\tau}{\tau}\Bigg]^{1/2}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\left|\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}{}_HJ_a^{\alpha}fg(t)+\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}I_a^{\alpha}fg(t)-{}_HJ_a^{\alpha}f(t)I_a^{\alpha}g(t)-I_a^{\alpha}f(t){}_HJ_a^{\alpha}g(t)\right|\\
\le&\left[\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}{}_HJ_a^{\alpha}f^2(t)+\frac{(\log\frac{t}{\tau})^{\alpha}}{\Gamma(\alpha+1)}I_a^{\beta}f^2(t)-2{}_HJ_a^{\alpha}f(t)I_a^{\beta}f(t)\right]^{1/2}\\
&\times\left[\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}{}_HJ_a^{\alpha}g^2(t)+\frac{(\log\frac{t}{\tau})^{\alpha}}{\Gamma(\alpha+1)}I_a^{\beta}g^2(t)-2{}_HJ_a^{\alpha}g(t)I_a^{\beta}g(t)\right]^{1/2}.
\end{aligned}$$

Thus, from Lemma 11.3, we have

$$\begin{aligned}
&\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}{}_HJ_a^{\alpha}f^2(t)+\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}I_a^{\beta}f^2(t)-2{}_HJ_a^{\alpha}f(t)I_a^{\beta}f(t)\\
&\le{}_HJ_a^{\alpha}(f-\varphi_1)(t)I_a^{\beta}(\varphi_2-f)(t)+{}_HJ_a^{\alpha}(\varphi_2-f)(t)I_a^{\beta}(f-\varphi_1)(t)\\
&+\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}{}_HJ_a^{\alpha}(\varphi_2f+\varphi_1f-\varphi_1\varphi_2)(t)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{(\log \frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)} I_a^{\beta}(\varphi_2 f + \varphi_1 f - \varphi_1\varphi_2)(t) \qquad (11.45)\\
&+ {}_HJ_a^{\alpha}\varphi_1(t) I_a^{\beta}(\varphi_2 - f)(t) + {}_HJ_a^{\alpha}\varphi_2(t) I_a^{\beta}(\varphi_1 - f)(t) - {}_HJ_a^{\alpha} f(t) I_a^{\beta}(\varphi_1 + \varphi_2)(t)\\
&= \frac{(t-a)^{\beta}}{\Gamma(\beta+1)} {}_HJ_a^{\alpha}(\varphi_2 f + \varphi_1 f - \varphi_1\varphi_2)(t)\\
&+ \frac{(\log \frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)} I_a^{\beta}(\varphi_2 f + \varphi_1 f - \varphi_1\varphi_2)(t) - 2{}_HJ_a^{\alpha} f(t) I_a^{\beta} f(t)\\
&= K_1(f, \varphi_1, \varphi_2),
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{(t-a)^{\beta}}{\Gamma(\beta+1)} {}_HJ_a^{\alpha} g^2(t) + \frac{(\log \frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)} I_a^{\beta} g^2(t) - 2{}_HJ_a^{\alpha} g(t) I_a^{\beta} g(t)\\
&\leq {}_HJ_a^{\alpha}(g - \psi_1)(t) I_a^{\beta}(\psi_2 - g)(t) + {}_HJ_a^{\alpha}(\psi_2 - g)(t) I_a^{\beta}(g - \psi_1)(t)\\
&+ \frac{(t-a)^{\beta}}{\Gamma(\beta+1)} {}_HJ_a^{\alpha}(\psi_2 g + \psi_1 g - \psi_1\psi_2)(t)\\
&+ \frac{(\log \frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)} I_a^{\beta}(\psi_2 g + \psi_1 g - \psi_1\psi_2)(t) \qquad (11.46)\\
&+ {}_HJ_a^{\alpha}\psi_1(t) I_a^{\beta}(\psi_2 - g)(t) + {}_HJ_a^{\alpha}\psi_2(t) I_a^{\beta}(\psi_1 - g)(t) - {}_HJ_a^{\alpha} g(t) I_a^{\beta}(\psi_1 + \psi_2)(t)\\
&= \frac{(t-a)^{\beta}}{\Gamma(\beta+1)} {}_HJ_a^{\alpha}(\varphi_2 g + \varphi_1 g - \varphi_1\varphi_2)(t)\\
&+ \frac{(\log \frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)} I_a^{\beta}(\varphi_2 g + \varphi_1 g - \varphi_1\varphi_2)(t) - 2{}_HJ_a^{\alpha} g(t) I_a^{\beta} g(t)\\
&= K_1(g, \psi_1, \psi_2).
\end{aligned}
$$

From (11.39), (11.45) and (11.46), we obtain the desired inequality in (11.43). □

Corollary 11.13 *If $K(f, \varphi_1, \varphi_2) = K(f, m, M)$ and $K(g, \psi_1, \psi_2) = K(g, p, P)$, $m, M, p, P \in \mathbb{R}$, then inequality (11.37) reduces to the following fractional integral inequality:*

$$
\begin{aligned}
&\left| \frac{(t-a)^{\beta}}{\Gamma(\beta+1)} {}_HJ_a^{\alpha} fg(t) + \frac{(\log \frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)} I_a^{\beta} fg(t) - {}_HJ_a^{\alpha} f(t) I_a^{\beta} g(t) - I_a^{\beta} f(t) {}_HJ_a^{\alpha} g(t) \right|\\
&\leq \frac{1}{4} \Bigg\{ \Bigg[\left({}_HJ_a^{\alpha} f(t) - I_a^{\beta} f(t) + M \frac{(t-a)^{\beta}}{\Gamma(\beta+1)} - m \frac{(\log \frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)} \right)^2
\end{aligned}
$$

$$+ \left(I_a^\beta f(t) - {}_H J_a^\alpha f(t) + M \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} - m \frac{(t-a)^\beta}{\Gamma(\beta+1)} \right)^2 \Bigg]^{1/2}$$

$$\times \left[\left({}_H J_a^\alpha g(t) - I_a^\alpha g(t) + P \frac{(t-a)^\beta}{\Gamma(\beta+1)} - p \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} \right)^2 \right.$$

$$\left. + \left({}_H J_a^\alpha g(t) - I_a^\beta g(t) + p \frac{(t-a)^\beta}{\Gamma(\beta+1)} - P \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} \right)^2 \right]^{1/2} \Bigg\}.$$

11.4.1 Applications

In this subsection, we demonstrate a method for constructing four bounding functions, and use them to give some estimates for Chebyshev type inequalities involving Riemann-Liouville and Hadamard fractional integrals for two unknown functions.

From the definitions, for $0 < a = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$, we define two notations of sub-integrals for Riemann-Liouville and Hadamard fractional integrals as

$$I_{t_j,t_{j+1}}^\alpha f(T) = \frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} (T-\tau)^{\alpha-1} f(\tau) d\tau, \quad j = 0, 1, \ldots, p. \tag{11.47}$$

and

$${}_H J_{t_j,t_{j+1}}^\alpha f(T) = \frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} \left(\log \frac{T}{\tau} \right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau}, \quad j = 0, 1, \ldots, p. \tag{11.48}$$

Note that

$$\begin{aligned} I_a^\alpha f(T) &= \sum_{j=0}^{p} I_{t_j,t_{j+1}}^\alpha f(T) \\ &= \frac{1}{\Gamma(\alpha)} \int_a^{t_1} (T-\tau)^{\alpha-1} f(\tau) d\tau + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (T-\tau)^{\alpha-1} f(\tau) d\tau \\ &\quad + \cdots + \frac{1}{\Gamma(\alpha)} \int_{t_p}^{T} (T-\tau)^{\alpha-1} f(\tau) d\tau, \end{aligned}$$

and

$$
\begin{aligned}
{}_H J_a^\alpha f(T) &= \sum_{j=0}^{p} J_{t_j,t_{j+1}}^\alpha f(T) \\
&= \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \left(\log \frac{T}{\tau}\right)^{\alpha-1} f(\tau)d\tau + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{T}{\tau}\right)^{\alpha-1} f(\tau)d\tau \\
&\quad + \cdots + \frac{1}{\Gamma(\alpha)} \int_{t_p}^{T} \left(\log \frac{T}{\tau}\right)^{\alpha-1} f(\tau)d\tau.
\end{aligned}
$$

Let u be a unit step function defined by

$$
u(t) = \begin{cases} 1, \ t > 0, \\ 0, \ t \leq 0, \end{cases} \tag{11.49}
$$

and let $u_a(t)$ be the Heaviside unit step function defined by

$$
u_a(t) = u(t-a) = \begin{cases} 1, \ t > a, \\ 0, \ t \leq a. \end{cases} \tag{11.50}
$$

Let φ_1 be a piecewise continuous functions on $[0, T]$ defined by

$$
\begin{aligned}
\varphi_1(t) &= m_1(u_0(t) - u_{t_1}(t)) + m_2(u_{t_1}(t) - u_{t_2}(t)) + \ldots + m_{p+1}u_{t_p}(t) \\
&= m_1 u_0(t) + (m_2 - m_1)u_{t_1}(t) + \ldots + (m_{p+1} - m_p)u_{t_p}(t) \\
&= \sum_{j=0}^{p} (m_{j+1} - m_j)u_{t_j}(t),
\end{aligned} \tag{11.51}
$$

where $m_0 = 0$ and $0 < a = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$.

Analogously, we define the functions φ_2, ψ_1 and ψ_2 as

$$
\varphi_2(t) = \sum_{j=0}^{p} (M_{j+1} - M_j)u_{t_j}(t), \tag{11.52}
$$

$$
\psi_1(t) = \sum_{j=0}^{p} (n_{j+1} - n_j)u_{t_j}(t), \tag{11.53}
$$

$$
\psi_2(t) = \sum_{j=0}^{p} (N_{j+1} - N_j)u_{t_j}(t), \tag{11.54}
$$

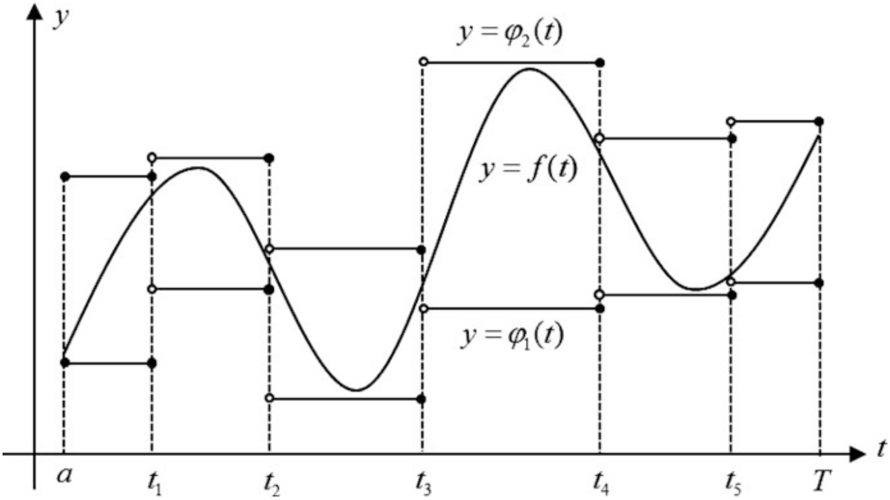

Fig. 11.1 Functions f, φ_1 and φ_2

where $n_0 = N_0 = M_0 = 0$. If there is an integrable function f on $[a, T]$ satisfying condition (11.10.1), then we get $m_{j+1} \leq f(t) \leq M_{j+1}$ for each $t \in (t_j, t_{j+1}]$, $j = 0, 1, 2, \ldots, p$. In particular, for $p = 4$, the time history of f has been shown in Fig. 11.1.

Proposition 11.1 *Let f and g be two integrable functions on $[a, T]$, $a > 0$. Assume that the functions $\varphi_1, \varphi_2, \psi_1$ and ψ_2 defined by (11.51), (11.52), (11.53) and (11.54) respectively, satisfy (11.10.1) and (11.14.1). Then, for $\alpha > 0$, the following inequality holds:*

$$\left|\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}{}_HJ_a^\alpha fg(t) + \frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha fg(t) - {}_HJ_a^\alpha f(t) I_a^\alpha g(t) - I_a^\alpha f(t){}_HJ_a^\alpha g(t)\right| \tag{11.55}$$

$$\leq |K^*(f, \varphi_1, \varphi_2)|^{1/2}\,|K^*(g, \psi_1, \psi_2)|^{1/2},$$

where

$$K^*(u, v, w)(T)$$

$$\leq \frac{(T-a)^\alpha}{\Gamma(\alpha+1)}\sum_{j=0}^{p}\left\{w\ {}_HJ_{t_i,t_{j+1}}^\alpha u(T) + v\ {}_HJ_{t_i,t_{j+1}}^\alpha u(T) - vw\left[\left(\log\frac{T}{t_j}\right)^\alpha - \left(\log\frac{T}{t_{j+1}}\right)^\alpha\right]\right\}$$

$$+ \frac{(\log\frac{T}{a})^\alpha}{\Gamma(\alpha+1)}\sum_{j=0}^{p}\left\{wI_{t_i,t_{j+1}}^\alpha u(T) + vI_{t_i,t_{j+1}}^\alpha u(T) - vw\left[(T-t_j)^\alpha - (T-t_{j+1})^\alpha\right]\right\}$$

$$-2\left(\sum_{j=0}^{p}{}_HJ_{t_i,t_{j+1}}^\alpha u(T)\right)\left(\sum_{j=0}^{p}I_{t_i,t_{j+1}}^\alpha u(T)\right).$$

Proof Since

$$
\begin{aligned}
I^{\alpha}_{t_j,t_{j+1}}(1)(T) &= \frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_{j+1}} (T-\tau)^{\alpha-1}\,d\tau \\
&= \frac{1}{\Gamma(\alpha+1)}\left[\left(T-t_j\right)^{\alpha} - \left(T-t_{j+1}\right)^{\alpha}\right], \\
{}_HJ^{\alpha}_{t_j,t_{j+1}}(1)(T) &= \frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_{j+1}} \left(\log\frac{T}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau} \\
&= \frac{1}{\Gamma(\alpha+1)}\left[\left(\log\frac{T}{t_j}\right)^{\alpha} - \left(\log\frac{T}{t_{j+1}}\right)^{\alpha}\right],
\end{aligned}
$$

we have

$$
\begin{aligned}
I^{\alpha}_{a}(\varphi_1\varphi_2)(T) &= \sum_{j=0}^{p}\frac{m_{j+1}M_{j+1}}{\Gamma(\alpha+1)}\left[\left(T-t_j\right)^{\alpha} - \left(T-t_{j+1}\right)^{\alpha}\right], \\
{}_HJ^{\alpha}_{t_j,t_{j+1}}(\psi_1\psi_2)(T) &= \sum_{j=0}^{p}\frac{n_{j+1}N_{j+1}}{\Gamma(\alpha+1)}\left[\left(\log\frac{T}{t_j}\right)^{\alpha} - \left(\log\frac{T}{t_{j+1}}\right)^{\alpha}\right].
\end{aligned}
$$

Therefore, two functionals $K^*(f,\varphi_1,\varphi_2)(T)$ and $K^*(g,\psi_1,\psi_2)(T)$ can be expressed as

$$
\begin{aligned}
K^*(f,\varphi_1,\varphi_2)(T) \le \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=0}^{p}\Bigg\{ & M_{j+1}J^{\alpha}_{t_i,t_{j+1}}f(T) + m_{j+1}J^{\alpha}_{t_i,t_{j+1}}f(T) \\
& - m_{j+1}M_{j+1}\left[\left(\log\frac{T}{t_j}\right)^{\alpha} - \left(\log\frac{T}{t_{j+1}}\right)^{\alpha}\right]\Bigg\} \\
& + \frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=0}^{p}\Bigg\{M_{j+1}I^{\alpha}_{t_i,t_{j+1}}f(T) + m_{j+1}I^{\alpha}_{t_i,t_{j+1}}f(T) \\
& - m_{j+1}M_{j+1}\left[(T-t_j)^{\alpha} - (T-t_{j+1})^{\alpha}\right]\Bigg\} \\
& - 2\left(\sum_{j=0}^{p}{}_HJ^{\alpha}_{t_i,t_{j+1}}f(T)\right)\left(\sum_{j=0}^{p}I^{\alpha}_{t_i,t_{j+1}}f(T)\right),
\end{aligned}
$$

and

$$K^*(g,\psi_1,\psi_2)(T) \le \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{p} \Bigg\{ N_{j+1} J^\alpha_{t_i,t_{j+1}} g(T) + n_{j+1} J^\alpha_{t_i,t_{j+1}} g(T)$$
$$- n_{j+1} M_{j+1} \left[\left(\log \frac{T}{t_j} \right)^\alpha - \left(\log \frac{T}{t_{j+1}} \right)^\alpha \right] \Bigg\}$$
$$+ \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{p} \Bigg\{ N_{j+1} I^\alpha_{t_i,t_{j+1}} g(T) + n_{j+1} I^\alpha_{t_i,t_{j+1}} g(T)$$
$$- n_{j+1} N_{j+1} \left[(T-t_j)^\alpha - (T-t_{j+1})^\alpha \right] \Bigg\}$$
$$- 2 \left(\sum_{j=0}^{p} {}_H J^\alpha_{t_i,t_{j+1}} g(T) \right) \left(\sum_{j=0}^{p} I^\alpha_{t_i,t_{j+1}} g(T) \right).$$

By applying Theorem 11.14, we obtain the required inequality (11.55). □

Proposition 11.2 *Let f and g be two integrable functions on $[a,T]$, $a > 0$. Assume that the functions $\varphi_1, \varphi_2, \psi_1$ and ψ_2 defined by (11.51), (11.52), (11.53) and (11.54), respectively, satisfy (11.10.1) and (11.14.1). Then, for $\alpha, \beta > 0$, the following inequality holds:*

$$\left| \frac{(t-a)^\beta}{\Gamma(\beta+1)} {}_H J^\alpha_a fg(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I^\beta_a fg(t) - {}_H J^\alpha_a f(t) I^\beta_a g(t) - I^\beta_a f(t) {}_H J^\alpha_a g(t) \right| \tag{11.56}$$
$$\le \left| K_1^*(f,\varphi_1,\varphi_2) \right|^{1/2} \left| K_1^*(g,\psi_1,\psi_2) \right|^{1/2},$$

where

$$K_1^*(u,v,w)(T)$$
$$\le \frac{(T-a)^\beta}{\Gamma(\beta+1)} \sum_{j=0}^{p} \left\{ w\, {}_H J^\alpha_{t_i,t_{j+1}} u(T) + v\, {}_H J^\alpha_{t_i,t_{j+1}} u(T) - vw \left[\left(\log \frac{T}{t_j} \right)^\alpha - \left(\log \frac{T}{t_{j+1}} \right)^\alpha \right] \right\}$$
$$+ \frac{(\log \frac{T}{a})^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{p} \left\{ w I^\beta_{t_i,t_{j+1}} u(T) + v I^\beta_{t_i,t_{j+1}} u(T) - vw \left[(T-t_j)^\beta - (T-t_{j+1})^\beta \right] \right\}$$
$$- 2 \left(\sum_{j=0}^{p} {}_H J^\alpha_{t_i,t_{j+1}} u(T) \right) \left(\sum_{j=0}^{p} I^\beta_{t_i,t_{j+1}} u(T) \right).$$

Proof By direct computations, we have

$$
\begin{aligned}
K_1^*(f,\varphi_1,\varphi_2)(T) \leq\ & \frac{(t-a)^\beta}{\Gamma(\beta+1)} \sum_{j=0}^{p} \Bigg\{ M_{j+1} J^{\alpha}_{t_i,t_{j+1}} f(T) + m_{j+1} J^{\alpha}_{t_i,t_{j+1}} f(T) \\
& - m_{j+1} M_{j+1} \left[\left(\log \frac{T}{t_j} \right)^\alpha - \left(\log \frac{T}{t_{j+1}} \right)^\alpha \right] \Bigg\} \\
& + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{p} \Bigg\{ M_{j+1} I^{\alpha}_{t_i,t_{j+1}} f(T) + m_{j+1} I^{\beta}_{t_i,t_{j+1}} f(T) \\
& - m_{j+1} M_{j+1} \left[(T-t_j)^\beta - (T-t_{j+1})^\beta \right] \Bigg\} \\
& - 2 \left(\sum_{j=0}^{p} {}_H J^{\alpha}_{t_i,t_{j+1}} f(T) \right) \left(\sum_{j=0}^{p} I^{\beta}_{t_i,t_{j+1}} f(T) \right),
\end{aligned}
$$

and

$$
\begin{aligned}
K_1^*(g,\psi_1,\psi_2)(T) \leq\ & \frac{(t-a)^\beta}{\Gamma(\beta+1)} \sum_{j=0}^{p} \Bigg\{ N_{j+1} J^{\alpha}_{t_i,t_{j+1}} g(T) + n_{j+1} J^{\alpha}_{t_i,t_{j+1}} g(T) \\
& - n_{j+1} M_{j+1} \left[\left(\log \frac{T}{t_j} \right)^\alpha - \left(\log \frac{T}{t_{j+1}} \right)^\alpha \right] \Bigg\} \\
& + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{p} \Bigg\{ N_{j+1} I^{\beta}_{t_i,t_{j+1}} g(T) + n_{j+1} I^{\beta}_{t_i,t_{j+1}} g(T) \\
& - n_{j+1} N_{j+1} \left[(T-t_j)^\beta - (T-t_{j+1})^\beta \right] \Bigg\} \\
& - 2 \left(\sum_{j=0}^{p} {}_H J^{\alpha}_{t_i,t_{j+1}} g(T) \right) \left(\sum_{j=0}^{p} I^{\beta}_{t_i,t_{j+1}} g(T) \right).
\end{aligned}
$$

By applying Theorem 11.15, the required inequality (11.56) follows. □

11.5 Certain Chebyshev Type Integral Inequalities Involving Hadamard's Fractional Operators

In this section, we obtain certain new integral inequalities which provide an estimation for the fractional integral of a product of the individual function fractional integrals, involving Hadamard fractional integral operators.

Theorem 11.16 *Let p be a positive function, and f and g be two differentiable functions on $[1,\infty)$. If $f' \in L_r([1,\infty)), g' \in L_s([1,\infty)), r > 1, r^{-1} + s^{-1} = 1$. Then, for all $t > 1$ and $\alpha > 0$,*

$$\begin{aligned}
&2\left|{}_HJ^{\alpha}\{p(t)\}{}_HJ^{\alpha}\{p(t)f(t)g(t)\} - {}_HJ^{\alpha}\{p(t)f(t)\}\,{}_HJ^{\alpha}\{p(t)g(t)\}\right| \\
&\leq \frac{||f'||_r\,||g'||_s}{\Gamma^2(\alpha)}\int_1^t\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\alpha-1}\frac{p(\tau)p(\rho)}{\tau\,\rho}|\tau-\rho|\,d\tau d\rho \\
&\leq \left|\left|f'\right|\right|_r\,\left|\left|g'\right|\right|_s t\left({}_HJ^{\alpha}\{p(t)\}\right)^2.
\end{aligned} \tag{11.57}$$

Proof We define

$$\mathscr{H}(\tau,\rho) = (f(\tau)-f(\rho))\,(g(\tau)-g(\rho))\,, \tag{11.58}$$

and

$$F(t,\tau) = \frac{\left(\log\frac{t}{\tau}\right)^{\alpha-1}}{\tau\,\Gamma\,(\alpha)},\quad \tau\in(1,t)\,,\quad t>1. \tag{11.59}$$

Notice that the function $F(t,\tau)$ remains positive, for all $\tau \in (1,t), t > 1$. Multiplying both sides of (11.58) by $F(t,\tau)p(\tau)$ and integrating with respect to τ from 1 to t, we get

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{p(\tau)}{\tau}\mathscr{H}(\tau,\rho)d\tau \\
&= {}_HJ^{\alpha}\{p(t)f(t)g(t)\} - f(\rho){}_HJ^{\alpha}\{p(t)g(t)\} \\
&-g(\rho){}_HJ^{\alpha}\{p(t)f(t)\} + f(\rho)g(\rho){}_HJ^{\alpha}\{p(t)\}\,.
\end{aligned} \tag{11.60}$$

Next, multiplying both sides of (11.60) by $F(t,\rho)p(\rho)$, and integrating with respect to ρ from 1 to t, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma^2(\alpha)}\int_1^t\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\alpha-1}\frac{p(\tau)p(\rho)}{\tau\,\rho}\mathscr{H}(\tau,\rho)d\tau d\rho \\
&= 2\left({}_HJ^{\alpha}\{p(t)\}{}_HJ^{\alpha}\{p(t)f(t)g(t)\} - {}_HJ^{\alpha}\{p(t)f(t)\}{}_HJ^{\alpha}\{p(t)g(t)\}\right).
\end{aligned} \tag{11.61}$$

In view of (11.58), we have

$$\mathscr{H}(\tau,\rho)=\int_{\tau}^{\rho}\int_{\tau}^{\rho}f'(y)g'(z)dydz.$$

Using the following Hölder's inequality for $r>1$ and $r^{-1}+s^{-1}=1$:

$$\left|\int_{\tau}^{\rho}\int_{\tau}^{\rho}f(y)g(z)dydz\right|\leq\left|\int_{\tau}^{\rho}\int_{\tau}^{\rho}|f(y)|^{r}dydz\right|^{r^{-1}}\left|\int_{\tau}^{\rho}\int_{\tau}^{\rho}|g(z)|^{s}dydz\right|^{s^{-1}},$$

we obtain

$$|\mathscr{H}(\tau,\rho)|\leq\left|\int_{\tau}^{\rho}\int_{\tau}^{\rho}\left|f'(y)\right|^{r}dydz\right|^{r^{-1}}\left|\int_{\tau}^{\rho}\int_{\tau}^{\rho}\left|g'(z)\right|^{s}dydz\right|^{s^{-1}}. \tag{11.62}$$

Since

$$\left|\int_{\tau}^{\rho}\int_{\tau}^{\rho}\left|f'(y)\right|^{r}dydz\right|^{r^{-1}}=|\tau-\rho|^{r^{-1}}\left|\int_{\tau}^{\rho}\left|f'(y)\right|^{r}dy\right|^{r^{-1}}$$

and

$$\left|\int_{\tau}^{\rho}\int_{\tau}^{\rho}\left|g'(z)\right|^{s}dydz\right|^{s^{-1}}=|\tau-\rho|^{s^{-1}}\left|\int_{\tau}^{\rho}\left|g'(z)\right|^{s}dz\right|^{s^{-1}},$$

the inequality (11.62) reduces to

$$|\mathscr{H}(\tau,\rho)|\leq|\tau-\rho|\left|\int_{\tau}^{\rho}\left|f'(y)\right|^{r}dy\right|^{r^{-1}}\left|\int_{\tau}^{\rho}\left|g'(z)\right|^{s}dz\right|^{s^{-1}}. \tag{11.63}$$

Again, it follows from (11.61) that

$$\begin{aligned}
&\frac{1}{\Gamma^{2}(\alpha)}\int_{1}^{t}\int_{1}^{t}\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\alpha-1}\frac{p(\tau)p(\rho)}{\tau\;\rho}|\mathscr{H}(\tau,\rho)|\,d\tau d\rho\\
&\leq\frac{1}{\Gamma^{2}(\alpha)}\int_{1}^{t}\int_{1}^{t}\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\alpha-1}\frac{p(\tau)p(\rho)}{\tau\;\rho}|\tau-\rho|\times\\
&\quad\times\left|\int_{\tau}^{\rho}\left|f'(y)\right|^{r}dy\right|^{r^{-1}}\left|\int_{\tau}^{\rho}\left|g'(z)\right|^{s}dz\right|^{s^{-1}}d\tau d\rho.
\end{aligned} \tag{11.64}$$

Applying Hölder's inequality on the right-hand side of (11.64), we get

$$\frac{1}{\Gamma^{2}(\alpha)}\int_{1}^{t}\int_{1}^{t}\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\alpha-1}\frac{p(\tau)p(\rho)}{\tau\;\rho}|\mathscr{H}(\tau,\rho)|\,d\tau d\rho$$

$$\leq \left[\frac{1}{\Gamma^r(\alpha)} \int_1^t \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \left(\log \frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)p(\rho)}{\tau\ \rho} |\tau - \rho| \ \left|\int_\tau^\rho |f'(y)|^r dy\right| d\tau d\rho\right]^{r^{-1}} \times$$

$$\times \left[\frac{1}{\Gamma^s(\alpha)} \int_1^t \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \left(\log \frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)p(\rho)}{\tau\ \rho} |\tau - \rho| \ \left|\int_\tau^\rho |g'(z)|^s dz\right| d\tau d\rho\right]^{s^{-1}}.$$

In view of the fact that

$$\left|\int_\tau^\rho |f(y)|^p dy\right| \leq ||f||_p^p,$$

we get

$$\begin{aligned} &\frac{1}{\Gamma^2(\alpha)} \int_1^t \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \left(\log \frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)p(\rho)}{\tau\ \rho} |\mathscr{H}(\tau,\rho)| \, d\tau d\rho \\ &\leq \left[\frac{||f'||_r^r}{\Gamma^r(\alpha)} \int_1^t \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \left(\log \frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)p(\rho)}{\tau\ \rho} |\tau - \rho| \ d\tau d\rho\right]^{r^{-1}} \times \\ &\quad \times \left[\frac{||g'||_s^s}{\Gamma^s(\alpha)} \int_1^t \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \left(\log \frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)p(\rho)}{\tau\ \rho} |\tau - \rho| \ d\tau d\rho\right]^{s^{-1}}. \end{aligned} \tag{11.65}$$

From (11.65), we have

$$\begin{aligned} &\frac{1}{\Gamma^2(\alpha)} \int_1^t \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \left(\log \frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)p(\rho)}{\tau\ \rho} |\mathscr{H}(\tau,\rho)| \, d\tau d\rho \\ &\leq \frac{||f'||_r \, ||g'||_s}{\Gamma^2(\alpha)} \left[\int_1^t \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \left(\log \frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)p(\rho)}{\tau\ \rho} |\tau - \rho| \ d\tau d\rho\right]^{r^{-1}} \times \\ &\quad \times \left[\int_1^t \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \left(\log \frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)p(\rho)}{\tau\ \rho} |\tau - \rho| \ d\tau d\rho\right]^{s^{-1}}. \end{aligned} \tag{11.66}$$

Using the relation $r^{-1} + s^{-1} = 1$, the above inequality takes the form:

$$\begin{aligned} &\frac{1}{\Gamma^2(\alpha)} \int_1^t \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \left(\log \frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)p(\rho)}{\tau\ \rho} |\mathscr{H}(\tau,\rho)| \, d\tau d\rho \\ &\quad \leq \frac{||f'||_r \, ||g'||_s}{\Gamma^2(\alpha)} \int_1^t \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \left(\log \frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)p(\rho)}{\tau\ \rho} |\tau - \rho| \ d\tau d\rho. \end{aligned} \tag{11.67}$$

On the other hand, (11.61) gives

$$\begin{aligned} &2 \left| {}_HJ^\alpha \{p(t)\} \ {}_HJ^\alpha \{p(t)f(t)g(t)\} - {}_HJ^\alpha \{p(t)f(t)\} \ {}_HJ^\alpha \{p(t)g(t)\}\right| \\ &\quad \leq \frac{1}{\Gamma^2(\alpha)} \int_1^t \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \left(\log \frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)p(\rho)}{\tau\ \rho} |\mathscr{H}(\tau,\rho)| \, d\tau d\rho. \end{aligned} \tag{11.68}$$

On making use of (11.67) and (11.68), the left-hand side of the inequality (11.57) follows in a straightforward manner.

To establish the right-hand side of the inequality (11.57), we observe that $1 \leq \tau \leq t, \quad 1 \leq \rho \leq t$, and that

$$0 \leq |\tau - \rho| \leq t.$$

Evidently, from (11.67), we have

$$\frac{1}{\Gamma^2(\alpha)} \int_1^t \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \left(\log \frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)p(\rho)}{\tau \ \rho} |\mathscr{H}(\tau,\rho)| \, d\tau d\rho$$

$$\leq \frac{||f'||_r \, ||g'||_s \ t}{\Gamma^2(\alpha)} \int_1^t \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \left(\log \frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)p(\rho)}{\tau \ \rho} \, d\tau d\rho$$

$$= \left\|f'\right\|_r \left\|g'\right\|_s t \left({}_HJ^{\alpha} \{p(t)\}\right)^2,$$

which completes the proof of Theorem 11.16. □

Now, we establish the following integral inequality, which may be regarded as a generalization of Theorem 11.16.

Theorem 11.17 *Let p be a positive function and f and g be two differentiable functions on* $[1, \infty)$. *If* $f' \in L_r([1,\infty)), g' \in L_s([1,\infty)), r > 1, r^{-1} + s^{-1} = 1$, *then*

$$\Big|{}_HJ^{\alpha} \{p(t)\} \ {}_HJ^{\beta} \{p(t)f(t)g(t)\} + {}_HJ^{\beta} \{p(t)\} \ {}_HJ^{\alpha} \{p(t)f(t)g(t)\}$$

$$- {}_HJ^{\alpha} \{p(t)f(t)\} \ {}_HJ^{\beta} \{p(t)g(t)\} - {}_HJ^{\beta} \{p(t)f(t)\} \, {}_HJ^{\alpha} \{p(t)g(t)\}\Big|$$

$$\leq \frac{||f'||_r \, ||g'||_s}{\Gamma(\alpha)\Gamma(\beta)} \int_1^t \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \left(\log \frac{t}{\rho}\right)^{\beta-1} \frac{p(\tau)p(\rho)}{\tau \ \rho} |\tau - \rho| \ d\tau d\rho$$

$$\leq \left\|f'\right\|_r \left\|g'\right\|_s t \, {}_HJ^{\alpha} \{p(t)\} \ {}_HJ^{\beta} \{p(t)\},$$

for all $t > 1, \ \alpha > 0$ *and* $\beta > 0$.

Proof The inequality (11.60) plays a pivotal role in proving this result. Multiplying both sides of (11.60) by $\left(\log \frac{t}{\rho}\right)^{\beta-1} p(\rho)/(\rho \ \Gamma(\beta)), \ \rho \in (1,t), t > 1$, which remains positive, and integrating with respect to ρ from 1 to t, we get

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_1^t \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \left(\log \frac{t}{\rho}\right)^{\beta-1} \frac{p(\tau)p(\rho)}{\tau \ \rho} \mathscr{H}(\tau,\rho) d\tau d\rho \\ &= {}_HJ^{\alpha} \{p(t)\} \ {}_HJ^{\beta} \{p(t)f(t)g(t)\} + {}_HJ^{\beta} \{p(t)\} \ {}_HJ^{\alpha} \{p(t)f(t)g(t)\} \\ &\quad - {}_HJ^{\alpha} \{p(t)f(t)\} \ {}_HJ^{\beta} \{p(t)g(t)\} - {}_HJ^{\beta} \{p(t)f(t)\} \ {}_HJ^{\alpha} \{p(t)g(t)\}. \end{aligned} \tag{11.69}$$

Now using (11.63) in (11.69), we get

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_1^t\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\beta-1}\frac{p(\tau)p(\rho)}{\tau\ \rho}\left|\mathscr{H}(\tau,\rho)\right|d\tau d\rho\\&\leq\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_1^t\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\beta-1}\frac{p(\tau)p(\rho)}{\tau\ \rho}\left|\tau-\rho\right|\times\\&\quad\times\left|\int_\tau^\rho\left|f'(y)\right|^r dy\right|^{r^{-1}}\left|\int_\tau^\rho\left|g'(z)\right|^s dz\right|^{s^{-1}}d\tau d\rho.\end{aligned}\tag{11.70}$$

Applying the Hölder's inequality on the right-hand side of (11.70), we obtain

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_1^t\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\beta-1}\frac{p(\tau)p(\rho)}{\tau\ \rho}\left|\mathscr{H}(\tau,\rho)\right|d\tau d\rho\\&\leq\left[\frac{1}{\Gamma^r(\alpha)}\int_1^t\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\beta-1}\frac{p(\tau)p(\rho)}{\tau\ \rho}\left|\tau-\rho\right|\left|\int_\tau^\rho\left|f'(y)\right|^r dy\right|d\tau d\rho\right]^{r^{-1}}\\&\times\left[\frac{1}{\Gamma^s(\beta)}\int_1^t\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\beta-1}\frac{p(\tau)p(\rho)}{\tau\ \rho}\left|\tau-\rho\right|\left|\int_\tau^\rho\left|g'(z)\right|^s dz\right|d\tau d\rho\right]^{s^{-1}},\end{aligned}$$

or

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_1^t\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\beta-1}\frac{p(\tau)p(\rho)}{\tau\ \rho}\left|\mathscr{H}(\tau,\rho)\right|d\tau d\rho\\&\leq\frac{\|f'\|_r\ \|g'\|_s}{\Gamma(\alpha)\Gamma(\beta)}\int_1^t\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\beta-1}\frac{p(\tau)p(\rho)}{\tau\ \rho}\left|\tau-\rho\right|\ d\tau d\rho.\end{aligned}\tag{11.71}$$

In view of (11.69) and (11.71), and the properties of modulus, one can easily arrive at the left-sided inequality of Theorem 11.17. Moreover, we have $1\leq\tau\leq t,\ \ 1\leq\rho\leq t$, and hence

$$0\leq|\tau-\rho|\leq t.$$

Therefore, from (11.71), we get

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_1^t\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\beta-1}\frac{p(\tau)p(\rho)}{\tau\ \rho}\left|\mathscr{H}(\tau,\rho)\right|d\tau d\rho\\&\leq\frac{\|f'\|_r\ \|g'\|_s}{\Gamma(\alpha)\Gamma(\beta)}\int_1^t\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\times\left(\log\frac{t}{\rho}\right)^{\beta-1}\frac{p(\tau)p(\rho)}{\tau\ \rho}\left|\tau-\rho\right|\ d\tau d\rho\\&\leq\left\|f'\right\|_r\ \left\|g'\right\|_s t\ {}_HJ^\alpha\left\{p(t)\right\}_H J^\beta\left\{p(t)\right\},\end{aligned}$$

which completes the proof of Theorem 11.17.

Remark 11.2 For $\beta=\alpha$, Theorem 11.17 immediately reduces to Theorem 11.16.

11.5.1 Special Cases

As implications of our main results, we consider some consequent results of Theorems 11.16 and 11.17 by suitably choosing the function $p(t)$. For instance, taking $p(t) = (\log t)^{\lambda}(\lambda \in [0,\infty), t \in (1,\infty))$, the following results follow from Theorems 11.16 and 11.17 respectively.

Corollary 11.14 *Let f and g be two differentiable functions on $[1,\infty)$. If $f' \in L_r([1,\infty))$, $g' \in L_s([1,\infty))$, $r > 1, r^{-1} + s^{-1} = 1$, then for all $t > 1$, $\lambda \in [0,\infty)$ and $\alpha > 0$,*

$$
\begin{aligned}
&2\Bigg|\frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda+\alpha)}(\log t)^{\lambda+\alpha}\ {}_HJ^{\alpha}\left\{(\log t)^{\lambda}f(t)g(t)\right\}\\
&\qquad\qquad - {}_HJ^{\alpha}\left\{(\log t)^{\lambda}f(t)\right\}\ {}_HJ^{\alpha}\left\{(\log t)^{\lambda}g(t)\right\}\Bigg|\\
&\le \frac{||f'||_r\,||g'||_s}{\Gamma^2(\alpha)}\int_1^t\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\alpha-1}\frac{(\log\tau)^{\lambda}(\log\rho)^{\lambda}}{\tau\ \rho}\,|\tau-\rho|\ d\tau d\rho\\
&\le ||f'||_r\,||g'||_s\, t\frac{\Gamma^2(1+\lambda)}{\Gamma^2(1+\lambda+\alpha)}(\log t)^{2\lambda+2\alpha}.
\end{aligned}
$$

Corollary 11.15 *Let f and g be two differentiable functions on $[1,\infty)$. If $f' \in L_r([1,\infty))$, $g' \in L_s([1,\infty))$, $r > 1, r^{-1} + s^{-1} = 1$, then*

$$
\begin{aligned}
&\Bigg|\frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda+\alpha)}(\log t)^{\lambda+\alpha}\ {}_HJ^{\beta}\left\{(\log t)^{\lambda}f(t)g(t)\right\}\\
&\qquad + \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda+\beta)}(\log t)^{\lambda+\beta}\ {}_HJ^{\alpha}\left\{(\log t)^{\lambda}f(t)g(t)\right\}\\
&- {}_HJ^{\alpha}\left\{(\log t)^{\lambda}f(t)\right\}\ {}_HJ^{\beta}\left\{(\log t)^{\lambda}g(t)\right\} - {}_HJ^{\beta}\left\{(\log t)^{\lambda}f(t)\right\}_H J^{\alpha}\left\{(\log t)^{\lambda}g(t)\right\}\Bigg|\\
&\le \frac{||f'||_r\,||g'||_s}{\Gamma(\alpha)\Gamma(\beta)}\int_1^t\int_1^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\beta-1}\frac{(\log\tau)^{\lambda}(\log\rho)^{\lambda}}{\tau\ \rho}\,|\tau-\rho|\ d\tau d\rho\\
&\le ||f'||_r\ \ ||g'||_s\, t\,\frac{\Gamma^2(1+\lambda)}{\Gamma(1+\lambda+\alpha)\Gamma(1+\lambda+\beta)}(\log t)^{2\lambda+\alpha+\beta},
\end{aligned}
$$

for all $t > 1$, $\alpha > 0$, $\beta > 0$ and $\lambda \in [0,\infty)$.

Further, if we put $\lambda = 0$ in Corollaries 11.14 and 11.15 (or set $p(t) = 1$ in Theorems 11.16 and 11.17), we obtain the following results:

Corollary 11.16 *Let f and g be two differentiable functions on $[1,\infty)$. If $f' \in L_r([1,\infty))$, $g' \in L_s([1,\infty))$, $r > 1, r^{-1} + s^{-1} = 1$, then for all $t > 1$ and $\alpha > 0$,*

$$
2\left|\frac{(\log t)^{\alpha}}{\Gamma(1+\alpha)}\ {}_HJ^{\alpha}\left\{(f(t)g(t)\right\} - {}_HJ^{\alpha}\left\{f(t)\right\}\ {}_HJ^{\alpha}\left\{(g(t)\right\}\right|
$$

$$\leq \frac{||f'||_r \, ||g'||_s}{\Gamma^2(\alpha)} \int_1^t \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \left(\log \frac{t}{\rho}\right)^{\alpha-1} \frac{|\tau-\rho|}{\tau\,\rho}\, d\tau d\rho$$

$$\leq \left\| f' \right\|_r \left\| g' \right\|_s \frac{t\,(\log t)^{2\alpha}}{\Gamma^2(1+\alpha)}.$$

Corollary 11.17 *Let f and g be two differentiable functions on $[1,\infty)$. If $f' \in L_r([1,\infty)),\ g' \in L_s([1,\infty)),\ r > 1, r^{-1} + s^{-1} = 1$, then*

$$\left| \frac{(\log t)^{\alpha}}{\Gamma(1+\alpha)}\, {}_HJ^{\beta}\, \{(f(t)g(t)\} + \frac{(\log t)^{\beta}}{\Gamma(1+\beta)}\, {}_HJ^{\alpha}\, \{f(t)g(t)\} \right.$$

$$\left. -\, {}_HJ^{\alpha}\, \{f(t)\}\, {}_HJ^{\beta}\, \{(g(t)\} -\, {}_HJ^{\beta}\, \{f(t)\}_H\, J^{\alpha}\, \{g(t)\} \right|$$

$$\leq \frac{||f'||_r \, ||g'||_s}{\Gamma(\alpha)\Gamma(\beta)} \int_1^t \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \left(\log \frac{t}{\rho}\right)^{\beta-1} \frac{|\tau-\rho|}{\tau\,\rho}\, d\tau d\rho$$

$$\leq \left\| f' \right\|_r \left\| g' \right\|_s \frac{t\,(\log t)^{\alpha+\beta}}{\Gamma(1+\alpha)\Gamma(1+\beta)},$$

for all $t > 1,\ \alpha > 0$ and $\beta > 0$.

11.6 Integral Inequalities with "maxima" and Their Applications to Hadamard Type Fractional Differential Equations

Differential equations with "maxima" are a special type of differential equations that contain the maximum of the unknown function over a previous interval. Several integral inequalities have been established in the case when maxima of the unknown scalar function is involved in the integral, for instance, see [39, 92] and references cited therein.

Recently in [158], some new types of integral inequalities on time scales with "maxima" are established, which can be used as a handy tool in the investigation of making estimates for bounds of solutions of dynamic equations on time scales with "maxima". In this section, we establish some new integral inequalities with "maxima" involving Hadamard's integral. The significance of our work lies in the fact that "maxima" are taken on intervals $[\beta t, t]$ which have non-constant length, where $0 < \beta < 1$. In many papers, the "maxima" on $[t-h, t]$, where $h > 0$, is a given constant.

11.6.1 Useful Lemmas

Throughout this subsection, we take $t_0 > 0$. The following results in Lemmas 11.4 and 11.5 are obtained by reducing the time scale $\mathbb{T} = \mathbb{R}, f(t) = g(t) \equiv 1$, and $a(t) = b(t) \equiv 0$ for all $t \in (t_0, T)$ in Theorems 3.3 and 3.2 ([160], pp. 8 and 6), respectively.

Lemma 11.4 ([160]) *Let the following conditions be satisfied:*

(11.4.1) *the functions p and q* $\in C((t_0, T), \mathbb{R}^+)$;
(11.4.2) *the function* $\phi \in C\left([\beta t_0, T), \mathbb{R}^+\right)$ *with* $\max_{s\in[\beta t_0,t_0]} \phi(s) > 0$, *where* $0 < \beta < 1$;
(11.4.3) *the function* $u \in C\left([\beta t_0, T), \mathbb{R}^+\right)$ *and satisfies the inequalities*

$$u(t) \leq \phi(t) + \int_{t_0}^{t} \left[p(s)u(s) + q(s) \max_{\xi\in[\beta s,s]} u(\xi)\right] ds, \quad t \in (t_0, T),$$
$$u(t) \leq \phi(t), \quad t \in [\beta t_0, t_0].$$

Then the following inequality holds:

$$u(t) \leq \phi(t) + h(t) \exp\left(\int_{t_0}^{t} [p(s) + q(s)]ds\right), \qquad t \in (t_0, T),$$

where

$$h(t) = \max_{s\in[\beta t_0,t_0]} \phi(s) + \int_{t_0}^{t} \left[p(s)\phi(s) + q(s) \max_{\xi\in[\beta s,s]} \phi(\xi)\right] ds, \quad t \in (t_0, T).$$

By splitting the initial function ϕ into two functions, we deduce the following corollary.

Corollary 11.18 *Let the following conditions be satisfied:*

(11.18.1) *the functions p, q and v* $\in C((t_0, T), \mathbb{R}^+)$;
(11.18.2) *the function* $w \in C\left([\beta t_0, t_0], \mathbb{R}^+\right)$ *with* $\max_{s\in[\beta t_0,t_0]} w(s) > 0$ *and* $w(t_0) = v(t_0)$, *where* $0 < \beta < 1$;
(11.18.3) *the function* $u \in C\left([\beta t_0, T), \mathbb{R}^+\right)$ *satisfies the inequalities:*

$$u(t) \leq v(t) + \int_{t_0}^{t} \left[p(s)u(s) + q(s) \max_{\xi\in[\beta s,s]} u(\xi)\right] ds, \quad t \in (t_0, T),$$
$$u(t) \leq w(t), \quad t \in [\beta t_0, t_0].$$

Then

$$u(t) \leq v(t) + h(t) \exp\left(\int_{t_0}^{t} [p(s) + q(s)]ds\right), \qquad t \in (t_0, T),$$

where

$$h(t) = \max_{s\in[\beta t_0,t_0]} w(s) + \int_{t_0}^{t} \left[p(s)v(s) + q(s) \max_{\xi\in[\beta s,s]} m(\xi) \right] ds, \quad t \in (t_0, T),$$

with

$$m(t) = \begin{cases} v(t), & t \in (t_0, T), \\ w(t), & t \in [\beta t_0, t_0]. \end{cases}$$

Lemma 11.5 ([160]) *Let the condition (11.4.1) of Lemma 11.4 be satisfied. In addition, assume that:*

(11.5.1) the function $k \in C((t_0, T), (0, \infty))$ is nondecreasing;
(11.5.2) the function $\phi \in C\left([\beta t_0, t_0), \mathbb{R}^+\right)$ for $0 < \beta < 1$;
(11.5.3) the function $u \in C\left([\beta t_0, T), \mathbb{R}^+\right)$ satisfies the inequalities:

$$u(t) \le k(t) + \int_{t_0}^{t} \left[p(s)u(s) + q(s) \max_{\xi\in[\beta s,s]} u(\xi) \right] ds, \quad t \in (t_0, T),$$
$$u(t) \le \phi(t), \quad t \in [\beta t_0, t_0].$$

Then

$$u(t) \le Nk(t) \exp\left(\int_{t_0}^{t} [p(s) + q(s)] ds \right), \qquad t \in (t_0, T),$$

where

$$N = \max\left\{ 1, \frac{\max_{s\in[\beta t_0,t_0]} \phi(s)}{k(t_0)} \right\}.$$

The following lemma is a consequence of Jensen's inequality which can be found in [104].

Lemma 11.6 ([104]) *Let $n \in N$, and let $x_1, \ldots, x_n$ be non-negative real numbers. Then, for $\sigma > 1$,*

$$\left(\sum_{i=1}^{n} x_i \right)^{\sigma} \le n^{\sigma-1} \sum_{i=1}^{n} x_i^{\sigma}.$$

11.6.2 Main Results

Theorem 11.18 *Suppose that the following conditions are satisfied:*

(11.18.1) the functions p and $r \in C((t_0, T), \mathbb{R}^+)$;

(11.18.2) the function $\phi \in C\left([\beta t_0, t_0], \mathbb{R}^+\right)$ *with* $\max_{s\in[\beta t_0,t_0]} \phi(s) > 0$, *where* $0 < \beta < 1$;
(11.18.3) for $\alpha > 0$, *the function* $u \in C\left([\beta t_0, T), \mathbb{R}^+\right)$ *with*

$$u(t) \le r(t) + \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} p(s) \max_{\xi\in[\beta s,s]} u(\xi) \frac{ds}{s}, t \in (t_0, T), \quad (11.72)$$

$$u(t) \le \phi(t), \quad t \in [\beta t_0, t_0]. \quad (11.73)$$

Then, the following assertions hold:

(*i*) *If* $\alpha > \frac{1}{2}$, *then*

$$u(t) \le t\left[c_1 r^2(t) + h_1(t) \exp\left(\frac{2\Gamma(2\alpha-1)}{t}\int_{t_0}^{t} p^2(s)ds\right)\right]^{1/2}, \quad t \in (t_0, T), \quad (11.74)$$

where

$$c_1 = \max\left\{2t_0^{-2}, (\beta t_0)^{-2}\right\}, \quad (11.75)$$

and

$$h_1(t) = c_1 \max_{s\in[\beta t_0,t_0]} \phi^2(s) + \frac{2c_1\Gamma(2\alpha-1)}{t} \times \int_{t_0}^{t} p^2(s) \max_{\xi\in[\beta s,s]} m_1^2(\xi)ds, \quad t \in (t_0, T), \quad (11.76)$$

with

$$m_1(t) = \begin{cases} r(t), & t \in (t_0, T), \\ \phi(t), & t \in [\beta t_0, t_0]. \end{cases} \quad (11.77)$$

In addition, if $r \in C((t_0, T), (0, \infty))$ *is a nondecreasing function, then*

$$u(t) \le \sqrt{c_1 N_1}\, t r(t) \exp\left(\frac{\Gamma(2\alpha-1)}{t}\int_{t_0}^{t} p^2(s)ds\right), \quad t \in (t_0, T), \quad (11.78)$$

where

$$N_1 = \max\left\{1, \frac{\max_{s\in[\beta t_0,t_0]} \phi^2(s)}{r^2(t_0)}\right\}. \quad (11.79)$$

(*ii*) *If* $0 < \alpha \le \dfrac{1}{2}$, *then*

$$u(t) \le t\left[c_2 r^b(t) + h_2(t)\exp\left(\frac{\left[2\Gamma(\alpha^2)\right]^{\frac{1}{\alpha}}}{t}\int_{t_0}^{t} p^b(s)ds\right)\right]^{1/b}, \quad t \in (t_0, T), \tag{11.80}$$

where $b = 1 + \dfrac{1}{\alpha}$,

$$c_2 = \max\left\{2^{\frac{1}{\alpha}} t_0^{-b}, (\beta t_0)^{-b}\right\}, \tag{11.81}$$

and

$$h_2(t) = c_2 \max_{s\in[\beta t_0, t_0]} \phi^b(s) + \frac{c_2\left[2\Gamma(\alpha^2)\right]^{1/\alpha}}{t} \times \int_{t_0}^{t} p^b(s) \max_{\xi\in[\beta s, s]} m_1^b(\xi)ds, \quad t \in (t_0, T). \tag{11.82}$$

Moreover, if $r \in C((t_0, T), (0, \infty))$ *is a nondecreasing function, then*

$$u(t) \le (c_2 N_2)^{\frac{1}{b}} t r(t)\exp\left(\frac{\left[2\Gamma(\alpha^2)\right]^{1/\alpha}}{bt}\int_{t_0}^{t} p^b(s)ds\right), \quad t \in (t_0, T), \tag{11.83}$$

where

$$N_2 = \max\left\{1, \frac{\max_{s\in[\beta t_0, t_0]} \phi^b(s)}{r^b(t_0)}\right\}. \tag{11.84}$$

Proof (*i*) $\alpha > \dfrac{1}{2}$. For $t \in (t_0, T)$, by using the Cauchy-Schwarz inequality in (11.72), we get

$$u(t) \le r(t) + \left[\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{2\alpha-2} ds\right]^{1/2}\left[\int_{t_0}^{t} p^2(s)\left(\max_{\xi\in[\beta s, s]} u(\xi)\right)^2 \frac{ds}{s^2}\right]^{1/2}. \tag{11.85}$$

Observe that

$$\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{2\alpha-2} ds = t\int_0^{\log\frac{t}{t_0}} \tau^{2\alpha-2}e^{-\tau}d\tau < \Gamma(2\alpha-1)t. \tag{11.86}$$

Substituting (11.86) in (11.85), we obtain

$$u(t) \le r(t) + [\Gamma(2\alpha-1)t]^{1/2}\left[\int_{t_0}^{t} p^2(s)\left(\max_{\xi\in[\beta s,s]} u(\xi)\right)^2 \frac{ds}{s^2}\right]^{1/2}.$$

Applying Lemma 11.6 with $n = 2$, $\sigma = 2$, we get the estimate

$$u^2(t) \le 2r^2(t) + 2\Gamma(2\alpha-1)t\int_{t_0}^{t} p^2(s)\left(\max_{\xi\in[\beta s,s]} u(\xi)\right)^2 \frac{ds}{s^2},\quad t\in(t_0,T).$$

Setting $v(t) = t^{-2}u^2(t)$ for $t \in (t_0, T)$, we have

$$\begin{aligned} v(t) &\le 2t^{-2}r^2(t) + \frac{2\Gamma(2\alpha-1)}{t}\int_{t_0}^{t} p^2(s)\left(\max_{\xi\in[\beta s,s]} u(\xi)\right)^2 \frac{ds}{s^2} \\ &\le 2t_0^{-2}r^2(t) + \frac{2\Gamma(2\alpha-1)}{t}\int_{t_0}^{t} p^2(s)\max_{\xi\in[\beta s,s]}\left(\xi^{-2}u^2(\xi)\right)ds \\ &\le c_1 r^2(t) + \frac{2\Gamma(2\alpha-1)}{t}\int_{t_0}^{t} p^2(s)\max_{\xi\in[\beta s,s]} v(\xi)ds, \end{aligned} \tag{11.87}$$

and for $t \in [\beta t_0, t_0]$,

$$v(t) \le t^{-2}\phi^2(t) \le (\beta t_0)^{-2}\phi^2(t) \le c_1\phi^2(t). \tag{11.88}$$

An application of Corollary 11.18 to (11.87) and (11.88) leads to

$$v(t) \le c_1 r^2(t) + h_1(t)\exp\left(\frac{2\Gamma(2\alpha-1)}{t}\int_{t_0}^{t} p^2(s)ds\right),\quad t\in(t_0,T),$$

where c_1 and h_1 are defined by (11.75) and (11.76), respectively. Therefore, we obtain the desired bound in (11.74).

Now, if $r \in C((t_0, T), (0, \infty))$ is a nondecreasing function, then, by Lemma 11.5 with (11.87) and (11.88), it follows that

$$v(t) \le c_1 N_1 r^2(t)\exp\left(\frac{2\Gamma(2\alpha-1)}{t}\int_{t_0}^{t} p^2(s)ds\right),\quad t\in(t_0,T),$$

where N_1 is defined by (11.79). Thus, we get the inequality in (11.78). This completes the proof of the first part.

(*ii*) For the case $0 < \alpha \le \frac{1}{2}$, let $a = 1+\alpha$ and $b = 1 + \frac{1}{\alpha}$. It is obvious that $\frac{1}{a} + \frac{1}{b} = 1$. Using the Hölder's inequality in (11.72), for $t \in (t_0, T)$, we obtain

$$u(t) \le r(t) + \left[\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{a(\alpha-1)} ds\right]^{1/a}\left[\int_{t_0}^{t} p^b(s)\left(\max_{\xi\in[\beta s,s]} u(\xi)\right)^b \frac{ds}{s^b}\right]^{1b}. \tag{11.89}$$

For the first integral in (11.89), repeating the process used to get (11.86), we obtain

$$\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{a(\alpha-1)}ds < \Gamma\left(1-a(1-\alpha)\right)t. \tag{11.90}$$

Obviously, $1-a(1-\alpha)=\alpha^2>0$ and $\Gamma\left(1-a(1-\alpha)\right)\in\mathbb{R}$. Substituting (11.90) in (11.89), we get

$$u(t)\le r(t)+\left[\Gamma(\alpha^2)t\right]^{1/a}\left[\int_{t_0}^{t}p^b(s)\left(\max_{\xi\in[\beta s,s]}u(\xi)\right)^b\frac{ds}{s^b}\right]^{1/b}.$$

Applying Lemma 11.6 with $n=2$, $\sigma=b$, we get the following estimate

$$\begin{aligned}u^b(t)&\le 2^{b-1}r^b(t)+2^{b-1}\left[\Gamma(\alpha^2)t\right]^{b/a}\int_{t_0}^{t}p^b(s)\left(\max_{\xi\in[\beta s,s]}u(\xi)\right)^b\frac{ds}{s^b}\\&=2^{\frac{1}{\alpha}}r^b(t)+\left[2\Gamma(\alpha^2)t\right]^{1/\alpha}\int_{t_0}^{t}p^b(s)\left(\max_{\xi\in[\beta s,s]}u(\xi)\right)^b\frac{ds}{s^b},\quad t\in(t_0,T).\end{aligned}$$

By taking $v(t)=t^{-b}u^b(t)$, we have

$$v(t)\le c_2r^b(t)+\frac{\left[2\Gamma(\alpha^2)\right]^{1/\alpha}}{t}\int_{t_0}^{t}p^b(s)\max_{\xi\in[\beta s,s]}v(\xi)ds,\quad t\in(t_0,T), \tag{11.91}$$

and

$$v(t)\le c_2\phi^b(t),\quad t\in[\beta t_0,t_0]. \tag{11.92}$$

An application of Corollary 11.18 to (11.91) and (11.92) yields

$$v(t)\le c_2r^b(t)+h_2(t)\exp\left(\frac{\left[2\Gamma(\alpha^2)\right]^{1/\alpha}}{t}\int_{t_0}^{t}p^b(s)ds\right),\quad t\in(t_0,T),$$

where c_2 and h_2 are defined by (11.81) and (11.82), respectively. Thus, we get the required inequality in (11.80).

Furthermore, if $r\in C((t_0,T),(0,\infty))$ is a nondecreasing function, then, by applying Lemma 11.5 to (11.92) and (11.93), we get

$$v(t)\le c_2N_2r^b(t)\exp\left(\frac{\left[2\Gamma(\alpha^2)\right]^{1/\alpha}}{t}\int_{t_0}^{t}p^b(s)ds\right),\quad t\in(t_0,T),$$

where N_2 is defined by (11.84). Therefore, the desired inequality in (11.83) is established. This completes the proof. ☐

Theorem 11.19 *Suppose that the conditions (11.18.1) and (11.18.2) are satisfied. In addition, we assume that:*

(11.19.1) the function $q \in C((t_0, T), \mathbb{R}^+)$;
(11.19.2) the function $u \in C\left([\beta t_0, T), \mathbb{R}^+\right)$ *with*

$$u(t) \leq r(t) + \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} \left[p(s)u(s) + q(s) \max_{\xi \in [\beta s, s]} u(\xi)\right] \frac{ds}{s}, \quad t \in (t_0, T), \tag{11.93}$$

$$u(t) \leq \phi(t), \quad t \in [\beta t_0, t_0], \tag{11.94}$$

where $\alpha > 0$.

Then, the following assertions hold:

(*a*) *If* $\alpha > \dfrac{1}{2}$, *then*

$$u(t) \leq t\left[c_3 r^2(t) + h_3(t) \exp\left(\frac{3\Gamma(2\alpha-1)}{t} \int_{t_0}^{t} \left[p^2(s) + q^2(s)\right] ds\right)\right]^{1/2}, t \in (t_0, T), \tag{11.95}$$

where

$$c_3 = \max\left\{3t_0^{-2}, (\beta t_0)^{-2}\right\}, \tag{11.96}$$

and

$$h_3(t) = c_3 \max_{s \in [\beta t_0, t_0]} \phi^2(s) + \frac{3c_3 \Gamma(2\alpha-1)}{t} \times \int_{t_0}^{t} \left[p^2(s) r^2(s) + q^2(s) \max_{\xi \in [\beta s, s]} m_1^2(\xi)\right] ds, \quad t \in (t_0, T), \tag{11.97}$$

with m_1 *defined by (11.77).*

Furthermore, if $r \in C((t_0, T), (0, \infty))$ *is a nondecreasing function, then*

$$u(t) \leq \sqrt{c_3 N_1}\, t r(t) \exp\left(\frac{3\Gamma(2\alpha-1)}{2t} \int_{t_0}^{t} \left[p^2(s) + q^2(s)\right] ds\right), \quad t \in (t_0, T), \tag{11.98}$$

where N_1 *is defined by (11.79).*

(*b*) *If* $0 < \alpha \leq \frac{1}{2}$, *then*

$$u(t) \leq t\left[c_4 r^b(t) + h_4(t)\exp\left(\frac{\left[3\Gamma(\alpha^2)\right]^{1/\alpha}}{t}\int_{t_0}^{t}\left[p^b(s)+q^b(s)\right]ds\right)\right]^{1/b}, t \in (t_0, T), \tag{11.99}$$

where $b = 1 + \frac{1}{\alpha}$,

$$c_4 = \max\left\{3^{1/\alpha} t_0^{-b}, (\beta t_0)^{-b}\right\}, \tag{11.100}$$

and

$$h_4(t) = c_4 \max_{s\in[\beta t_0, t_0]} \phi^b(s) + \frac{c_4\left[3\Gamma(\alpha^2)\right]^{1/\alpha}}{t} \times \int_{t_0}^{t}\left[p^b(s)r^b(s) + q^b(s)\max_{\xi\in[\beta s, s]} m_1^b(\xi)\right]ds, t \in (t_0, T). \tag{11.101}$$

In addition, if $r \in C((t_0, T), (0, \infty))$ *is a nondecreasing function, then*

$$u(t) \leq (c_4 N_2)^{\frac{1}{b}} t r(t)\exp\left(\frac{\left[3\Gamma(\alpha^2)\right]^{1/\alpha}}{bt}\int_{t_0}^{t}\left[p^b(s)+q^b(s)\right]ds\right), \quad t \in (t_0, T), \tag{11.102}$$

where N_2 *is defined by (11.84).*

Proof (*a*) $\alpha > \frac{1}{2}$. Using the Cauchy-Schwarz inequality in (11.93), for $t \in (t_0, T)$, we have

$$\begin{aligned} u(t) &\leq r(t) + \left[\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{2\alpha-2}ds\right]^{1/2}\left[\int_{t_0}^{t} p^2(s)u^2(s)\frac{ds}{s^2}\right]^{1/2} \\ &\quad + \left[\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{2\alpha-2}ds\right]^{1/2}\left[\int_{t_0}^{t} q^2(s)\left(\max_{\xi\in[\beta s, s]} u(\xi)\right)^2\frac{ds}{s^2}\right]^{1/2} \\ &\leq r(t) + [\Gamma(2\alpha-1)t]^{1/2}\left\{\left[\int_{t_0}^{t} p^2(s)u^2(s)\frac{ds}{s^2}\right]^{1/2}\right. \\ &\quad \left. + \left[\int_{t_0}^{t} q^2(s)\left(\max_{\xi\in[\beta s, s]} u(\xi)\right)^2\frac{ds}{s^2}\right]^{1/2}\right\}. \end{aligned}$$

From Lemma 11.6 with $n = 3$, $\sigma = 2$, we get

$$u^2(t) \le 3r^2(t) + 3\Gamma(2\alpha - 1)t\Bigg[\int_{t_0}^{t} p^2(s)u^2(s)\frac{ds}{s^2} + \int_{t_0}^{t} q^2(s)\left(\max_{\xi\in[\beta s,s]} u(\xi)\right)^2 \frac{ds}{s^2}\Bigg], \quad t \in (t_0, T).$$

Setting $v(t) = t^{-2}u^2(t)$, we obtain

$$v(t) \le c_3 r^2(t) + \frac{3\Gamma(2\alpha - 1)}{t}\left[\int_{t_0}^{t} p^2(s)v(s)ds + \int_{t_0}^{t} q^2(s) \max_{\xi\in[\beta s,s]} v(\xi)ds\right], \; t \in (t_0, T), \tag{11.103}$$

and

$$v(t) \le c_3\phi^2(t), \quad t \in [\beta t_0, t_0]. \tag{11.104}$$

Using Corollary 11.18 for (11.103) and (11.104), it follows that

$$v(t) \le c_3 r^2(t) + h_3(t)\exp\left(\frac{3\Gamma(2\alpha - 1)}{t}\int_{t_0}^{t}\left[p^2(s) + q^2(s)\right]ds\right), \quad t \in (t_0, T),$$

where c_3 and h_3 are defined by (11.96) and (11.97), respectively. Therefore, we get the desired inequality in (11.95).

As a special case, if $r \in C((t_0, T), (0, \infty))$ is a nondecreasing function, then by applying Lemma 11.5 with (11.103) and (11.104), we have

$$v(t) \le c_3 N_1 r^2(t)\exp\left(\frac{3\Gamma(2\alpha - 1)}{t}\int_{t_0}^{t}\left[p^2(s) + q^2(s)\right]ds\right), \quad t \in (t_0, T).$$

where N_1 is defined by (11.79). Thus, we get the required inequality in (11.98). This completes the proof of the first part.

(*b*) $0 < \alpha \le \frac{1}{2}$. Let $a = 1 + \alpha$ and $b = 1 + \frac{1}{\alpha}$. Using the Hölder's inequality in (11.93) for $t \in (t_0, T)$, we obtain

$$u(t) \le r(t) + \left[\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{a(\alpha-1)} ds\right]^{1/a}\left\{\left[\int_{t_0}^{t} p^b(s)u^b(s)\frac{ds}{s^b}\right]^{1/b} + \left[\int_{t_0}^{t} q^b(s)\left(\max_{\xi\in[\beta s,s]} u(\xi)\right)^b \frac{ds}{s^b}\right]^{1/b}\right\}$$

$$\leq r(t) + \left[\Gamma(\alpha^2)t\right]^{1/a} \left\{ \left[\int_{t_0}^{t} p^b(s)u^b(s)\frac{ds}{s^b}\right]^{1/b} + \left[\int_{t_0}^{t} q^b(s)\left(\max_{\xi\in[\beta s,s]} u(\xi)\right)^b \frac{ds}{s^b}\right]^{1/b} \right\}.$$

By Lemma 11.6 with $n = 3$, $\sigma = b$, we get

$$u^b(t) \leq 3^{\frac{1}{\alpha}} r^b(t) + \left[3\Gamma(\alpha^2)t\right]^{1/\alpha} \left[\int_{t_0}^{t} p^b(s)u^b(s)\frac{ds}{s^b} + \int_{t_0}^{t} q^b(s)\left(\max_{\xi\in[\beta s,s]} u(\xi)\right)^b \frac{ds}{s^b}\right], \quad t \in (t_0, T).$$

Taking $v(t) = t^{-b}u^b(t)$, we obtain

$$v(t) \leq c_4 r^b(t) + \frac{\left[3\Gamma(\alpha^2)\right]^{1/\alpha}}{t}\left[\int_{t_0}^{t} p^b(s)v(s)ds + \int_{t_0}^{t} q^b(s) \max_{\xi\in[\beta s,s]} v(\xi)ds\right], \ t \in (t_0, T), \tag{11.105}$$

and

$$v(t) \leq c_4\phi^b(t), \quad t \in [\beta t_0, t_0]. \tag{11.106}$$

Applying Corollary 11.18 to (11.105) and (11.106), we have the following estimate

$$v(t) \leq c_4 r^b(t) + h_4(t)\exp\left(\frac{\left[3\Gamma(\alpha^2)\right]^{1/\alpha}}{t}\int_{t_0}^{t}\left[p^b(s)d + q^b(s)\right]ds\right), \ t \in (t_0, T),$$

where c_4 and h_4 are defined by (11.100) and (11.101), respectively. Hence, the result (11.99) is proved.

As a special case, if $r \in C((t_0, T), (0, \infty))$ is a nondecreasing function, then by using Lemma 11.5 together with (11.105) and (11.106), we get

$$v(t) \leq c_4 N_2 r^b(t)\exp\left(\frac{\left[3\Gamma(\alpha^2)\right]^{1/\alpha}}{t}\int_{t_0}^{t}\left[p^b(s)d + q^b(s)\right]ds\right), \ t \in (t_0, T),$$

where N_2 is defined by (11.84). Thus, the required inequality in (11.102) is proved. This completes the proof. □

11.6.3 Applications to Hadamard Fractional Differential Equations with "maxima"

In this subsection, the dependence of solutions on the given orders and the bound for solutions of an initial value problem of Hadamard fractional differential equations are investigated. We consider the following fractional differential equation with "maxima":

$$ {}_HD^{\alpha}y(t) = f\left(t, y(t), \max_{s\in[\beta t,t]} y(s)\right), \quad t \in I = (t_0, T), \tag{11.107} $$

$$ {}_HD^{\alpha-k}y(t)\big|_{t=t_0^+} = \eta_k, \quad k = 1, 2, \ldots, n, \quad n = -[-\alpha], \tag{11.108} $$

and initial function

$$ y(t) = \phi(t), \qquad t \in [\beta t_0, t_0], \tag{11.109} $$

where ${}_HD^{\alpha}$ represents the Hadamard fractional derivative of order α ($\alpha > 0$), $f \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, ϕ is a given continuous function on $[\beta t_0, t_0]$, $0 < \beta < 1$ and $\eta_k,\ k = 1, 2, \ldots, n$ are constants.

The problem (11.107)–(11.109) describes a fractional order model in which some parameters are often involved. The values of these parameters can be measured only up to certain degree of accuracy. Hence, in (11.107)–(11.109), the orders of fractional differential equation α and the initial conditions $\alpha - k$ may be subject to some errors either by necessity or for convenience. Thus, it is important to know how the solution of (11.107)–(11.109) changes when the values of α and $\alpha - k$ are slightly altered.

Theorem 11.20 *Let $\alpha > 0$ and $\delta > 0$ be such that $0 \le n - 1 < \alpha - \delta < \alpha \le n$. Also, let $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption:*

(11.20.1) there exist constants $L_1, L_2 > 0$ such that $|f(t, u_1, u_2) - f(t, v_1, v_2)| \le L_1|u_1 - v_1| + L_2|u_2 - v_2|$, for each $t \in I$ and $u_1, u_2, v_1, v_2 \in \mathbb{R}$.

If y and z are the solutions of the initial value problems (11.107)–(11.109) and

$$ {}_HD^{\alpha-\delta}z(t) = f\left(t, z(t), \max_{s\in[\beta t,t]} z(s)\right), \quad t \in I, \tag{11.110} $$

$$ {}_HD^{\alpha-\delta-k}z(t)\big|_{t=t_0^+} = \overline{\eta}_k, \quad k = 1, 2, \ldots, n, \quad n = -[-(\alpha-\delta)], \tag{11.111} $$

with initial function

$$ z(t) = \overline{\phi}(t), \qquad t \in [\beta t_0, t_0], \tag{11.112} $$

respectively, where $\overline{\eta}_k$ *are constants and* $\overline{\phi}$ *is a given continuous function on* $[\beta t_0, t_0]$ *such that* $\phi(t) \not\equiv \overline{\phi}(t)$ *for all* $t \in [\beta t_0, t_0]$*, then the following estimates hold for* $t_0 < t \leq h < T$*:*

(*I*) *For* $\alpha - \delta > \dfrac{1}{2}$ *and* $t \in I$,

$$|z(t) - y(t)| \leq t\Bigg[c_5 A^2(t) + h_5(t) \times \exp\left(\frac{3\Gamma(2\alpha - 2\delta - 1)(L_1^2 + L_2^2)(t - t_0)}{\Gamma^2(\alpha)t}\right)\Bigg]^{1/2}. \quad (11.113)$$

(*II*) *For* $0 < \alpha - \delta \leq \dfrac{1}{2}$ *and* $t \in I$,

$$|z(t) - y(t)| \leq t\Bigg[c_6 A^b(t) + h_6(t) \times \exp\left(\frac{\left[3\Gamma\left((\alpha - \delta)^2\right)\right]^{\frac{1}{\alpha-\delta}}(L_1^b + L_2^b)(t - t_0)}{\Gamma^b(\alpha)t}\right)\Bigg]^{1/b}, \quad (11.114)$$

where

$$\begin{aligned} A(t) = &\left|\sum_{j=1}^{n} \frac{\overline{\eta}_j}{\Gamma(\alpha - \delta - j + 1)}\left(\log\frac{t}{t_0}\right)^{\alpha-\delta-j} - \sum_{j=1}^{n} \frac{\eta_j}{\Gamma(\alpha - j + 1)}\left(\log\frac{t}{t_0}\right)^{\alpha-j}\right| \\ &+ \left|\left(\log\frac{t}{t_0}\right)^{\alpha-\delta}\left(\frac{1}{\Gamma(\alpha - \delta + 1)} - \frac{1}{(\alpha - \delta)\Gamma(\alpha)}\right)\right| \|f\| \\ &+ \left|\frac{1}{(\alpha - \delta)\Gamma(\alpha)}\left(\log\frac{t}{t_0}\right)^{\alpha-\delta} - \frac{1}{\Gamma(\alpha + 1)}\left(\log\frac{t}{t_0}\right)^{\alpha}\right| \|f\|, \end{aligned} \quad (11.115)$$

$$\|f\| = \sup_{t_0 \leq t \leq h}\left|f\left(t, y(t), \max_{s\in[\beta t, t]} y(s)\right)\right|,$$

$$b = 1 + \frac{1}{\alpha - \delta}, \quad c_5 = \max\left\{3t_0^{-2}, (\beta t_0)^{-2}\right\}, \quad c_6 = \max\left\{3^{\frac{1}{\alpha-\delta}} t_0^{-b}, (\beta t_0)^{-b}\right\},$$

$$\begin{aligned} h_5(t) = &\ c_5 \max_{s\in[\beta t_0, t_0]}\left|\overline{\phi}(s) - \phi(s)\right|^2 \\ &+ \frac{3c_5\Gamma(2\alpha - 2\delta - 1)}{\Gamma^2(\alpha)t}\int_{t_0}^{t}\left(L_1^2 A^2(s) + L_2^2 \max_{\xi\in[\beta s, s]} m_2^2(\xi)\right) ds, \end{aligned}$$

and

$$h_6(t) = c_6 \max_{s\in[\beta t_0,t_0]} \left|\overline{\phi}(s) - \phi(s)\right|^b + \frac{c_6\left[3\Gamma\left((\alpha-\delta)^2\right)\right]^{\frac{1}{\alpha-\delta}}}{\Gamma^b(\alpha)t} \int_{t_0}^{t} \left(L_1^b A^b(s) + L_2^b \max_{\xi\in[\beta s,s]} m_2^b(\xi)\right) ds,$$

with a continuous function $m_2(t)$ defined by

$$m_2(t) = \begin{cases} A(t), & t\in I, \\ \left|\overline{\phi}(t) - \phi(t)\right|, & t\in[\beta t_0, t_0]. \end{cases}$$

Proof The solutions y and z of the initial value problems (11.107)–(11.109) and (11.110)–(11.112) satisfy the equations

$$y(t) = \sum_{j=1}^{n} \frac{\eta_j}{\Gamma(\alpha-j+1)} \left(\log\frac{t}{t_0}\right)^{\alpha-j} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log\frac{t}{s}\right)^{\alpha-1} f\left(s, y(s), \max_{\xi\in[\beta s,s]} y(\xi)\right) \frac{ds}{s},$$

and

$$z(t) = \sum_{j=1}^{n} \frac{\overline{\eta}_j}{\Gamma(\alpha-\delta-j+1)} \left(\log\frac{t}{t_0}\right)^{\alpha-\delta-j} + \frac{1}{\Gamma(\alpha-\delta)} \int_{t_0}^{t} \left(\log\frac{t}{s}\right)^{\alpha-\delta-1} f\left(s, z(s), \max_{\xi\in[\beta s,s]} z(\xi)\right) \frac{ds}{s},$$

respectively. By the assumption (11.20.1), it follows that

$$\begin{aligned} &|z(t) - y(t)| \\ \le &\left|\sum_{j=1}^{n} \frac{\overline{\eta}_j}{\Gamma(\alpha-\delta-j+1)} \left(\log\frac{t}{t_0}\right)^{\alpha-\delta-j} - \sum_{j=1}^{n} \frac{\eta_j}{\Gamma(\alpha-j+1)} \left(\log\frac{t}{t_0}\right)^{\alpha-j}\right| \\ &+ \left|\frac{1}{\Gamma(\alpha-\delta)} \int_{t_0}^{t} \left(\log\frac{t}{s}\right)^{\alpha-\delta-1} f\left(s, z(s), \max_{\xi\in[\beta s,s]} z(\xi)\right) \frac{ds}{s}\right. \\ &\left. - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log\frac{t}{s}\right)^{\alpha-\delta-1} f\left(s, z(s), \max_{\xi\in[\beta s,s]} z(\xi)\right) \frac{ds}{s}\right| \end{aligned}$$

$$
\begin{aligned}
&+\left|\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\alpha-\delta-1} f\left(s, z(s), \max_{\xi\in[\beta s,s]} z(\xi)\right)\frac{ds}{s}\right.\\
&\left.-\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\alpha-\delta-1} f\left(s, y(s), \max_{\xi\in[\beta s,s]} y(\xi)\right)\frac{ds}{s}\right|\\
&+\left|\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\alpha-\delta-1} f\left(s, y(s), \max_{\xi\in[\beta s,s]} y(\xi)\right)\frac{ds}{s}\right.\\
&\left.-\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\alpha-1} f\left(s, y(s), \max_{\xi\in[\beta s,s]} y(\xi)\right)\frac{ds}{s}\right|\\
&\le A(t)+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\alpha-\delta-1}\\
&\quad\times\left(L_1\,|z(s)-y(s)|+L_2\left|\max_{\xi\in[\beta s,s]} z(\xi)-\max_{\xi\in[\beta s,s]} y(\xi)\right|\right)\frac{ds}{s}\\
&\le A(t)+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\log\frac{t}{s}\right)^{\alpha-\delta-1}\\
&\quad\times\left(L_1\,|z(s)-y(s)|+L_2\max_{\xi\in[\beta s,s]}|z(\xi)-y(\xi)|\right)\frac{ds}{s},\quad t\in I,
\end{aligned}
$$

where $A(t)$ is defined by (11.115), and

$$|z(t)-y(t)| = \left|\overline{\phi}(t)-\phi(t)\right|, \qquad t\in[\beta t_0, t_0].$$

Applying Theorem 11.19 yields the desired inequalities (11.113) and (11.114). This completes the proof. □

In the following theorem, we present the upper bounds for the solution of the problem (11.107)–(11.109).

Theorem 11.21 *Assume that:*

(11.21.1) there exist functions $\mu, \nu \in C(I, \mathbb{R}^+)$ *such that for* $t \in I$, $u_1, u_2 \in \mathbb{R}$,

$$\left|f\left(t, u_1, u_2\right)\right| \le \mu(t)\,|u_1| + \nu(t)\,|u_2|. \tag{11.116}$$

If y is solution of the initial value problem (11.107)–(11.109) such that $\phi(t) \not\equiv 0$ for all $t \in [\beta t_0, t_0]$, then the following estimates hold:

(*III*) *Let* $\alpha > \frac{1}{2}$. *Then, for* $t \in I$,

$$|y(t)| \leq t\Bigg[c_3\left(\sum_{j=1}^{n}\frac{|\eta_j|}{\Gamma(\alpha-j+1)}\left(\log\frac{t}{t_0}\right)^{\alpha-j}\right)^2 + h_7(t)\exp\left(\frac{3\Gamma(2\alpha-1)}{\Gamma^2(\alpha)t}\int_{t_0}^{t}\left[\mu^2(s)+\nu^2(s)\right]ds\right)\Bigg]^{1/2}. \quad (11.117)$$

(*IV*) *Let* $0 < \alpha \leq \frac{1}{2}$. *Then, for* $t \in I$,

$$|y(t)| \leq t\Bigg[\frac{c_4|\eta_1|^b}{\Gamma^b(\alpha)}\left(\log\frac{t}{t_0}\right)^{b(\alpha-1)} + h_8(t)\exp\left(\frac{\left[3\Gamma(\alpha^2)\right]^{1/\alpha}}{\Gamma^b(\alpha)t}\int_{t_0}^{t}\left[\mu^2(s)+\nu^2(s)\right]ds\right)\Bigg]^{1/b}, \quad (11.118)$$

where b, c_3, c_4 *are defined as in Theorem 11.19,*

$$h_7(t) = c_3\max_{s\in[\beta t_0,t_0]}\phi^2(s) + \frac{3c_3\Gamma(2\alpha-1)}{\Gamma^2(\alpha)t} \times\int_{t_0}^{t}\left[\mu^2(s)\left(\sum_{j=1}^{n}\frac{|\eta_j|}{\Gamma(\alpha-j+1)}\left(\log\frac{s}{t_0}\right)^{\alpha-j}\right)^2 + \nu^2(s)\max_{\xi\in[\beta s,s]}m_3^2(\xi)\right]ds,$$

and

$$h_8(t) = c_4\max_{s\in[\beta t_0,t_0]}|\phi(s)|^b + \frac{c_4\left[3\Gamma(\alpha^2)\right]^{1/\alpha}}{\Gamma^b(\alpha)t}\int_{t_0}^{t}\left[\frac{|\eta_1|^b\mu^b(s)}{\Gamma^b(\alpha)}\left(\log\frac{s}{t_0}\right)^{b(\alpha-1)} + \nu^b(s)\max_{\xi\in[\beta s,s]}m_3^b(\xi)\right]ds,$$

with a continuous function $m_3(t)$ *defined by*

$$m_3(t) = \begin{cases}\displaystyle\sum_{j=1}^{n}\frac{|\eta_j|}{\Gamma(\alpha-j+1)}\left(\log\frac{t}{t_0}\right)^{\alpha-j}, & t \in I,\\ |\phi(t)|, & t \in [\beta t_0, t_0].\end{cases}$$

Proof The solution y of the initial value problem (11.107)–(11.109) satisfies the following equations

$$\begin{aligned} y(t) &= \sum_{j=1}^{n} \frac{\eta_j}{\Gamma(\alpha - j + 1)} \left(\log \frac{t}{t_0}\right)^{\alpha - j} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha - 1} f\left(s, y(s), \max_{\xi \in [\beta s, s]} y(\xi)\right) \frac{ds}{s}, \ t \in I, \\ y(t) &= \phi(t), \quad t \in [\beta t_0, t_0]. \end{aligned}$$

For $\alpha > 0$, by using the assumption (11.21.1), it follows that

$$\begin{aligned} |y(t)| &\leq \sum_{j=1}^{n} \frac{|\eta_j|}{\Gamma(\alpha - j + 1)} \left(\log \frac{t}{t_0}\right)^{\alpha - j} \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left(\log \frac{t}{s}\right)^{\alpha - 1} \left(\mu(s)|y(s)| + \nu(s) \max_{\xi \in [\beta s, s]} |y(\xi)|\right) \frac{ds}{s}, \quad t \in I, \\ |y(t)| &= |\phi(t)|, \quad t \in [\beta t_0, t_0]. \end{aligned}$$

Then a direct application of Theorem 11.19 yields the estimates in inequalities (11.117) and (11.118). This completes the proof. □

11.7 Notes and Remarks

We have obtained some integral inequalities involving Hadamard fractional integral for integrable functions bounded by integrable functions. Some new inequalities of mixed type for Riemann-Liouville and Hadamard fractional integrals are also established. Then we switch our focus to fractional integral inequalities of Chebyshev type for functions and integrals expressible in product form. This follows new integral inequalities with "maxima" involving Hadamard integral. The papers [124, 148, 150] and [161] are the sources of the work presented in this chapter.

References

1. Agarwal, R.P., Y. Zhou, and Y. He. 2010. Existence of fractional neutral functional differential equations. *Computers and Mathematics with Applications* 59:1095–1100.
2. Agarwal, R.P., and D. O'Regan. 2001. *Infinite Interval Problems for Differential, Difference and Integral Equations*. Dordrecht: Kluwer Academic Publisher.
3. Agarwal, R.P., Y. Zhou, J.R. Wang, and X. Luo. 2011. Fractional functional differential equations with causal operators in Banach spaces. *Mathematical and Computer Modelling* 54:1440–1452.
4. Ahmad, B., and S. Sivasundaram. 2008. Some basic results for fractional functional integro-differential equations. *Communications in Applied Analysis* 12:467–477.
5. Ahmad, B., and P.W. Eloe. 2010. A nonlocal boundary value problem for a nonlinear fractional differential equation with two indices. *Communications on Applied Nonlinear Analysis* 17:69–80.
6. Ahmad, B., and J.J. Nieto. 2011. Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions. *Boundary Value Problems* 2011:36, 9 pp.
7. Ahmad, B., J.J. Nieto, A. Alsaedi, and M. El-Shahed. 2012. A study of nonlinear Langevin equation involving two fractional orders in different intervals. *Nonlinear Analysis: Real World Applications* 13:599–606.
8. Ahmad, B., S.K. Ntouyas, and A. Alsaedi. 2011. New existence results for nonlinear fractional differential equations with three-point integral boundary conditions. *Advances in Difference Equations* Art. ID 107384:11pp.
9. Ahmad, B., S.K. Ntouyas, and A. Alsaedi. 2013. A study of nonlinear fractional differential equations of arbitrary order with Riemann-Liouville type multistrip boundary conditions. *Mathematical Problems in Engineering* Art. ID 320415:9 pp.
10. Ahmad, B., and S.K. Ntouyas. 2014. Initial value problems for hybrid Hadamard fractional differential equations. *Electronic Journal of Differential Equations* 2014(161):1–8.
11. Ahmad, B., and J.J. Nieto. 2009. Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions. *Computers and Mathematics with Applications* 58:1838–1843.
12. Ahmad, B., S.K. Ntouyas, and A. Alsaedi. 2013. New results for boundary value problems of Hadamard-type fractional differential inclusions and integral boundary conditions. *Boundary Value Problems* 2013:275.
13. Ahmad, B., and S.K. Ntouyas. 2016. Hadamard-type fractional functional differential equations and inclusions with retarded and advanced arguments. *Advances in Difference Equations* 2016:80

B. Ahmad et al., *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, DOI 10.1007/978-3-319-52141-1

14. Ahmad, B., and S.K. Ntouyas. 2014. An existence theorem for fractional hybrid differential inclusions of Hadamard type with Dirichlet boundary conditions. *Abstract and Applied Analysis* 2014:Article ID 705809:7 pp.
15. Ahmad, B., and S.K. Ntouyas. 2014. On three-point Hadamard-type fractional boundary value problems. *International Electronic Journal of Pure and Applied Mathematics* 8:31–42.
16. Ahmad, B., and S.K. Ntouyas. 2014. An existence theorem for fractional hybrid differential inclusions of Hadamard type. *Dissc. Math. Diff. Incl. Contr. Optim.* 34:207–218.
17. Ahmad, B., and S.K. Ntouyas. 2015. Initial value problems of fractional order Hadamard-type functional differential equations. *Electronic Journal of Differential Equations* 2015(77):1–9.
18. Ahmad, B., and S.K. Ntouyas. 2015. Nonlocal boundary value problems for hybrid fractional differential equations and inclusions of Hadamard type. *Fractional Differential Calculus* 5:107–123.
19. Ahmad, B., and S.K. Ntouyas. 2016. Initial value problems for functional and neutral functional Hadamard type fractional differential inclusions. *Miskolc Mathematical Notes* 17:15–27.
20. Ahmad, B., and S.K. Ntouyas. 2014. A fully Hadamard-type integral boundary value problem of a coupled system of fractional differential equations. *Fractional Calculus and Applied Analysis* 17:348–360.
21. Ahmad, B., and S.K. Ntouyas. 2014. An existence theorem for fractional hybrid differential inclusions of Hadamard type with Dirichlet boundary conditions. *Abstract and Applied Analysis* 2014:Article ID 705809:7 pp.
22. Ahmad, B., and S.K. Ntouyas. 2015. On Hadamard fractional integro-differential boundary value problems. *Journal of Applied Mathematics and Computing* 47:119–131.
23. Ahmad, B., and S.K. Ntouyas. Boundary value problems of Hadamard-type fractional differential equations and inclusions with nonlocal conditions. *Vietnam Journal of Mathematics* DOI:10.1007/s10013-016-0213-z.
24. Ahmad, B., S.K. Ntouyas, and A. Alsaedi. 2014. Existence theorems for nonlocal multivalued Hadamard fractional integro-differential boundary value problems. *Journal of Inequalities and Applications* 2014:454.
25. Ahmad, B., S.K. Ntouyas, and J. Tariboon. 2015. Existence results for mixed Hadamard and Riemann-Liouville fractional integro-differential equations. *Advances in Difference Equations* 2015:293.
26. Ahmad, B., S.K. Ntouyas, and J. Tariboon. 2016. Existence results for mixed Hadamard and Riemann-Liouville fractional integro-differential inclusions. *Journal Nonlinear Science and Applications* 9:6333–6347.
27. Ahmad, B., S.K. Ntouyas, and J. Tariboon. 2016. A study of mixed Hadamard and Riemann-Liouville fractional integro-differential inclusions via endpoint theory. *Applied Mathematics Letters* 52:9–14.
28. Ahmad, B., S.K. Ntouyas, and A. Alsaedi. 2016. On a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions. *Chaos Solitons Fractals* 83:234–241.
29. Aljoudi, S., B. Ahmad, J.J. Nieto, and A. Alsaedi. 2016. A coupled system of Hadamard type sequential fractional differential equations with coupled strip conditions. *Chaos Solitons Fractals* 91:39–46.
30. Alsaedi, A., S.K. Ntouyas, B. Ahmad, and A. Hobiny. 2015. Nonlinear Hadamard fractional differential equations with Hadamard type nonlocal non-conserved conditions. *Advances in Difference Equations* 2015:285.
31. Altafini, C., A. Speranzon, and K.H. Johansson. 2002. Hybrid control of a truck and trailer vehicle. In *Hybrid Systems: Computation and Control*. Lecture Notes in Computer Science, ed. C.J. Tomlin, and M.R. Greenstreet, vol. 2289. New York: Springer.
32. Alur, R., C. Courcoubetis, and D. Dill. 1990. Model checking for real-time systems. In *Proceedings of the 5th Annual IEEE Symposium on Logic in Computer Science*, Philadelphia, PA, pp. 414–425.

33. Amini-Harandi, A. 2010. Endpoints of set-valued contractions in metric spaces. *Nonlinear Analysis* 72:132–134.
34. Ammi, M., E. El Kinani, and D. Torres. 2012. Existence and uniqueness of solutions to functional integro-differential fractional equations. *Electronic Journal of Differential Equations* 2012(103), 1–9.
35. Anastassiou, G.A. 2009. *Fractional Differentiation Inequalities*. New York: Springer Publishing Company.
36. Atangana, A. 2014. Exact solution of the time-fractional groundwater flow equation within a leaky aquifer equation. *Journal of Vibration and Control* DOI:1077546314544514.
37. Atangana, A., and D. Baleanu. 2014. Modelling the advancement of the impurities and the melted oxygen concentration within the scope of fractional calculus. *International Journal of Non-Linear Mechanics* 67:278–284.
38. Avalishvili, G., M. Avalishvili, and D. Gordeziani. 2011. On integral nonlocal boundary value problems for some partial differential equations. *Bulletin of the Georgian National Academy of Sciences* 5:31–37.
39. Bainov, D., and S. Hristova. 2011. *Differential Equations with Maxima*. Pure and Applied Mathematics. New York: Chapman and Hall/CRC.
40. Baleanu, D., K. Diethelm, E. Scalas, and J.J. Trujillo. 2012. *Fractional Calculus Models and Numerical Methods. Series on Complexity, Nonlinearity and Chaos*. Boston: World Scientific.
41. Baleanu, D., O.G. Mustafa, and R.P. Agarwal. 2011. On L^p-solutions for a class of sequential fractional differential equations. *Applied Mathematics and Computation* 218:2074–2081.
42. Balluchi, A., L. Benvenutti, M. Di Benedetto, C. Pinello, and A. Sangiovanni-Vincentelli. 2000. Automotive engine control and hybrid systems: challenges and opportunities. *Proceedings of the IEEE* 7:888–912.
43. Balluchi, A., P. Soueres, and A. Bicchi. 2001. Hybrid feedback control for path tracking by a bounded-curvature vehicle. In *Hybrid Systems: Computation and Control*. Lecture Notes in Computer Science, ed. M. Di Benedetto, and A.L. Sangiovanni-Vincentelli, vol. 2034. New York: Springer.
44. Belarbi, S., and Z. Dahmani. 2009. On some new fractional integral inequalities. *Journal of Inequalities in Pure and Applied Mathematics* 10(3):Article 86:5 pp.
45. Benchohra, M., J. Henderson, S.K. Ntouyas, and A. Ouahab. 2008. Existence results for fractional order functional differential equations with infinite delay. *Journal of Mathematical Analysis and Applications* 338:1340–1350.
46. Benchohra, M., J. Henderson, S.K. Ntouyas, and A. Ouahab. 2008. Existence results for fractional functional differential inclusions with infinite delay and applications to control theory. *Fractional Calculus and Applied Analysis* 11:35–56.
47. Benchohra, M., J. Henderson, S.K. Ntouyas. 2006. *Impulsive Differential Equations and Inclusions*, vol. 2. New York: Hindawi Publishing Corporation.
48. Bouziani, A. 2001. On the solvability of nonlocal pluriparabolic problems. *Electronic Journal of Differential Equations* 2001(21), 1–16.
49. Boyd, D.W., and J.S.W. Wong. 1969. On nonlinear contractions. *Proceedings of the American Mathematical Society* 20:458–464.
50. Bressan, A., and G. Colombo. 1988. Extensions and selections of maps with decomposable values. *Studia Mathematica* 90:69–86.
51. Butzer, P.L., A.A. Kilbas, and J.J. Trujillo. 2002. Compositions of Hadamard-type fractional integration operators and the semigroup property. *Journal of Mathematical Analysis and Applications* 269:387–400.
52. Butzer, P.L., A.A. Kilbas, and J.J. Trujillo. 2002. Fractional calculus in the Mellin setting and Hadamard-type fractional integrals. *Journal of Mathematical Analysis and Applications* 269:1–27.
53. Butzer, P.L., A.A. Kilbas, and J.J. Trujillo. 2002. Mellin transform analysis and integration by parts for Hadamard-type fractional integrals. *Journal of Mathematical Analysis and Applications* 270:1–15.

54. Byszewski, L., and V. Lakshmikantham. 1991. Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space. *Applicable Analysis* 40:11–19.
55. Byszewski, L. 1991. Theorems about existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. *Journal of Mathematical Analysis and Applications* 162:494–505.
56. Cannon, J.R. 1963. The solution of the heat equation, subject to the specification of energy. *Quarterly of Applied Mathematics* 21:155–160.
57. Castaing, C., and M. Valadier. 1977. *Convex Analysis and Measurable Multifunctions*. Lecture Notes in Mathematics, vol. 580. Berlin-Heidelberg-New York: Springer.
58. Chegis, R.Yu., and R.Yu. Chegis. 1984. Numerical solution of a heat conduction problem with an integral boundary condition. *Litovsk. Mat. Sb.* 24:209–215.
59. Chinchane, V.L., and D.B. Pachpatte. 2013. A note on some integral inequalities via Hadamard integral. *Journal of Fractional Calculus and Applications* 4:1–5.
60. Chinchane, V.L., and D.B. Pachpatte. 2012. On some integral inequalities using Hadamard fractional integral. *Malaya Journal of Matematik* 1:62–66.
61. Čiegis, R., and A. Bugajev. 2012. Numerical approximation of one model of the bacterial self-organization. *Nonlinear Analysis: Modelling and Control* 17:253–270.
62. Coffey, W.T., Yu.P. Kalmykov, and J.T. Waldron. 2004. *The Langevin Equation*, 2nd ed. Singapore: World Scientific.
63. Conti, R. 1967. Recent trends in the theory of boundary value problems for ordinary differential equations. *Bollettino dell'Unione Matematica Italiana* 22:135–178.
64. Covitz, H., and S.B. Nadler Jr. 1970. Multivalued contraction mappings in generalized metric spaces. *Israel Journal of Mathematics* 8:5–11.
65. Dahmani, Z. 2010. On Minkowski and Hermit-Hadamard integral inequalities via fractional integration. *Annals of Functional Analysis* 1:51–58.
66. Dahmani, Z. 2010. New inequalities in fractional integrals. *International Journal of Nonlinear Science* 9:493–497.
67. Dahmani, Z. 2011. The Riemann-Liouville operator to genarate some new inequalities. *International Journal of Nonlinear Science* 12:452–455.
68. Darwish, M.A., and S.K. Ntouyas. 2008. Existence results for a fractional functional differential equation of mixed type. *Communications on Applied Nonlinear Analysis* 15:47–55.
69. Deimling, K. 1992. *Multivalued Differential Equations*. Berlin-New York: Walter De Gruyter.
70. Deimling, K. 1985. *Nonlinear Functional Analysis*. Berlin/Heidelberg: Springer.
71. Denisov, S.I., H. Kantz, and P. Hänggi. 2010. Langevin equation with super-heavy-tailed noise. *Journal of Physics A: Mathematical and Theoretical* 43:10. Art. ID 285004.
72. Denton, Z., and A.S. Vatsala. 2010. Fractional integral inequalities and application. *Computers and Mathematics with Applications* 59:1087–1094.
73. Dhage, B. 2006. Existence results for neutral functional differential inclusions in Banach algebras. *Nonlinear Analysis* 64:1290–1306.
74. Dhage, B.C. 2006. A nonlinear alternative with applications to nonlinear perturbed differential equations. *Nonlinear Studies* 13(4):343–354.
75. Diethelm, K. 2010. *The Analysis of Fractional Differential Equations*. Lecture Notes in Mathematics. Berlin/Heidelberg: Springer.
76. Egbunonu, P., and M. Guay. 2007. Identification of switched linear systems using subspace and integer programming techniques. *Nonlinear Analysis: Hybrid Systems* 1:577–592.
77. El Borai, M., and M. Abbas. 2007. On some integro-differential equations of fractional orders involving Carathéodory nonlinearities. *International Journal of Mathematics* 2:41–52.
78. El-Sayed, A.M.A. 1998. Nonlinear functional differential equations of arbitrary orders. *Nonlinear Analysis* 33:181–186.
79. Engell, S., G. Frehse, and E. Schnieder. 2002. *Lecture Notes In Control and Information Sciences; Modelling, Analysis and Design of Hybrid Systems*. Heidelberg: Springer.

80. Engell, S., S. Kowalewski, C. Schultz, and O. Strusberg. 2000. Continuous-discrete interactions in chemical process plants. *Proceedings of the IEEE* 7:1050–1068.
81. Faieghi, M., S. Kuntanapreeda, H. Delavari, and D. Baleanu. 2013. LMI-based stabilization of a class of fractional-order chaotic systems. *Nonlinear Dynamics* 72:301–309.
82. Frigon, M. 1995. Théorèmes d'existence de solutions d'inclusions différentielles. In *Topological Methods in Differential Equations and Inclusions*. NATO ASI Series C, ed. A. Granas, and M. Frigon, vol. 472. Boston: Kluwer Academic Publishers, pp. 51–87.
83. Gambo, Y.Y., F. Jarad, D. Baleanu, and T. Abdeljawad. 2014. On Caputo modification of the Hadamard fractional derivative. *Advances in Difference Equations* 2014:10.
84. Ge, Z.M., and C.Y. Ou. 2008. Chaos synchronization of fractional order modified Duffing systems with parameters excited by a chaotic signal. *Chaos Solitons Fractals* 35:705–717.
85. Gorniewicz, L. 1999. *Topological Fixed Point Theory of Multivalued Mappings, Mathematics and its Applications*, vol. 495. Dordrecht: Kluwer Academic Publishers.
86. Granas, A., and J. Dugundji. 2003. *Fixed Point Theory*. New York: Springer.
87. Goufo, E.F.D. 2015. A biomathematical view on the fractional dynamics of cellulose degradation. *Fractional Calculus and Applied Analysis* 18:554–564.
88. Guo, D., and V. Lakshmikantham. 1988. *Nonlinear Problems in Abstract Cones*. San Diego: Academic Press.
89. Hadamard, J. 1892. Essai sur l'etude des fonctions donnees par leur developpment de Taylor. *J. Mat. Pure Appl. Ser.* 8:101–186.
90. Hale, J., and S.M. Verduyn Lunel. 1993. *Introduction to Functional Differential Equations*. Applied Mathematical Sciences, vol. 99. New York: Springer.
91. Hilfer, R. 2000. *Applications of Fractional calculus in Physics*. Singapore: World Scientific.
92. Hristova, S., and K. Stefanova. 2011. Some integral inequalities with maximum of the unknown functions. *Advances in Dynamical Systems and Applications* 6:57–69.
93. Hu, Sh., and N. Papageorgiou. 1997. *Handbook of Multivalued Analysis, Theory I*. Dordrecht: Kluwer.
94. Jarad, F., T. Abdeljawad, and D. Baleanu. 2012. Caputo-type modification of the Hadamard fractional derivatives. *Advances in Difference Equations* 2012:142.
95. Jiang, F., and F. Meng. 2007. Explicit bounds on some new nonlinear integral inequalities with delay. *Journal of Computational and Applied Mathematics* 205:479–486.
96. Kilbas, A.A., H.M. Srivastava, and J.J. Trujillo. 2006. *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, vol. 204. Amsterdam: Elsevier Science B.V.
97. Kilbas, A.A. 2001. Hadamard-type fractional calculus. *Journal of the Korean Mathematical Society* 38:1191–1204.
98. Kilbas, A.A., J.J. Trujillo. 2003. Hadamard-type integrals as G-transforms. *Integral Transforms and Special Functions* 14:413–427.
99. Kisielewicz, M. 1991. *Differential Inclusions and Optimal Control*. Dordrecht/The Netherlands: Kluwer.
100. Kolmanovskii, V., and A. Myshkis. 1999. *Introduction to the Theory and Applications of Functional-Differential Equations*. Mathematics and its Applications, vol. 463. Dordrecht: Kluwer Academic Publishers.
101. Krasnoselskii, M.A. 1955. Two remarks on the method of successive approximations. *Uspekhi Matematicheskikh Nauk* 10:123–127.
102. Krasnoselskii, M.A., and P.P. Zabreiko. 1984. *Geometrical Methods of Nonlinear Analysis*. New York: Springer.
103. Krogh, B., and N. Lynch, eds. 2000. *Hybrid Systems: Computation and Control*. Lecture Notes in Computer Science, vol. 1790. New York: Springer.
104. Kuczma, M. 2009. *An Introduction to the Theory of Functional Equations and Inequalities: Cauchy's Equation and Jensen's Inequality*. Boston: Birkhäuser.
105. Lakshmikantham, V., D.D. Bainov, and P.S. Simeonov. 1989. *Theory of Impulsive Differential Equations*. Singapore-London: World Scientific.

106. Lakshmikantham, V. 2008. Theory of fractional functional differential equations. *Nonlinear Analysis* 69:3337–3343.
107. Lakshmikantham, V., S. Leela, and J.V. Devi. 2009. *Theory of Fractional Dynamic Systems*. Cambridge: Cambridge Academic Publishers.
108. Lasota, A., and Z. Opial. 1965. An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations. *Bulletin L'Académie Polonaise des Science, Série des Sciences Mathématiques, Astronomiques et Physiques* 13:781–786.
109. Leggett, R.W., and L.R. Williams. 1980. A fixed point theorem with application to an infectious disease model. *Journal of Mathematical Analysis and Applications* 76:91–97.
110. Liang, S., and J. Zhang. 2011. Existence of multiple positive solutions for m-point fractional boundary value problems on an infinite interval. *Mathematical and Computer Modelling* 54:1334–1346.
111. Lim, S.C., M. Li, and L.P. Teo. 2008. Langevin equation with two fractional orders. *Physics Letters A* 372:6309–6320.
112. Lim, S.C., and L.P. Teo. 2009. The fractional oscillator process with two indices. *Journal of Physics A: Mathematical and Theoretical* 42:34. Art. ID 065208.
113. Lipovan, O. 2000. A retarded Gronwall-like inequality and its applications. *Journal of Mathematical Analysis and Applications* 252:389–401.
114. Liu, X., M. Jia, and W. Ge. 2013. Multiple solutions of a p-Laplacian model involving a fractional derivative. *Advances in Difference Equations* 2013:126.
115. Lizana, L., T. Ambjörnsson, A. Taloni, E. Barkai, and M.A. Lomholt. 2010. Foundation of fractional Langevin equation: Harmonization of a many-body problem. *Physical Review E* 81:8. Art. ID 051118.
116. Lozinski, A., R.G. Owens, and T.N. Phillips. 2011. *The Langevin and Fokker-Planck Equations in Polymer Rheology*. Handbook of Numerical Analysis, vol. 16(C), pp. 211–303.
117. Lygeros, J., D.N. Godbole, and S. Sastry. 1998. A verified hybrid controller for automated vehicles. *IEEE Transactions on Automatic Control* 43:522–539.
118. Mainardi, F., Yu. Luchko, and G. Pagnini. 2001. The fundamental solution of the space-time fractional diffusion equation. *Fractional Calculus and Applied Analysis* 4:153–192.
119. Magin, R.L. 2006. *Fractional Calculus in Bioengineering*. Connecticut: Begell House Publishers.
120. Mesloub, S., and S.A. Messaoudi. 2003. A nonlocal mixed semilinear problem for second-order hyperbolic equations. *Electronic Journal of Differential Equations* 30:1–17.
121. Miller, K.S., and B. Ross. 1993. *An Introduction to the Fractional Calculus and Differential Equations*. New York: Wiley.
122. Nicoud, F., and T. Schfonfeld. 2002. Integral boundary conditions for unsteady biomedical CFD applications. *International Journal for Numerical Methods in Fluids* 40:457–465.
123. Ntouyas, S.K., and M. Obaid. 2012. A coupled system of fractional differential equations with nonlocal integral boundary conditions. *Advances in Difference Equations* 2012:130.
124. Ntouyas, S.K., S.l D. Purohit, and J. Tariboon. 2014. Certain Chebyshev type integral inequalities involving the Hadamard's fractional operators. *Abstract and Applied Analysis* 2014:Article ID 249091:7 pp.
125. Ntouyas, S.K., J. Tariboon, and Ch. Thaiprayoon. 2016. Nonlocal boundary value problems for Riemann-Liouville fractional differential inclusions with Hadamard fractional integral boundary conditions. *Taiwanese Journal Of Mathematics* 20:91–107.
126. Ntouyas, S.K., J. Tariboon, and W. Sudsutad. Boundary value problems for Riemann-Liouville fractional differential inclusions with nonlocal Hadamard fractional integral conditions. *Mediterranean Journal of Mathematics* DOI 10.1007/s00009-015-0543-1.
127. Ntouyas, S.K., J. Tariboon, and W. Sudsutad. Fractional problems for Riemann-Liouville differential equations and inclusions with nonlocal Hadamard fractional integral boundary conditions, Preprint.
128. Ntouyas, S.K., and J. Tariboon. 2016. Fractional integral problems for Hadamard-Caputo fractional Langevin differential inclusions. *Journal of Applied Mathematics and Computing* 51:13–33.

129. Ntouyas, S.K., J. Tariboon, and P. Thiramanus. 2016. Mixed problems of fractional coupled systems of Riemann-Liouville differential equations and Hadamard integral conditions. *Journal Of Applied Analysis* 21:813–828.
130. Povstenko, Y.Z. 2015. *Fractional Thermoelasticity*. New York: Springer.
131. O'Regan, D. 1996. Fixed-point theory for the sum of two operators. *Applied Mathematics Letters* 9:1–8.
132. O'Regan, D. 1994. *Theory of Singular Boundary Value Problems*. River Edge: World Scientific.
133. O'Regan, D., and S. Stanek. 2013. Fractional boundary value problems with singularities in space variables. *Nonlinear Dynamics* 71:641–652.
134. Pachpatte, B.G. 1995. On some generalizations of Bellman's lemma. *Journal of Mathematical Analysis and Applications* 5:141–150.
135. Pachpatte, B.G. 1998. *Inequalities for Differential and Integral Equations*. New York: Academic.
136. Pachpatte, B.G. 2002. Explicit bounds on certain integral inequalities. *Journal of Mathematical Analysis and Applications* 267:48–61.
137. Pachpatte, B.G. 2006. *Integral and Finite Difference Inequalities and Applications*. Netherlands: Elsevier.
138. Pepyne, D.L., and C.G. Cassandras. 2000. Optimal control of hybrid systems in manufacturing. *Proceedings of the IEEE* 7:1108–1123.
139. Petrusel, A. 2001. Fixed points and selections for multivalued operators. *Seminar on Fixed Point Theory Cluj-Napoca* 2:3–22.
140. Petryshyn, W.V., and P.M. Fitzpatric. 1974. A degree theory, fixed point theorems, and mapping theorems for multivalued noncompact maps. *Transactions of the American Mathematical Society* 194:1–25.
141. Podlubny, I. 1999. *Fractional Differential Equations*. San Diego: Academic.
142. Povstenko, Y.Z. 2015. *Fractional Thermoelasticity*. New York: Springer.
143. Pul'kina, L.S. 2004. A nonlocal problem with integral conditions for a hyperbolic equation. *Difference Equations* 40:947–953.
144. Sadovskii, B.N. 1967. On a fixed point principle. *Functional Analysis and its Applications* 1:74–76.
145. Samko, S.G., A.A. Kilbas, and O.I. Marichev. *Fractional Integrals and Derivatives, Theory and Applications*. Yverdon: Gordon and Breach.
146. Samoilenko, A.M., and N.A. Perestyuk. 1995. *Impulsive Differential Equations*. Singapore: World Scientific.
147. Sroysang, B. 2013. A study on Hadamard fractional integral. *International Journal of Mathematical Analysis* 7:1903–1906.
148. Sudsutad, W., S.K. Ntouyas, and J. Tariboon. 2014. Fractional integral inequalities via Hadamard's fractional integral. *Abstract and Applied Analysis* 2014:Article ID 563096:11 pp.
149. Sudsutad, W., S.K. Ntouyas, and J. Tariboon. 2015. Systems of fractional Langevin equation via Riemann-Liouville and Hadamard types and their fractional integral conditions. *Advances in Difference Equations* 2015:235.
150. Sudsutad, W., S.K. Ntouyas, and J. Tariboon. 2016. On mixed type Riemann-Liouville and Hadamard fractional integral inequalities. *Journal of Computational Analysis and Applications* 21:299–314.
151. Su, X. 2009. Boundary value problem for a coupled system of nonlinear fractional differential equations. *Applied Mathematics Letters* 22:64–69.
152. Sun, J., Y. Liu, and G. Liu. 2012. Existence of solutions for fractional differential systems with antiperiodic boundary conditions. *Computers and Mathematics with Applications* 64:1557–1566.
153. Sun, L., J. Sun, and G. Wang. 2013. Generalizations of fixed-point theorems of Altman and Rothe types. *Abstract and Applied Analysis* Article ID 639030:4 pp.

154. Tariboon, J., S.K. Ntouyas, and W. Sudsutad. 2014. Nonlocal Hadamard fractional integral conditions for nonlinear Riemann-Liouville fractional differential equations. *Boundary Value Problems* 2014:253.
155. Tariboon, J., S.K. Ntouyas, and Ch. Thaiprayoon. 2014. Nonlinear Langevin equation of Hadamard-Caputo type fractional derivatives with nonlocal fractional integral conditions. *Advances in Mathematical Physics* 2014:Article ID 372749:15 pp.
156. Tariboon, J., S.K. Ntouyas, and W. Sudsutad. 2016. Coupled systems of Riemann-Liouville fractional differential equations with Hadamard fractional integral boundary conditions. *The Journal of Nonlinear Science and Applications* 9:295–308.
157. Tariboon, J., S.K. Ntouyas, and P. Thiramanus. 2016. Riemann-Liouville fractional differential equations with Hadamard fractional integral conditions. *International Journal of Applied Mathematics and Statistical* 54:119–134.
158. Tariboon, J., P. Thiramanus, and S.K. Ntouyas. 2013. Dynamic integral inequalities on time scales with 'maxima'. *Journal of Inequalities and Applications* 2013:564.
159. Taylor, C., T. Hughes, and C. Zarins. 1998. Finite element modeling of blood flow in arteries. *Computer Methods in Applied Mechanics and Engineering* 158:155–196.
160. Thaiprayoon, C., J. Tariboon, and S.K. Ntouyas. 2014. Separated boundary value problems for second-order impulsive q-integro-difference equations. *Advances in Difference Equations* 2014:88.
161. Thiramanus, P., J. Tariboon, and S.K. Ntouyas. 2014. Integral inequalities with "maxima" and their applications to Hadamard type fractional differential equations. *Journal of Inequalities and Applications* 2014:398.
162. Thiramanus, P., S.K. Ntouyas, and J. Tariboon. 2014. Existence and uniqueness results for Hadamard-type fractional differential equations with nonlocal fractional integral boundary conditions. *Abstract and Applied Analysis* 2014:Article ID 902054:9 pp.
163. Thiramanus, P., S.K. Ntouyas, and J. Tariboon. 2015. Existence of solutions for Riemann-Liouville fractional differential equations with nonlocal Erdélyi-Kober integral boundary conditions on the half-line. *Boundary Value Problems* 2015:196.
164. Thiramanus, P., S.K. Ntouyas, and J. Tariboon. 2016. Positive solutions for Hadamard fractional differential equations on infinite domain. *Advances in Difference Equations* 2016:83.
165. Tolstonogov, A.A. 2000. *Differential Inclusions in a Banach Space*. Dordrecht: Kluwer Academic Publishers.
166. Uranagase, M., and T. Munakata. 2010. Generalized Langevin equation revisited: mechanical random force and self-consistent structure. *Journal of Physics A: Mathematical and Theoretical* 43:11. Art. ID 455003.
167. Vaandrager, F., and J. Van Schuppen, eds. 1999. *Hybrid Systems: Computation and Control*. Lecture Notes in Computer Science, vol. 1569. New York: Springer.
168. Varaiya, P. 1993. Smart cars on smart roads: problems of control. *IEEE Transactions on Automatic Control* 38:195–207.
169. Wang, J., H. Xiang, and Z. Liu. 2010. Positive solution to nonzero boundary values problem for a coupled system of nonlinear fractional differential equations. *International Journal of Difference Equations* 2010:Article ID 186928:12 pp.
170. Wang, G., S. Liu, D. Baleanu, and L. Zhang. 2014. A new impulsive multi-orders fractional differential equation involving multipoint fractional integral boundary conditions. *Abstract and Applied Analysis* 2014:Article ID 932747:10 pp.
171. Wang, J., and Y. Zhang. 2015. On the concept and existence of solutions for fractional impulsive systems with Hadamard derivatives. *Applied Mathematics Letters* 39:85–90.
172. Webb, J.R.L., and G. Infante. 2006. Positive solutions of nonlocal boundary value problems: a unified approach. *Journal of the London Mathematical Society* 74:673–693.
173. Webb, J.R.L., G. Infante, and D. Franco. 2008. Positive solutions of nonlinear fourth order boundary value problems with local and nonlocal boundary conditions. *Proceedings of the Royal Society of Edinburgh Section A* 138:427–446.
174. Womersley, J.R. 1955. Method for the calculation of velocity, rate of flow and viscous drag in arteries when the pressure gradient is known. *The Journal of Physiology* 127:553–563.

175. Yu, C., and G. Gao. 2004. Some results on a class of fractional functional differential equations. *Communications on Applied Nonlinear Analysis* 11:67–75.
176. Yukunthorn, W., S. Suanta, S.K. Ntouyas, and J. Tariboon. 2015. Boundary value problems for impulsive multi-order Hadamard fractional differential equations. *Boundary Value Problems* 2015:148.
177. Yukunthorn, W., B. Ahmad, S.K. Ntouyas, and J. Tariboon. 2016. On Caputo-Hadamard type fractional impulsive hybrid systems with nonlinear fractional integral conditions. *Nonlinear Analysis: Hybrid Systems* 19:77–92.
178. Zeidler, E. 1986. *Nonlinear Functional Analysis and Its Application: Fixed Point-Theorems*, vol. 1. New York: Springer.
179. Zhang, F., G. Chen, C. Li, and J. Kurths. 2013. Chaos synchronization in fractional differential systems. *Philosophical Transactions of the Royal Society A* 371:20120155.
180. Zhang, L., B. Ahmad, G. Wang, and R.P. Agarwal. 2013. Nonlinear fractional integro-differential equations on unbounded domains in a Banach space. *Journal of Computational and Applied Mathematics* 249:51–56.
181. Zhang, L., B. Ahmad, and G. Wang. 2015. Successive iterations for positive extremal solutions of nonlinear fractional differential equations on a half-line. *Bulletin of the Australian Mathematical Society* 91:116–128.
182. Zhang, L., B. Ahmad, and G. Wang. 2016. Existence and approximation of positive solutions for nonlinear fractional integro-differential boundary value problems on an unbounded domain. *Applied and Computational Mathematics* 15:149–158.
183. Zaslavsky, G.M. 2002. Chaos, fractional kinetics, and anomalous transport. *Physics Reports* 371:461–580.
184. Zhang, L., B. Ahmad, G. Wang, R. Agarwal, M. Al-Yami, and W. Shammakh. 2013. Nonlocal integrodifferential boundary value problem for nonlinear fractional differential equations on an unbounded domain. *Abstract and Applied Analysis* 2013:Article ID 813903:5 pp.
185. Zhao, X., and W. Ge. 2010. Unbounded solutions for a fractional boundary value problems on the infinite interval. *Acta Applicandae Mathematicae* 109:495–505.
186. Zhao, Y., S. Sun, Z. Han, and Q. Li. 2011. Theory of fractional hybrid differential equations. *Computers and Mathematics with Applications* 62:1312–1324.
187. Zhou, Y., C. Ionescu, and J.A.T. Machado. 2015. Fractional dynamics and its applications. *Nonlinear Dynamics* 80:1661–1664.

Index

B. Ahmad et al., *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, DOI 10.1007/978-3-319-52141-1