Homework 1 of Optimization Theory and Algorithm (2th Week)-2024"

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Reference Textbook: [1] "Stephen Boyd and Lieven Vandenberghe, "Convex optimization", 2th Edition, 2013.

[2] 刘浩洋, 户将, 李勇锋, 文再文, 最优化: 建模, 算法与理论, 高等教育出版社,2020.

1. (30 points)(Exercise 2.10 of the textbook [1]) Solution set of a quadratic inequality. Let $C \subseteq \mathbf{R}^n$ be the solution set of a quadratic inequality,

$$C = \left\{x \in \mathbf{R}^n \mid x^T A x + b^T x + c \leq 0\right\},$$

with $A \in \mathbf{S}^n, b \in \mathbf{R}^n$, and $c \in \mathbf{R}$. (a) Show that C is convex if $A \succeq 0$. (b) Show that the intersection of C and the hyperplane defined by $g^T x + h = 0$ (where $g \neq 0$) is convex if $A + \lambda g g^T \succeq 0$ for some $\lambda \in \mathbf{R}$. Are the converses of these statements true?

2.(30 points) (Exercise 2.18 of the textbook [1]) Invertible linear-fractional functions. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be the linear-fractional function

$$f(x) = (Ax + b)/\left(c^T x + d\right), \quad \operatorname{dom} f = \left\{x \mid c^T x + d > 0\right\}.$$

Suppose the matrix

$$Q = \begin{bmatrix} A & b \\ c^T & d \end{bmatrix}$$

is nonsingular. Show that f is invertible and that f^{-1} is a linear-fractional mapping. Give an explicit expression for f^{-1} and its domain in terms of A, b, c, and d. Hint. It may be easier to express f^{-1} in terms of Q.

3. (40 points) convex hull/Sparse Representation of a Polytope:

Please give the Sparse Representation of a Polytope $\mathcal{A} = \{\mathbf{x} \in \mathbb{R}^{\mathbf{n}} : \|\mathbf{x}\|_\infty \leq \theta, \|\mathbf{x}\|_{\mathbf{1}} \leq \mathbf{s}\theta\}$.

Reference is as follows:

[2]T. T. Cai and A. Zhang, Sparse representation of a polytope and recovery of sparse signals and low-rank matrices, IEEE Trans. Inform. Theory, 60 (2014), pp. 122–132.

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